NOTE ON SELF H -MAPS OF $SU(3)$

By

Ken-ichi MARUYAMA

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In [4] the author and Oka studied self H -maps of H -spaces E_K of type $(3, 7)$ and proved that if a self H -map is a homotopy equivalence then it is equivalent to the identity. Here we take $SU(3)$ the special unitary group, as an object. $SU(3)$ has two typical involutions, namely, C (complex conjugation) and V (inversion map). Though as in [4], using the K -theoretic method, we can see that V can not be a self H -map for any multiplication, we know that C is a self H -map for the usual (Lie group) multiplication θ . In fact, N. Sawashita [7] has proved that $\mathcal{E}_H(SU(3), \theta)$, the group of H -equivalences of $SU(3)$, is isomorphic to $\mathbb{Z}/2$ with generator C . However we shall prove the following theorem.

THEOREM. *There is a multiplication M on $SU(3)$ for which C is not an H -map. Then $\mathcal{E}_H(SU(3), M) = \{1\}$, the trivial group. Further, $\mathcal{E}_H(SU(3), \mu) = \{1\}$ or $\mathbb{Z}/2$ for any multiplication μ .*

REMARK. *That C is not an H -map still holds when localized at 7. Moreover, when $\mathcal{E}_H(SU(3), \mu) \neq \{1\}$, there are two possible generators of the group $\mathbb{Z}/2$, C and $\kappa(\beta)C$, where β is the element of order 2 in $\pi_8(SU(3)) = \mathbb{Z}/12$. It is still open whether or not there is a multiplication for which $\kappa(\beta)C$ is a self H -map.*

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§1. Constructing a multiplication

We recall the following theorems. Firstly, let l be a subset of all primes P and $\bar{l}$ be the complement of l in P . And let X be a CW complex and $X_{(l)}$ be the localization of X at l .

THEOREM 1.1 ([6], Cor. 4.2). *X is homotopy equivalent to a pull back*

$$X_{(l)} \times_{X_{(o)}} X_{(\bar{l})}.$$

Professor T. Kudo, to whom this volume is dedicated.

THEOREM 1.2 ([3]). *If X, μ is an H -space then the set of (homotopy classes) of multiplications on X is 1-1 correspondence with $M(X)=[X \wedge X, X]$. The correspondence is given by $\mu_w = \mu + w\Lambda$ for $w \in M(X)$, where $\Lambda: X \times X \rightarrow X \wedge X$ is a projection.*

We consider the principal bundle

$$SU(2)(=S^3) \longrightarrow SU(3) \longrightarrow S^5.$$

As the characteristic element of this bundle, $\eta_3 \in \pi_4(S^3)$, is of order 2, $SU(3)_{(p)} = S^3_{(p)} \times S^5_{(p)}$, for an odd prime p . Moreover, if $p \geq 5$, $S^3_{(p)}$ and $S^5_{(p)}$ has a homotopy associative and homotopy commutative multiplication [1]. We call these ones m and n respectively. As we define M_0 to be a product multiplication of m and n (on $SU(3)_{(p)}$), $M(SU(3)_{(p)}) = M(S^3_{(p)} \times S^5_{(p)})$ can be considered as a (commutative) group.

Note that there is a cofibration

$$S^3_{(p)} \vee S^5_{(p)} \longrightarrow S^3_{(p)} \times S^5_{(p)} \xrightarrow{j} S^3_{(p)} \wedge S^5_{(p)},$$

then the following map is easily seen to be monic,

$$(j \wedge j)^* : [(S^3_{(p)} \wedge S^5_{(p)}) \wedge (S^3_{(p)} \wedge S^5_{(p)}), SU(3)_{(p)}] \longrightarrow M(SU(3)_{(p)}).$$

(see [9] 1.3.5 Lemma).

The left hand side of the above is isomorphic to $\pi_{16}(S^3)_{(p)} + \pi_{16}(S^5)_{(p)}$. $\pi_{16}(S^5)_{(7)} = \mathbb{Z}/7$ is generated by the element $\alpha_1(5)$. (see [8].) Define a new multiplication on $S^3_{(p)} \times S^5_{(p)}$, by Th. 1.2, as $M_0 + \alpha\Lambda$, and call this M , where $\alpha = (j \wedge j)^*(\alpha_1(5))$.

Here we should note that $p_2 C_{(7)} \bar{M}(C_{(7)} \times C_{(7)}) = n(p_2 \times p_2) - \alpha\Lambda$, where $p_2: S^3_{(7)} \times S^5_{(7)} \rightarrow S^3_{(7)}$ is the projection, because $C_{(7)}: S^5_{(7)} \times S^3_{(7)} \rightarrow S^5_{(7)} \times S^3_{(7)}$ is homotopic to the diagonal map " 1×-1 ". Similarly, $p_2 \bar{M} = n(p_2 \times p_2) + \alpha\Lambda$. Now since $\alpha\Lambda \neq -\alpha\Lambda$, we can conclude that $\bar{M}$ can not make $C_{(7)}$ to be an H -map. Since $SU(3)_{(0)}$ has a unique multiplication by Th. 1.2, $\bar{M}$ coincides with θ at 0. By Th. 1.1, $SU(3)$ is equipped with the multiplication M which is $\bar{M}$ at 7 and $\theta_{(7)}$ at $\bar{7}$.

REMARK. *Above construction can be extended for $SU(n)$ ($n \geq 4$). (see [5])*

§ 2. Another informations of self H -maps of $SU(3)$

The cohomology ring of $SU(3)$ is an exterior algebra:

$$H^*(SU(3)) = \Lambda(x_3, x_5), \dim x_i = i.$$

For a map $f: SU(3) \rightarrow SU(3)$, we define $\deg([f]) = (q, r)$ if $f^*(x_3) = qx_3$, $f^*(x_5) = rx_5$, where $[f]$ denotes a homotopy class of f . Then $\deg$ is a homomorphism from the homotopy classes of self maps of $SU(3)$ to $Z + Z$;

$$\deg: [SU(3), SU(3)] \longrightarrow Z + Z.$$

Let $[SU(3), SU(3)]_H$ be the set of homotopy classes of self H -maps with respect to a given multiplication μ .

PROPOSITION 2.1. *For any multiplication μ , $\deg[SU(3), SU(3)]_H \subset \{(q, r) | q \equiv r \pmod{2} \text{ and } q \equiv 0, 1 \pmod{4}\}$.*

PROOF. There is a cofibration

$$SU(3) * SU(3) \xrightarrow{\mu'} \Sigma SU(3) \xrightarrow{i} P \xrightarrow{j} \Sigma SU(3) \wedge \Sigma SU(3).$$

where $*$ denotes the join, Σ denotes the (reduced) suspension, μ' is the Hopf construction of μ , and P is called the projective plane of $SU(3)$, μ .

Now we assume that $f: SU(3) \rightarrow SU(3)$ is an H -map, then there exists a self map g of P making the following diagram homotopy commute.

$$(2.2) \quad \begin{array}{ccccccc} SU(3) * SU(3) & \xrightarrow{\mu'} & \Sigma SU(3) & \xrightarrow{i} & P & \xrightarrow{j} & \Sigma SU(3) \wedge \Sigma SU(3) \\ \downarrow f * f & & \downarrow \Sigma f & & \downarrow g & & \downarrow \Sigma f \wedge \Sigma f \\ SU(3) * SU(3) & \xrightarrow{\mu'} & \Sigma SU(3) & \xrightarrow{i} & P & \xrightarrow{j} & \Sigma SU(3) \wedge \Sigma SU(3) \end{array}$$

We see the following Lemma on the K -cohomology of P and the Adams operation ψ^2 as in [4].

LEMMA 2.3. *$\tilde{K}^0(P)$ has a basis u, v, u^2, v^2, uv, w , as a free abelian group, such that $\psi^2(u) = 4u + hu^2 + 2y$, where h is an odd integer, y is a linear combination of v^2, uv and w .*

Apply $\tilde{K}^0(\)$ to (2.2) to get the commutative diagram of exact sequences. Then, comparing $g^*\psi^2(u)$ and $\psi^2g^*(u)$, we see the proposition as in [4].

§3. The group of H -equivalences of $SU(3)$

Let $\mathcal{E}(SU(3))$ be the group of self homotopy equivalences of $SU(3)$. By Sawashita [7], $\mathcal{E}(SU(3)) = G \times Z/2$, where $Z/2$ is generated by C and G is the following split extension.

$$0 \longrightarrow Z/12 \xrightarrow{\kappa} G \longrightarrow Z/2 \longrightarrow 1$$

(i.e., G is isomorphic to the dihedral group D_{12}) with V giving splitting.

The fact that $Im \kappa$ does not contain H -maps except the identity is also known to Sawashita [7, Lemma 1.5]. Now on the map $\deg$ we see

$$\deg(\kappa(\beta)) = (1, 1) \quad \text{for any } \beta \in \mathbb{Z}/12,$$

$$\deg(V) = (-1, -1),$$

$$\deg(C) = (1, -1).$$

By Prop. 2.1 and the above facts, elements which have a form $\kappa(\beta)C$ are the only one candidate for H -map for a given μ . But if this is an H -map then $\{\kappa(\beta)C\} \cdot \{\kappa(\beta)C\} = \kappa(2\beta)$ (C commutes with others) is also an H -map. So, $2\beta = 0$ from the above comment. Therefore $\mathcal{E}_H(SU(3), \mu) = [SU(3), SU(3)]_H \cap \mathcal{E}(SU(3))$ is one of $\{1\}$, $\{1, C\}$, $\{1, \kappa(\beta)C\}$, where β is the element of order 2 in $\pi_8(SU(3))$, this proves the latter half of Theorem.

As the homomorphism κ is natural with localization, we have $\kappa(\beta)C_{(7)} = \kappa(\beta_{(7)}C_{(7)}) = C_{(7)}$, for, the order of β is 2 or 0. By §1, C and $\kappa(\beta)C$ are not H -maps respect to M .

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Department of Mathematics
Kyushu University 33
Hakozaki, Fukuoka 812
Japan