A DOMAIN EXHAUSTED BY COMPLETE KÄHLER DOMAINS OVER A STEIN MANIFOLD

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§1. Introduction

In this paper we shall prove the following theorem (Theorem 3).

THEOREM. *A domain D exhausted by complete Kähler domains over a Stein manifold is a Stein manifold.*

This is one of the characterization of Stein manifolds by their geometric properties. Since every weakly 1-complete manifold over a Stein manifold is exhausted by complete Kähler domains, this is also a generalization of the Levi's problem over a Stein manifold.

T. Yasuoka [6] gave this result when D is a domain in a Stein manifold. Using the idea of [6], we shall give the same result when D is a domain over $\mathbb{C}^n$ (Theorem 2). Using the method of F. Docquier-H. Grauert [2], we shall prove Theorem 3.

§2. Preliminaries

Let M be a complex manifold. We say that (D, Φ) is a *domain over M* if Φ is a holomorphic mapping of a complex manifold D into M such that Φ is locally biholomorphic.

A complex manifold X is said to be *exhausted by complete Kähler domains* if there exists a sequence $\{X_i; i=1, 2, 3, \dots\}$ of open submanifolds of X such that $X = \bigcup_{i=1}^{\infty} X_i$, $X_i \Subset X_{i+1}$ for every i , and each X_i has a complete C^∞ Kähler metric.

We obtain the following proposition by fundamental arguments.

PROPOSITION 1. (i) *Let X be a complex manifold. Then X is exhausted*

by complete Kähler domains if and only if every component of X is exhausted by complete Kähler domains.

(ii) Let X_1 and X_2 be open submanifolds of a complex manifold. If X_1 and X_2 are exhausted by complete Kähler domains, then $X_1 \cap X_2$ is exhausted by complete Kähler domains.

(iii) Let X be a complex manifold exhausted by complete Kähler domains, and Y be a closed submanifold of X . Then Y is exhausted by complete Kähler domains.

Let X be a complex manifold of dimension n , and $D_{(p,q)}(X)$ be the space of $C^\infty(p, q)$ -forms on X with compact support. If ϕ is a C^∞ strictly plurisubharmonic function on X ,

$$d\sigma^2 = \sum_{j,k=1}^n \frac{\partial^2 \phi}{\partial z^j \partial \bar{z}^k} dz^j d\bar{z}^k$$

defines a Kähler metric on X , where $(z^1, \dots, z^n)$ denotes a local coordinate system of X . We denote by $L_{(p,q)}^q(X)$ the completion of $D_{(p,q)}(X)$ with respect to the norm

$$|f|_\sigma^2 = \int_X f \wedge \bar{*} f \quad f \in D_{(p,q)}(X),$$

where $\bar{*}$ denotes the conjugate of the star operator $*$ associated to $d\sigma^2$.

The differential operator $\bar{\partial}: D_{(p,q)}(X) \rightarrow D_{(p,q+1)}(X)$ is defined by

$$\bar{\partial}f = (-1)^p \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} \frac{\partial f_{i_1 \dots i_p j_1 \dots j_q}}{\partial \bar{z}^j} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^j \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q},$$

where

$$f = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} f_{i_1 \dots i_p j_1 \dots j_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}.$$

We can regard $\bar{\partial}$ as a densely defined closed linear operator from $L_{(p,q)}^q(X)$ to $L_{(p,q+1)}^q(X)$. The following lemma is given in [5].

LEMMA 1. Let X be a complete C^∞ Kähler manifold of dimension n and ϕ be a C^∞ bounded strictly plurisubharmonic function on X . Then for any $f \in L_{(n,1)}^q(X)$ with $\bar{\partial}f=0$, there exists a $g \in L_{(n,0)}^q(X)$ such that $\bar{\partial}g=f$.

Let Ω be an open subset of $\mathbb{C}^n$. We denote by $L^2(\Omega)$ the space of functions in Ω which are square integrable with respect to the Lebesgue measure. Let

$W^s(\Omega)$, where s is a non-negative integer, denote the space of functions in Ω whose derivatives of order $\leq s$ are in $L^2(\Omega)$.

The following lemma is given in [3].

LEMMA 2. *If $f \in L^2(\mathbb{C}^n)$ has compact support and $\partial f / \partial \bar{z}^j \in L^2(\mathbb{C}^n)$ for $j = 1, \dots, n$, then $f \in W^1(\mathbb{C}^n)$.*

Let X be a complex manifold of dimension n . We assume that there exists a C^∞ strictly plurisubharmonic function ϕ on X . We define a Kähler metric on X by

$$d\sigma^2 = \sum_{j,k} \frac{\partial^2 \phi}{\partial z^j \partial \bar{z}^k} dz^j d\bar{z}^k.$$

Let Y be a relatively compact open submanifold of X . We assume that Y has a complete C^∞ Kähler metric.

LEMMA 3. *Let X , Y , and $d\sigma^2$ be as above. For every $f \in C_{(n,1)}^\infty(X)$ with $\bar{\partial}f = 0$ on X , there exists a $g \in C_{(n,0)}^\infty(Y)$ such that $\bar{\partial}g = f$ on Y .*

PROOF. Since $f \in C_{(n,1)}^\infty(X)$ and Y is relatively compact in X , $f \in L_{(n,1)}^q(Y)$. Then there exists a $g \in L_{(n,0)}^q(Y)$ such that $\bar{\partial}g = f$ on Y from Lemma 1. For an arbitrary point $p \in Y$, let U be a coordinate neighborhood of p in Y with coordinate $(z^1, \dots, z^n)$. We can express f by

$$f = \sum_{j=1}^n f_j dZ^N \wedge d\bar{z}^j$$

and g by

$$g = g' dZ^N,$$

where we set $dZ^N = dz^1 \wedge \dots \wedge dz^n$. Then we have $\partial g' / \partial \bar{z}^j = f_j$ on U , $j = 1, \dots, n$. Let V and W be neighborhoods of p with $W \subset V \subset U$. We can find $\eta \in D(V)$ such that $\eta \equiv 1$ on W . We may regard $\eta g' \in L^2(\mathbb{C}^n)$ and has compact support in $\mathbb{C}^n$. The smoothness of f implies that $f_j \in W^{s+2n}(V)$, $j = 1, \dots, n$, for every positive integer s . Suppose that $g' \in W^t(V)$ for some integer t , $0 \leq t \leq s + 2n$. Then we have

$$\frac{\partial(\eta g')}{\partial \bar{z}^j} = \frac{\partial \eta}{\partial \bar{z}^j} g' + f_j \eta \in W^t(V).$$

Since $\partial(\eta g') / \partial \bar{z}^j$ has compact support in V , we may regard

$$\frac{\partial(\eta g')}{\partial \bar{z}^j} \in W^t(\mathbb{C}^n).$$

Therefore by Lemma 2, $\eta g' \in W^{t+1}(\mathbb{C}^n)$. Since $g' \in L^s(V) = W^0(V)$, by repeating the above argument, we have $\xi g' \in W^{s+2n+1}(\mathbb{C}^n)$ for some $\xi \in D(V)$ such that $\xi \equiv 1$ on a neighborhood W_1 of p . By the Sobolev lemma we have $\xi g' \in C^{s+1}(\mathbb{C}^n)$, that is, $g' \in C^{s+1}(W_1)$. Hence $g \in C_{(n,0)}^\infty(Y)$, because we may choose s arbitrarily.

§3. Cousin-I pairs

Let (D_1, Φ_1) and (D_2, Φ_2) be domains over a complex manifold M . We say that (D_1, Φ_1) is a *domain over* (D_2, Φ_2) and write $(D_1, \Phi_1) < (D_2, \Phi_2)$ if there exists a holomorphic mapping τ of D_1 in D_2 such that $\Phi_1 = \Phi_2 \circ \tau$. We consider a sequence $\{(D_n, \Phi_n); n=1, 2, 3, \dots\}$ of domains over M such that $(D_n, \Phi_n) < (D_{n+1}, \Phi_{n+1})$ for $n \geq 1$ and call it a *monotonously increasing sequence of domains over* M . Then there exists a holomorphic mapping τ_m^n of D_n in D_m such that $\Phi_n = \Phi_m \circ \tau_m^n$ for any $m \geq n$. Let (D', Φ') be a domain over M such that $(D_n, \Phi_n) < (D', \Phi')$ ($n \geq 1$) with a holomorphic mapping τ'_n of D_n in D' satisfying $\tau'_n = \tau'_m \circ \tau_m^n$ for $m \geq n$. In this case $\{(D_n, \Phi_n); n=1, 2, 3, \dots\}$ is called a *monotonously increasing sequence over* (D', Φ') . There exists a domain (D, Φ) over M called the *limit of a monotonously increasing sequence* $\{(D_n, \Phi_n); n=1, 2, 3, \dots\}$ of domains over M which has the following two properties (cf. J. Kajiwaru [4]).

- i) $\{(D_n, \Phi_n); n=1, 2, 3, \dots\}$ is a monotonously increasing sequence over (D, Φ) .
- ii) For any (D', Φ') over which $\{(D_n, \Phi_n); n=1, 2, 3, \dots\}$ is a monotonously increasing sequence, $(D, \Phi) < (D', \Phi')$ holds.

Let $(D_1, \Phi_1) < (D_2, \Phi_2)$. There exists a holomorphic mapping τ of D_1 in D_2 such that $\Phi_1 = \Phi_2 \circ \tau$. We say that $\{(D_1, \Phi_1), (D_2, \Phi_2), \tau\}$ is a *Cousin-I pair* if for any Cousin-I distribution $\{(m_i, V_i); i \in I\}$ in D_2 there exists a solution of the Cousin-I distribution $\{(m_i \circ \tau, \tau^{-1}(V_i)); i \in I\}$ in D_1 .

The following proposition is a slight modification of Proposition 1 of J. Kajiwaru [4].

PROPOSITION 2. *Let $\{(D_n, \Phi_n); n=1, 2, 3, \dots\}$ be a monotonously increasing sequence of domains over a Stein manifold M . If $\{(D_n, \Phi_n), (D_{n+1}, \Phi_{n+1})\}$ is a Cousin-I pair for any positive integer n , then the limit D of $\{(D_n, \Phi_n)\}$ is a Cousin-I manifold.*

PROOF. Let $\{(m_i, V_i); i \in I\}$ be a Cousin-I distribution in D . Then $\{(m_i \circ \tau_n, \tau_n^{-1}(V_i)); i \in I\}$ is a Cousin-I distribution in D_n . If we put $f_{ij} = m_i - m_j \in H^0(V_i \cap V_j, \mathcal{O})$ and $f_{ij}^n = m_i \circ \tau_n - m_j \circ \tau_n \in H^0(\tau_n^{-1}(V_i \cap V_j), \mathcal{O})$, then $\{f_{ij}^n\} \in Z^1(\tau_n^{-1}(B), \mathcal{O})$ is the canonical image of $\{f_{ij}\} \in Z^1(B, \mathcal{O})$ where $B = \{V_i; i \in I\}$ and $\tau_n^{-1}(B) = \{\tau_n^{-1}(V_i); i \in I\}$. Since $\{(D_n, \Phi_n), (D_{n+1}, \Phi_{n+1})\}$ is a Cousin-I pair for any n , $f_{ij}^{n+1} \circ \tau_{n+1}^n \in B^1(\tau_n^{-1}(B), \mathcal{O})$ for any n . Since $f_{ij}^{n+1} \circ \tau_{n+1}^n = f_{ij}^n$, there exists a holomorphic function f_i in V_i for any i such that $f_{ij} = f_i - f_j$ in $V_i \cap V_j$ from Lemma 6 of J. Kajiwaru [4]. If we put $m = m_i - f_i$ in V_i , m is well-defined and a solution of the distribution $\{(m_i, V_i); i \in I\}$.

§4. Domains over $\mathbb{C}^n$

THEOREM 1. Let (D, π) be a domain over $\mathbb{C}^n$. If D is exhausted by complete Kähler domains $\{X_i; i = 1, 2, 3, \dots\}$, then D is a Cousin-I manifold.

PROOF. Let $\pi = (\pi_1, \dots, \pi_n)$. Then $|\pi_1|^2 + \dots + |\pi_n|^2$ is a C^∞ strictly pluri-subharmonic function on D . By Lemma 3, for every $f \in C_{(n,1)}^\infty(X_{i+1})$ with $\bar{\partial}f = 0$ on X_{i+1} , there exists a $g \in C_{(n,0)}^\infty(X_i)$ such that $\bar{\partial}g = f$ on X_i for every positive integer i . Then for every $f \in C_{(0,1)}^\infty(X_{i+1})$ with $\bar{\partial}f = 0$ on X_{i+1} , there exists a $g \in C^\infty(X_i)$ such that $\bar{\partial}g = f$ on X_i , since local coordinate systems of D are given by $(\pi_1, \dots, \pi_n)$. This implies that (X_i, X_{i+1}) is a Cousin-I pair. By Proposition 2, D is a Cousin-I manifold.

The following lemma is given in M. Abe-Y. Abe [1].

LEMMA 4. Let (D, π) be a domain over $\mathbb{C}^n$. If the following two conditions are satisfied, then D is a Stein manifold.

- i) D is a Cousin-I manifold.
- ii) For every non constant linear function f of n variables without constant term, the set $\{y \in D \mid (f \circ \pi)(y) = 0\}$ is a Stein submanifold of D .

THEOREM 2. Let (D, π) be a domain over $\mathbb{C}^n$. If D is exhausted by complete Kähler domains, then D is a Stein manifold.

PROOF. We shall prove the theorem by the induction on the dimension of $\mathbb{C}^n$. When $n = 1$, D is a Stein manifold. Then we assume that $n \geq 2$ and that the theorem holds for $k < n$. Let f be a non constant linear function of n variables without constant term. Then $\tilde{D} = \{y \in D \mid (f \circ \pi)(y) = 0\}$ is a closed submanifold of D , so $\tilde{D}$ is exhausted by complete Kähler domains by Proposition 1. Every

component of $\tilde{D}$ is a domain over $\mathbb{C}^{n-1}$, so by the assumption of the induction $\tilde{D}$ is a Stein manifold. Therefore by Lemma 4 and Theorem 1, D is a Stein manifold.

§5. Domains over a Stein manifold

Using Theorem 2, we can prove the following theorem by the method developed in F. Docquier-H. Grauert [2].

THEOREM 3. *Let (G, Φ) be a domain over a Stein manifold S . If G is exhausted by complete Kähler domains, then G is a Stein manifold.*

PROOF. By Remmert's imbedding theorem, there exists a holomorphic imbedding $\mu: S \rightarrow \mathbb{C}^k$. By F. Docquier-H. Grauert ([2], Satz 3), we can find a domain (B, π) over $\mathbb{C}^k$, a biholomorphic mapping λ from S onto a closed submanifold M of B , and a holomorphic mapping $\rho: B \rightarrow M$, so that B is a Stein manifold, $\pi \circ \lambda = \mu$, and $\rho|_M$ is identity. We put $\alpha = \lambda \circ \Phi$, and define the set X by

$$X = \{(x, b) \in G \times B \mid \alpha(x) = \rho(b)\}.$$

Putting $\rho'(x, b) = x$, $\alpha'(x, b) = b$, (X, α') becomes a domain over B with respect to the topology induced from the product topology of $G \times B$, and both ρ' and α' are holomorphic. Since B is a Stein manifold, B is exhausted by complete Kähler domains. Therefore $G \times B$ is exhausted by complete Kähler domains. Since X is a closed submanifold of $G \times B$, X is also exhausted by complete Kähler domains by Proposition 1. X is a domain over $\mathbb{C}^k$, so it is a Stein manifold by Theorem 2. We put $\tilde{\lambda}(x) = (x, \lambda \circ \Phi(x))$ for any $x \in G$. $\tilde{\lambda}$ is a biholomorphic mapping from G onto a closed submanifold of X . Therefore G is a Stein manifold.

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