AUTOMATA IN CATEGORIES AND REGULAR LANGUAGES

By

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§ 0. Introduction

Since the investigations on categorical automata theory by M. A. Arbib and E. G. Manes [2], H. Ehrig et al. [6], and many other authors, category theory has been playing more and more important roles in several fields of computer science. Arbib and Manes [2] have successfully unified sequential machines, linear systems, tree automata, stochastic automata and so on, by means of a notion of input processes, and as an application of their frameworks of categorical automata theory they have proposed a few postulates for the (coequalizer-reachable) minimal realization problem of input process machines via categorical observations of Nerode equivalences.

In this paper we consider automata in general categories with mechanisms accepting languages over a given alphabet and we study classes of languages accepted by some classes of these automata. In §1, we define automata in categories and some elementary notions needed later, and we list examples which we discuss later in detail. In §2, we show a fundamental fact (Lemma 2.1) that, whenever there is an adjunction $F \dashv G: \mathbf{X} \rightarrow \mathbf{A}$, two categories $\mathbf{X}$ and $\mathbf{A}$ have equivalent automata each other. This is a generalization of a principle of so-called subset-construction assigning a nondeterministic automaton to a deterministic automaton with same ability. Based on a different formulation of the principle, Ehrig et al. [6] have extensively developed theory of automata in pseudoclosed categories. In §3, we devote to study languages accepted by automata in the category $\mathbf{Ab}$ of abelian groups. In the first half of the section, we particularly consider languages accepted by automata with cyclic groups as their state objects and in the second half we determine the class of languages accepted by automata with finitely generated abelian groups as their state objects.

§1. Automata in categories

Let Σ be an alphabet, that is, a finite set and it is fixed throughout this paper. Let $\mathbf{K}$ be a category and let I, Y be fixed objects of $\mathbf{K}$, called an *input object* and an *output object*, respectively. An *automaton* M in $\mathbf{K}$ is a quartet $(Q, \delta_\Sigma, \tau, \beta)$ such that Q is an object of $\mathbf{K}$ called the *state object* of M , δ_Σ is a family $\{\delta_\sigma: Q \rightarrow Q; \sigma \in \Sigma\}$ of endomorphisms of Q called the Σ -*dynamics* of M , τ is a morphism $\tau: I \rightarrow Q$ called the *input morphism* of M , and β is a morphism $\beta: Q \rightarrow Y$ called the *output morphism* of M . An automaton M in $\mathbf{K}$ is rendered by the diagram

$$M: \quad \begin{array}{c} \delta_\sigma \\ \curvearrowright \\ I \xrightarrow{\tau} Q \xrightarrow{\beta} Y \end{array} \quad (\sigma \in \Sigma).$$

Let Σ^* be the set of all finite strings of symbols of Σ , that is, the free monoid generated by Σ . For an automaton M in $\mathbf{K}$, we define an endomorphism $\delta_w: Q \rightarrow Q$ ($w \in \Sigma^*$) to be the identity morphism 1_Q if w is the null string $\lambda \in \Sigma^*$ and to be the composite $\delta_{\sigma_n} \cdots \delta_{\sigma_2} \cdot \delta_{\sigma_1}: Q \rightarrow Q$ if $w = \sigma_1 \sigma_2 \cdots \sigma_n$ with $\sigma_i \in \Sigma$ ($i = 1, 2, \dots, n$). We denote the class of all objects of $\mathbf{K}$ by $|\mathbf{K}|$ and let $\mathcal{F}$ be a subclass of $|\mathbf{K}|$. We say an automaton M in $\mathbf{K}$ to be an $\mathcal{F}$ -*automaton* in $\mathbf{K}$ if the state object Q of M is in $\mathcal{F}$. We assume $\mathbf{K}$ is equipped with a specified morphism $t: I \rightarrow Y$ of the input object I to the output object Y , called a *final morphism* of $\mathbf{K}$. Then the *language* $T_t(M)$ accepted by an automaton $M = (Q, \delta_\Sigma, \tau, \beta)$ in $\mathbf{K}$ is defined as follows:

$$T_t(M) = \{w \in \Sigma^*; \beta \cdot \delta_w \cdot \tau = t\}.$$

We define $L_\Sigma(\mathbf{K}, \mathcal{F}, t)$ to be the set of all languages L over Σ such that $L = T_t(M)$ for some $\mathcal{F}$ -automaton M in $\mathbf{K}$, and define $L_\Sigma(\mathbf{K}, t) = L_\Sigma(\mathbf{K}, |\mathbf{K}|, t)$. A general problem is to determine the sets $L_\Sigma(\mathbf{K}, t)$ and $L_\Sigma(\mathbf{K}, \mathcal{F}, t)$ for a given subclass $\mathcal{F}$ of $|\mathbf{K}|$ and a given final morphism $t: I \rightarrow Y$ in a category $\mathbf{K}$. In this paper we give a few sufficient conditions for $L_\Sigma(\mathbf{K}, \mathcal{F}, t)$ to be identical with the family $\text{Reg}(\Sigma)$ of all regular languages [8] over Σ and for $L_\Sigma(\mathbf{K}, t)$ to be identical with the family $L(\Sigma)$ of all languages over Σ . (Namely, $L(\Sigma)$ is the power set 2^{Σ^*} of Σ^* .)

Let L be a language over Σ and $u, v \in \Sigma^*$. A language $u^{-1}Lv^{-1}$ is defined by $u^{-1}Lv^{-1} = \{w \in \Sigma^*; uwv \in L\}$. The following proposition is one of the most general property of $L_\Sigma(\mathbf{K}, \mathcal{F}, t)$.

PROPOSITION 1.1. If $L \in L_{\Sigma}(\mathbf{K}, \mathcal{F}, t)$, then $u^{-1}Lv^{-1} \in L_{\Sigma}(\mathbf{K}, \mathcal{F}, t)$ for any $u, v \in \Sigma^*$.

PROOF. Let $L = T_t(M)$ for some $\mathcal{F}$ -automaton $M = (Q, \delta_{\Sigma}, \tau, \beta)$ in $\mathbf{K}$, i.e.

Consider a new $\mathcal{F}$ -automaton $M' = (Q, \delta_{\Sigma}, \tau', \beta')$ in $\mathbf{K}$ with $\tau' = \delta_u \cdot \tau$ and $\beta' = \beta \cdot \delta_v$, i.e.

Then it follows that

$$\begin{aligned} w &\in u^{-1}Lv^{-1} \\ \iff uvw &\in L \\ \iff \beta \cdot \delta_{uvw} \cdot \tau &= t \quad (\text{since } L = T_t(M)) \\ \iff (\beta \cdot \delta_v) \cdot \delta_w \cdot (\delta_u \cdot \tau) &= t \\ \iff w &\in T_t(M'), \end{aligned}$$

which proves that $u^{-1}Lv^{-1} = T_t(M')$. This completes the proof. $\square$

Let M, M' be two automata in $\mathbf{K}$:

A simulation $f: M \rightarrow M'$ is a morphism $f: Q \rightarrow Q'$ of $\mathbf{K}$ making the diagram

commute for each $\sigma \in \Sigma$. The following proposition is immediate from the definitions.

PROPOSITION 1.2. *Let M, M' be two automata in $\mathbf{K}$ and $t: I \rightarrow Y$ a final morphism of $\mathbf{K}$. If there is a simulation $f: M \rightarrow M'$, then $T_t(M) = T_t(M')$. $\square$*

We now give some examples.

EXAMPLE (i). We denote by $\mathcal{F}_0$ the class of all finite sets and by **Set** the category of sets and maps. Let $I_0 = \{0\}$, $Y_0 = \{0, 1\}$ and let a final morphism $t_0: I_0 \rightarrow Y_0$ be a map t_0 with $t_0(0) = 1$. We then have $L_\Sigma(\mathbf{Set}, \mathcal{F}_0, t_0) = \text{Reg}(\Sigma)$ and $L_\Sigma(\mathbf{Set}, t_0) = L(\Sigma)$. This is a model of our theory.

EXAMPLE (ii). Let **1** be the category with only one object 0 and only one morphism, namely the identity morphism 1_0 of 0. Then a final morphism t is necessarily 1_0 and it is obvious that $L_\Sigma(\mathbf{1}, t) = \{\Sigma^*\}$, since only one automaton M in **1** accepts Σ^* .

EXAMPLE (iii). Let **A** be one of the following categories: **Top** the category of topological spaces and continuous maps, **CompHaus** the category of compact Hausdorff spaces and continuous maps, **Poset** the category of partially ordered sets and order-preserving maps, **Graph** the category of directed graphs, and so on. In these situations we call an object of **A** a finite object if its underlying set is finite. Let $\mathcal{F}$ be the class of all finite objects of **A** and a final morphism $t: I = \{0\} \rightarrow Y = \{0, 1\}$ a map with $t(0) = 1$. (Notice that I and Y can be regarded as objects of the category **A** concerned.) We will prove $L_\Sigma(\mathbf{A}, t) = L(\Sigma)$ by Corollary 2.3(a) and $L_\Sigma(\mathbf{A}, \mathcal{F}, t) = \text{Reg}(\Sigma)$ by Theorem 2.4.

EXAMPLE (iv). Let **Ab** be the category of abelian groups and homomorphisms, $\mathcal{F}$ the class of all finite abelian groups, and $\mathcal{F}'$ the class of all finitely generated abelian groups. Let I be the infinite cyclic group $\mathbb{Z}$, Y the cyclic group $\mathbb{Z}_2$ of order 2, and a final morphism $\pi_0: \mathbb{Z} \rightarrow \mathbb{Z}_2$ a (unique) epimorphism. We will prove $L_\Sigma(\mathbf{Ab}, \pi_0) = L(\Sigma)$ in Corollary 2.3(a), $L_\Sigma(\mathbf{Ab}, \mathcal{F}, \pi_0) = \text{Reg}(\Sigma)$ in Theorem 2.7 and $L_\Sigma(\mathbf{Ab}, \mathcal{F}', \pi_0) = \text{Reg}(\Sigma)$ in Theorem 3.3.

EXAMPLE (v). Let **Gr** be the category of groups and homomorphisms and $\mathcal{F}$ the class of all finite groups. Let $I, Y, t: I \rightarrow Y$ be the same as in Example (iv). It will be proved that $L_\Sigma(\mathbf{Gr}, \pi_0) = L(\Sigma)$ in Corollary 2.3(a) and $L_\Sigma(\mathbf{Gr}, \mathcal{F}, \pi_0) = \text{Reg}(\Sigma)$ in Corollary 2.10.

EXAMPLE (vi). Let **Toset** be the category of totally ordered sets and order-

preserving maps and $\mathcal{F}$ the class of all finite totally ordered sets. Let $I = \{0\}$, $Y = \{0, 1\}$ with $0 < 1$ and $t: I \rightarrow Y$ a map with $t(0) = 1$. We can easily show that if $\lambda \in T_t(M)$ for an automaton M in **Toset** then $T_t(M)^+ = T_t(M)$, that is, $v, w \in T_t(M)$ implies $vw \in T_t(M)$. (For it is clear that $\tau(0) < \delta_u \cdot \tau(0)$ in state object Q of M , since $\lambda \notin T_t(M)$, $\beta \cdot \delta_u \cdot \tau(0) = \beta \cdot \delta_v \cdot \tau(0) = 1$ and Q is totally ordered, and hence we have $1 = \beta \cdot \delta_v \cdot \tau(0) \leq \beta \cdot \delta_{uv} \cdot \tau(0)$ because $\beta \cdot \delta_v: I \rightarrow Q$ is monotone.) This clearly indicates that $L_X(\mathbf{Toset}, t) \subsetneq L(\Sigma)$ and $L_X(\mathbf{Toset}, \mathcal{F}, t) \subsetneq \text{Reg}(\Sigma)$, because a finite language is regular.

§2. Adjunction

Let $F \dashv G: \mathbf{X} \rightarrow \mathbf{A}$ be an adjunction [3, 9] from $\mathbf{X}$ to $\mathbf{A}$, that is, $F: \mathbf{X} \rightarrow \mathbf{A}$, $G: \mathbf{A} \rightarrow \mathbf{X}$ are a pair of functors with a natural bijection $\phi(X, A): \mathbf{A}(FX, A) \cong \mathbf{X}(X, GA)$.

Following to [4, 5], we use the convenient display

$$\frac{FX \xrightarrow{g} A}{X \xrightarrow{\psi} GA} \quad \text{or} \quad \frac{X \xrightarrow{\psi} GA}{FX \xrightarrow{g} A}$$

if and only if $g \in \mathbf{A}(FX, A)$ and $\psi \in \mathbf{X}(X, GA)$ correspond under $\phi(X, A)$. Also we write $\psi = \hat{g}$ and conversely $g = \check{\psi}$ if

$$\frac{FX \xrightarrow{g} A}{X \xrightarrow{\psi} GA}.$$

For an automaton

$$M: \quad FI \xrightarrow{\tau} Q \xrightarrow{\beta} Y \quad (\sigma \in \Sigma)$$

in $\mathbf{A}$, we define an automaton $\hat{M}$ in $\mathbf{X}$ by

$$\hat{M}: \quad I \xrightarrow{\tau} GQ \xrightarrow{G\beta} GY \quad (\sigma \in \Sigma).$$

For an automaton

$$N: \begin{array}{c} \delta_\sigma \\ \circlearrowleft \\ I \xrightarrow{\tau} Q \xrightarrow{\beta} GY \end{array} \quad (\sigma \in \Sigma)$$

in $\mathbf{X}$, we define an automaton $\tilde{N}$ in $\mathbf{A}$ by

$$\tilde{N}: \begin{array}{c} F\delta_\sigma \\ \circlearrowleft \\ FI \xrightarrow{F\tau} FQ \xrightarrow{\tilde{\beta}} Y \end{array} \quad (\sigma \in \Sigma).$$

LEMMA 2.1. Let $F \dashv G: \mathbf{X} \rightarrow \mathbf{A}$ be an adjunction and let $s: FI \rightarrow Y$ and $t: I \rightarrow GY$ be final morphisms of $\mathbf{A}$ and $\mathbf{X}$, respectively. If

$$\frac{FI \xrightarrow{s} Y}{I \xrightarrow{t} GY},$$

then $T_s(M) = T_t(\hat{M})$ for any automaton $M = (Q, \delta_\Sigma, \tau: FI \rightarrow Q, \beta: Q \rightarrow Y)$ in $\mathbf{A}$ and $T_t(N) = T_s(\tilde{N})$ for any automaton $N = (Q, \delta_\Sigma, \tau: I \rightarrow Q, \beta: Q \rightarrow GY)$ in $\mathbf{X}$.

$$\begin{array}{ll} \text{PROOF. (a)} & w \in T_s(M) \quad (b) \quad w \in T_t(N) \\ \iff \frac{\beta \cdot \delta_w \cdot \tau = s}{G\beta \cdot G\delta_w \cdot \hat{\tau} = t} & \iff \frac{\beta \cdot \delta_w \cdot \tau = t}{\tilde{\beta} \cdot F\delta_w \cdot F\tau = s} \\ \iff w \in T_t(\hat{M}). & \iff w \in T_s(\tilde{N}). \quad \square \end{array}$$

COROLLARY 2.2. Let $F \dashv G: \mathbf{X} \rightarrow \mathbf{A}$ be an adjunction. If $\mathcal{F}$ is a subclass of $|\mathbf{X}|$ and if

$$\frac{FY \xrightarrow{s} Y}{I \xrightarrow{t} GY},$$

then

- (a) $L_{\mathbf{X}}(\mathbf{A}, s: FI \rightarrow Y) = L_{\mathbf{X}}(\mathbf{X}, t: I \rightarrow GY),$
- (b) $L_{\mathbf{X}}(\mathbf{A}, G^{-1}\mathcal{F}, s: FI \rightarrow Y) \subset L_{\mathbf{X}}(\mathbf{X}, \mathcal{F}, t: I \rightarrow GY),$

where $G^{-1}\mathcal{F} = \{Q \in |\mathbf{A}|; GQ \in \mathcal{F}\}.$

PROOF. (a) It follows at once, since Lemma 2.1 indicates that categories $\mathbf{X}$ and $\mathbf{A}$ have equivalent automata each other.

(b) For any $G^{-1}\mathcal{F}$ -automaton $M = (Q, \delta_\Sigma, \tau: FI \rightarrow Q, \beta: Q \rightarrow Y)$, $\hat{M}$ is an $\mathcal{F}$ -automaton equivalent to M , that is, $T_s(M) = T_t(\hat{M})$ by Lemma 2.1. This proves (b). $\square$

COROLLARY 2.3. Let $F \dashv G: \mathbf{Set} \rightarrow \mathbf{A}$ be an adjunction and let $I = I_0$ and $\mathcal{F} = \mathcal{F}_0$ (Cf. Example (i)). If there exists an object Y of $\mathbf{A}$ with $GY = Y_0 = \{0, 1\}$ and if

$$\frac{FI_0 \xrightarrow{s} Y}{I_0 \xrightarrow{t_0} GY = Y_0},$$

then

- (a) $L_X(\mathbf{A}, s: FI_0 \rightarrow Y) = L(\Sigma)$,
- (b) $L_X(\mathbf{A}, G^{-1}\mathcal{F}_0, s: FI_0 \rightarrow Y) \subset \text{Reg}(\Sigma)$.

PROOF. This is a special case of the last corollary. $\square$

THEOREM 2.4. Let $F \dashv G: \mathbf{X} \rightarrow \mathbf{A}$ be an adjunction, $\mathcal{F}$ a subclass of $|\mathbf{A}|$ and assume that

$$\frac{FI \xrightarrow{s} Y}{I \xrightarrow{t} GY}.$$

If $F\mathcal{F} \subset G^{-1}\mathcal{F}$ (or equivalently $GFQ \in \mathcal{F}$ for any $Q \in \mathcal{F}$), then $L_X(\mathbf{A}, G^{-1}\mathcal{F}, s: FI \rightarrow Y) = L_X(\mathbf{X}, \mathcal{F}, t: I \rightarrow GY)$.

PROOF. It immediately follows from Lemma 2.1 and from the fact that, whenever N is an $\mathcal{F}$ -automaton in $\mathbf{X}$, $\tilde{N}$ is a $G^{-1}\mathcal{F}$ -automaton in $\mathbf{A}$ since $F\mathcal{F} \subset G^{-1}\mathcal{F}$. $\square$

COROLLARY 2.5. Let $F \dashv G: \mathbf{Set} \rightarrow \mathbf{A}$ be an adjunction and let $I = I_0$ and $\mathcal{F} = \mathcal{F}_0$ (Cf. Example (i)). If there exists an object Y of $\mathbf{A}$ with $GY = Y_0$,

$$\frac{FI_0 \xrightarrow{s} Y}{I_0 \xrightarrow{t_0} GY = Y_0},$$

and GFQ is a finite set for any finite set Q , then

$$L_X(\mathbf{A}, G^{-1}\mathcal{F}_0, s: FI_0 \rightarrow Y) = \text{Reg}(\Sigma).$$

PROOF. This is a special case of Theorem 2.4 (Cf. Example (i)). $\square$

EXAMPLE (vii). Let $\mathbf{Rel}$ be the category [9; p. 26] of sets and binary relations, $F: \mathbf{Set} \rightarrow \mathbf{Rel}$ the inclusion functor and $G: \mathbf{Rel} \rightarrow \mathbf{Set}$ the power set functor, i.e.

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \quad \text{in } \mathbf{Rel} \\
& \downarrow G & \\
2^A & \xrightarrow{2^\alpha} & 2^B \quad \text{in } \mathbf{Set}
\end{array}$$

where, for $S \subset A$, $2^\alpha(S) = S\alpha = \{b \in B; (s, b) \in \alpha \text{ for some } s \in S\}$. Let $A \in |\mathbf{Set}|$ and $B \in |\mathbf{Rel}|$, that is, A, B are sets. Then we have

$$\begin{aligned}
\mathbf{Rel}(FA, B) &= \mathbf{Set}(A \times B, 2) \\
&\cong \mathbf{Set}(A, 2^B) \\
&= \mathbf{Set}(A, GB),
\end{aligned}$$

which shows that $F \dashv G: \mathbf{Set} \rightarrow \mathbf{Rel}$ and it follows that

$$\frac{FI_0 \xrightarrow{1_{FI_0}} FI_0}{I_0 \xrightarrow{i_0} GFI_0 = Y_0}$$

where $GFI_0 = Y_0$ by $0 = \phi$ and $1 = \{0\}$ (Cf. Example (i)). Hence $L_\Sigma(\mathbf{Rel}, 1_{FI_0}: FI_0 \rightarrow FI_0) = L(\Sigma)$ by Corollary 2.3(a). Since it is trivial that GFQ is a finite set for any finite set Q , we obtain $L_\Sigma(\mathbf{Rel}, \mathcal{F}_0, 1_{FI_0}: FI_0 \rightarrow FI_0) = \text{Reg}(\Sigma)$ by Corollary 2.5. $\square$

We can also apply Corollary 2.3(a) to Example (iv) and (v), to conclude that $L_\Sigma(\mathbf{Ab}, \pi_0) = L_\Sigma(\mathbf{Gr}, \pi_0) = L(\Sigma)$ because of the existence of adjunctions $\mathbf{Set} \rightarrow \mathbf{Ab}$, $\mathbf{Set} \rightarrow \mathbf{Gr}$ ([9; p. 85]) and $GZ_2 = Y_0$. However we cannot apply Corollary 2.5 to Example (iv) and (v) to determine $L_\Sigma(\mathbf{Ab}, G^{-1}\mathcal{F}_0, \pi_0)$ and $L_\Sigma(\mathbf{Gr}, G^{-1}\mathcal{F}_0, \pi_0)$, because FS is not a finite group even if S is a finite set. The following lemma is a generalization of Lemma 2.1.

LEMMA 2.6. *Let $F: \mathbf{X} \rightarrow \mathbf{A}$ be a functor and $s: I \rightarrow Y$, $\mu: I \rightarrow F\bar{I}$, $v: F\bar{Y} \rightarrow Y$ morphisms of $\mathbf{A}$, where $\bar{I}, \bar{Y} \in |\mathbf{X}|$ and $I, Y \in |\mathbf{A}|$. If there exists a unique morphism $t: \bar{I} \rightarrow \bar{Y}$ of $\mathbf{X}$ which makes the diagram*

$$\begin{array}{ccc}
I & \xrightarrow{s} & Y \\
\mu \downarrow & & \uparrow v \\
F\bar{I} & \xrightarrow{Ft} & F\bar{Y}
\end{array}$$

commute, then the following automata M in $\mathbf{X}$ and $\bar{M}$ in $\mathbf{A}$ are equivalent, that is, $T_t(M) = T_s(\bar{M})$:

$$\begin{array}{c}
 M: \\
 \begin{array}{c}
 \delta_\sigma \\
 \curvearrowright \\
 I \xrightarrow{\tau} Q \xrightarrow{\beta} \bar{Y} \quad (\sigma \in \Sigma),
 \end{array} \\
 \\
 \bar{M}: \\
 \begin{array}{c}
 F\delta_\sigma \\
 \curvearrowright \\
 I \xrightarrow{\mu} F\bar{I} \xrightarrow{F\tau} FQ \xrightarrow{F\beta} F\bar{Y} \xrightarrow{v} Y \quad (\sigma \in \Sigma). \\
 \underbrace{\hspace{1.5cm}}_{\bar{\tau}} \quad \underbrace{\hspace{1.5cm}}_{\bar{\beta}}
 \end{array}
 \end{array}$$

$$\begin{aligned}
 \text{PROOF.} \quad w \in T_s(\bar{M}) &\iff \bar{\beta} \cdot F\delta_w \cdot \bar{\tau} = s \\
 &\iff v \cdot F(\beta \cdot \delta_w \cdot \tau) \cdot \mu = s \\
 &\iff \beta \cdot \delta_w \cdot \tau = t \quad (\text{by the assumption}) \\
 &\iff w \in T_t(M) \quad \square
 \end{aligned}$$

THEOREM 2.7. Let $F \dashv G: \mathbf{Set} \rightarrow \mathbf{A}$ be an adjunction. Then $L_x(\mathbf{A}, G^{-1}\mathcal{F}_0, s) = \text{Reg}(\Sigma)$ if the following conditions (1)~(5) are satisfied:

- (1) $\mathbf{A}$ has finite coproducts.
- (2) There exists an object Y of $\mathbf{A}$ with $GY = Y_0$.
- (3) $s: FI_0 \rightarrow Y$ is an epimorphism of $\mathbf{A}$, where

$$\frac{FI_0 \xrightarrow{s} Y}{I_0 \xrightarrow{i_0} GY = Y_0}.$$

- (4) There exists a morphism $\zeta: Y \rightarrow Y$ with $\zeta \neq 1_Y$.
- (5) For any finite set $S \in \mathcal{F}_0$, a coproduct $\bigsqcup_{x \in S} Y_{(x)}$ is in $G^{-1}\mathcal{F}_0$, where $Y_{(x)} = Y$ for each $x \in S$.

PROOF. It follows from Corollary 2.3(b) that $L_x(\mathbf{A}, G^{-1}\mathcal{F}_0, s: FI_0 \rightarrow Y) \subset \text{Reg}(\Sigma)$. Thus, for the proof of the theorem, it suffices to show that we can construct a $G^{-1}\mathcal{F}_0$ -automaton $\bar{M}$ equivalent to each finite automaton M in $\mathbf{Set}$. First of all we define a functor $\bar{F}: \mathcal{F}_0 \rightarrow \mathbf{A}$ (where $\mathcal{F}_0$ is a full subcategory of $\mathbf{Set}$ consisting of all finite sets) as follows: Let $f: S \rightarrow T$ be a map of a finite set S to a finite set T . Define $\bar{F}S = \bigsqcup_{x \in S} Y_{(x)}$ and define $\bar{F}f: \bar{F}S \rightarrow \bar{F}T$ to be a morphism $\bar{F}f: \bigsqcup_{x \in S} Y_{(x)} \rightarrow \bigsqcup_{y \in T} Y_{(y)}$ of $\mathbf{A}$ which makes the diagram

$$\begin{array}{c}
 Y \\
 \swarrow \quad \searrow \\
 \text{\scriptsize in_x} \quad \text{\scriptsize $in_{f(x)}$} \\
 \bar{F}S = \coprod_{x \in S} Y_{(x)} \xrightarrow{\bar{F}f} \coprod_{y \in T} Y_{(y)} = \bar{F}T
 \end{array}$$

commute for each $x \in S$, where $in_x: Y \rightarrow \coprod_{x \in S} Y_{(x)}$ and $in_y: Y \rightarrow \coprod_{y \in T} Y_{(y)}$ are x -th injection ($x \in S$) and y -th injection ($y \in T$) of coproducts, respectively.

Trivially we have $\bar{F}I_0 = Y$ and $\bar{F}Y_0 = Y \amalg Y$. Secondly define $\mu = s: FI_0 \rightarrow \bar{F}I_0$ and define $v = \zeta \perp 1_Y: \bar{F}Y_0 \rightarrow Y$, where $(\zeta \perp 1_Y) \cdot in_0 = \zeta$ and $(\zeta \perp 1_Y) \cdot in_1 = 1_Y$. Finally we prove that the data $\bar{F}$, μ and v satisfy the assumption of Lemma 8. There are only two maps $t_0, \neg t_0: I_0 \rightarrow Y_0$, where $t_0(0) = 1$, $\neg t_0(0) = 0$. Then we have $\bar{F}(t_0) = in_1$, $\bar{F}(\neg t_0) = in_0$, and $v \cdot \bar{F}(t_0) \cdot \mu = s$ and $v \cdot \bar{F}(\neg t_0) \cdot \mu = \zeta \cdot s$. We note that $\zeta \cdot s \neq s$ by (3), (4). This shows that t_0 is a unique map with $v \cdot \bar{F}(t_0) \cdot \mu = s$. Hence Lemma 8 is valid and we can construct a $G^{-1}\mathcal{F}_0$ -automaton $\bar{M}$ in $\mathbf{A}$ equivalent to each finite automaton M . Thus we can conclude the desired result. $\square$

COROLLARY 2.8. $L_X(\mathbf{Ab}, G^{-1}\mathcal{F}_0, \pi_0) = \text{Reg}(\Sigma)$. $\square$

It is well known that a coproduct $Z_2 \amalg Z_2$ in $\mathbf{Gr}$ is not a finite group and therefore we cannot directly apply the above theorem to determine $L_X(\mathbf{Gr}, G^{-1}\mathcal{F}_0, \pi_0)$.

PROPOSITION 2.9. Let $G: \mathbf{A} \rightarrow \mathbf{X}$, $H: \mathbf{B} \rightarrow \mathbf{X}$, $K: \mathbf{A} \rightarrow \mathbf{B}$ be functors with $G = HK$, $\mathcal{F}$ a subclass of $|\mathbf{X}|$ and $s: I \rightarrow Y$ a final morphism of $\mathbf{A}$. If K is a faithful functor, then $L_X(\mathbf{A}, G^{-1}\mathcal{F}, s: I \rightarrow Y) \subset L_X(\mathbf{B}, H^{-1}\mathcal{F}, Ks: KI \rightarrow KY)$.

PROOF. Let $M = (Q, \delta_s, \tau: I \rightarrow Q, \beta: Q \rightarrow Y)$ be a $G^{-1}\mathcal{F}$ -automaton in $\mathbf{A}$ and construct a new $H^{-1}\mathcal{F}$ -automaton $KM = (KQ, K\delta_s, K\tau: KI \rightarrow KQ, K\beta: KQ \rightarrow KY)$ in $\mathbf{B}$. We can then prove $T_s(M) = T_{Ks}(KM)$ without difficulties. This completes the proof. $\square$

COROLLARY 2.10. $L_X(\mathbf{Gr}, G^{-1}\mathcal{F}_0, \pi_0) = \text{Reg}(\Sigma)$.

PROOF. It follows at once since $L_X(\mathbf{Gr}, G^{-1}\mathcal{F}_0, \pi_0) \subset \text{Reg}(\Sigma)$ by Corollary 2.3(b) and $\text{Reg}(\Sigma) = L_X(\mathbf{Gr}, G^{-1}\mathcal{F}_0, \pi_0) \subset L_X(\mathbf{Ab}, G^{-1}\mathcal{F}_0, \pi_0)$ by the last proposition. $\square$

§3. Automata in $\mathbf{Ab}$

Let M be a (traditional) automata with only one state q_0 . It is clear that $T(M) = \Sigma^*$ if q_0 is a final state of M and $T(M) = \emptyset$ otherwise:

We now determine the language $T_{\pi_0}(M)$ over an alphabet Σ accepted by an automaton $M = (Q, \delta_\Sigma, \tau: Z \rightarrow Q, \beta: Q \rightarrow Z_2)$ in $\mathbf{Ab}$ with Q a cyclic group. (A cyclic group is a group with one generator.)

We denote by Z_n a quotient group Z/nZ for a nonnegative integer n , and by $\pi_n: Z \rightarrow Z_2$ a (unique) epimorphism if n is even and a zero homomorphism otherwise. We note the following trivial lemma.

LEMMA 3.1. *For any homomorphism $f: Z_n \rightarrow Z_2$, there is a (not necessarily unique) homomorphism $f': Z_2 \rightarrow Z_2$ with $f' \cdot \pi_n = f$. In particular, for any homomorphism $g: Z_n \rightarrow Z_m$, there is a homomorphism $g': Z_2 \rightarrow Z_2$ with $g' \cdot \pi_n = \pi_m \cdot g$. $\square$*

The above lemma easily concludes the following proposition, which determines the language $T_{\pi_0}(M)$ of an automaton M in $\mathbf{Ab}$ with a cyclic group as its state object.

PROPOSITION 3.2. *Let $M = (Z_n, \delta_\Sigma, \tau: Z \rightarrow Z_n, \beta: Z_n \rightarrow Z_2)$ be an automaton in $\mathbf{Ab}$ (Cf. Example (iv)).*

- (a) *If n is odd, then $T_{\pi_0}(M) = \emptyset$.*
- (b) *If n is even, then $T_{\pi_0}(M) = \emptyset$ or $T_{\pi_0}(M) = \Lambda^*$ for some subset Λ of Σ .*

PROOF. (a) Since n is odd, β is necessarily a zero homomorphism and so $T_{\pi_0}(M) = \emptyset$.

(b) Assume n is even. In the case that either τ or β are zero homomorphisms, we obviously have $T_{\pi_0}(M) = \emptyset$ as in (a). In the case that neither τ nor β are zero homomorphisms, we have $\beta = \pi_n$ and by Lemma 3.1 the following commutative diagram

$$\begin{array}{ccccc}
Z & & Z_n & \xrightarrow{\delta_\sigma} & Z_n \\
& \nearrow \tau & \downarrow \pi_n & & \downarrow \pi_n \\
& & Z_2 & \xrightarrow{\delta'_\sigma} & Z_2 \\
& \searrow \tau' = \pi_n \circ \tau & & & \nearrow \beta' = 1_{Z_2} \\
& & & & Z_2
\end{array}
\quad \begin{array}{l} \beta = \pi_n \\ \beta' = 1_{Z_2} \end{array}$$

for any $\sigma \in \Sigma$. Put $M' = (Z_2, \{\delta'_\sigma; \sigma \in \Sigma\}, \tau', \beta')$. Then it follows from Proposition 1.2 that $T_{\pi_0}(M') = T_{\pi_0}(M)$. If τ' is a zero homomorphism, then $T_{\pi_0}(M') = \phi$. If τ' is not a zero homomorphism, then we have $\tau' = \pi_0$ and $T_{\pi_0}(M') = \{\sigma \in \Sigma; \delta'_\sigma = 1_{Z_2}\}^*$. This completes the proof. $\square$

THEOREM 3.3. $L_\Sigma(\mathbf{Ab}, \mathcal{F}', \pi_0: Z \rightarrow Z_2) = \text{Reg}(\Sigma)$.

PROOF. (Cf. Example (iv).) Since FS is in $\mathcal{F}'$ for each finite set S , it is immediate from Lemma 1.3 that $L_\Sigma(\mathbf{Ab}, \mathcal{F}', \pi_0) \subset \text{Reg}(\Sigma)$. To prove the converse inclusion, we assume $M = (Q, \delta_\Sigma, \tau: Z \rightarrow Q, \beta: Q \rightarrow Z_2)$ is an $\mathcal{F}'$ -automaton in $\mathbf{Ab}$. By means of the fundamental theorem of abelian groups, we can decompose $Q \in \mathcal{F}'$ into a direct sum of a finite number of cyclic groups, namely,

$$Q \cong Z_{\theta_1} \oplus Z_{\theta_2} \oplus \cdots \oplus Z_{\theta_k}$$

for some nonnegative integers $\theta_1, \theta_2, \dots, \theta_k \geq 0$. Thus we can represent each δ_σ by a matrix (Cf. [3, 5])

$$\delta_\sigma = (\delta_{\sigma i, j})_{1 \leq i, j \leq k}$$

where $\delta_{\sigma i, j}: Z_{\theta_i} \rightarrow Z_{\theta_j}$ ($1 \leq i, j \leq k$). Similarly, β can be represented as

$$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$$

where $\beta_j: Z_{\theta_j} \rightarrow Z_2$ ($1 \leq j \leq k$). By Lemma 3.1, we can choose $\delta'_{\sigma i, j}: Z_2 \rightarrow Z_2$ ($1 \leq i, j \leq k$) and $\beta'_j: Z_2 \rightarrow Z_2$ ($1 \leq j \leq k$) making the diagrams

$$\begin{array}{ccc}
Z_{\theta_i} & \xrightarrow{\delta_{\sigma i, j}} & Z_{\theta_j} \\
\pi_{\theta_i} \downarrow & & \downarrow \pi_{\theta_j} \\
Z_2 & \xrightarrow{\delta'_{\sigma i, j}} & Z_2
\end{array}
\quad
\begin{array}{ccc}
Z_{\theta_j} & \xrightarrow{\beta_j} & Z_2 \\
\pi_{\theta_j} \downarrow & & \nearrow \beta'_j \\
Z_2 & &
\end{array}$$

commute. Put $\delta'_\sigma = (\delta'_{\sigma i, j})_{1 \leq i, j \leq k}$ and $\beta' = \begin{pmatrix} \beta'_1 \\ \vdots \\ \beta'_k \end{pmatrix}$. Then the following diagram

$$\begin{array}{ccccc}
& & Z_{\theta_1} \oplus \cdots \oplus Z_{\theta_k} & \xrightarrow{(\delta_{\sigma i, j})} & Z_{\theta_1} \oplus \cdots \oplus Z_{\theta_k} & \xrightarrow{\beta} & Z_2 \\
Z & \xrightarrow{\tau} & \downarrow \pi_{\theta_1} \oplus \cdots \oplus \pi_{\theta_k} & & \downarrow \pi_{\theta_1} \oplus \cdots \oplus \pi_{\theta_k} & & \nearrow \beta' \\
& & Z_2 \oplus \cdots \oplus Z_2 & \xrightarrow{(\delta'_{\sigma i, j})} & Z_2 \oplus \cdots \oplus Z_2 & &
\end{array}$$

$(\pi_{\theta_1} \oplus \cdots \oplus \pi_{\theta_k}) \cdot \tau$

is commutative. Therefore we have proved from Proposition 1.2 that $T_{\pi_0}(M)$ is a regular language. This completes the proof of the theorem. $\square$

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