ON A p -ADIC INTERPOLATION FUNCTION FOR THE EULER NUMBERS AND ITS DERIVATIVES

By

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§1. Introduction

The m -th Euler number $H^m(u)$ attached to an algebraic number $u \neq 1$ is defined by the generating function on an indeterminate t

$$\frac{1-u}{e^t-u} = e^{H(u)t} = \sum_{m=0}^{\infty} \frac{H^m(u)}{m!} t^m.$$

Let $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}$ and $\mathbb{C}_p$ be the rational p -adic number field, the integer ring of $\mathbb{Q}_p$, the ring of rational integers and the completion of the algebraic closure of $\mathbb{Q}_p$ respectively. The normalized valuation in $\mathbb{C}_p$ is denoted by $|\cdot|$ with $|p| = p^{-1}$.

It is known and can be easily proved that if $u \neq 0$ satisfies the condition $|1-u^{p^\rho}| \geq 1$ for any natural number ρ , then

$$H^m(u) = \frac{1-u}{u} \lim_{\rho \rightarrow \infty} \sum_{a=0}^{p^\rho-1} a^m \frac{u^{p^\rho-a}}{1-u^{p^\rho}}$$

holds in the field $\mathbb{Q}_p(u)$ [5].

We can define a finitely additive p -adic measure μ_u by setting

$$\mu_u(a + p^\rho \mathbb{Z}_p) = \frac{u^{p^\rho-a}}{1-u^{p^\rho}}$$

for any interval $a + p^\rho \mathbb{Z}_p$ with $0 \leq a < p^\rho$. Then this measure yields an integral

$$\int_{\mathbb{Z}_p} a^m d\mu_u(a) = \lim_{\rho \rightarrow \infty} \sum_{a=0}^{p^\rho-1} a^m \frac{u^{p^\rho-a}}{1-u^{p^\rho}},$$

and the formula above can be written as

$$(1) \quad \int_{\mathbb{Z}_p} a^m d\mu_u(a) = \frac{u}{1-u} H^m(u) \quad \text{for } m \geq 0.$$

Let Z_p^* be the group of units of Z_p , and for any $a \in Z_p^*$ we have the canonical decomposition $a = \omega(a)\langle a \rangle$ with the Teichmüller character ω and principal unit $\langle a \rangle$, where $\langle a \rangle \equiv 1 \pmod{q}$ with $q = p$ ($p > 2$) or $q = 4$ ($p = 2$).

We define a function $G_p(s, u)$ on Z_p by

$$\begin{aligned} G_p(s, u) &= \int_{Z_p^*} \langle a \rangle^{-s} d\mu_u^*(a) \\ &= \lim_{\rho \rightarrow \infty} \sum_{a=1}^{p^\rho} \langle a \rangle^{-s} \frac{u^{p^\rho - a}}{1 - u^{p^\rho}}, \end{aligned}$$

where $*$ means to take sum over all integers prime to p in the given range and thus $d\mu_u^*$ denotes the corresponding measure on Z_p^* to $d\mu_u$. Then this function is analytic in Z_p and regarded as an interpolation function for the Euler numbers. Indeed we have

$$(2) \quad G_p(-m, u) = \frac{u}{1-u} H^m(u) - p^m \frac{u^p}{1-u^p} H^m(u^p)$$

for any non-negative integer $m \equiv 0 \pmod{p-1}$ if $p > 2$ and $m \equiv 0 \pmod{2}$ if $p = 2$.

The continuity of $G_p(s, u)$ gives us the Frobenius congruences for Euler numbers.

In the present paper we calculate the derivatives of $G_p(s, u)$ at $s=0$, and from a relation between the p -adic L -functions $L_p(s, \chi)$ and the functions $G_p(s, u)$ derive an explicit formula for $L'_p(0, \chi)$ with any Dirichlet character χ , which includes the Ferrero-Greenberg [2], and give an explanation of Diamond's formula [1].

§ 2. Partial log gamma functions

Let $\mathcal{C}(Z_p, C_p)$ be the space of all continuous functions on Z_p with values in C_p and $\mathcal{UD}(Z_p, C_p)$ be its subspace of all uniformly differentiable functions. These become to C_p -Banach algebras with usual pointwise operations under certain norms.

LEMMA 1. For any $f \in \mathcal{C}(Z_p, C_p)$ we have $\lim_{\rho \rightarrow \infty} \sum_{i=0}^{p^\rho-1} f(i) = 0$.

This can be easily proved, and is useful.

LEMMA 2. For any $f \in \mathcal{C}(Z_p, C_p)$ there exists uniquely $F \in \mathcal{C}(Z_p, C_p)$ such that $\Delta F(x) = F(x+1) - F(x) = f(x)$, $F(0) = 0$. If f belongs to $\mathcal{UD}(Z_p, C_p)$, then the function F belongs to $\mathcal{UD}(Z_p, C_p)$.

PROOF. For $n \in \mathbb{N}$, the set of natural numbers, set $F(n) = \sum_{i=0}^{n-1} f(i)$. Then for any $\rho, k \in \mathbb{N}$ we have $F(n + p^\rho k) - F(n) = \sum_{i=n}^{n+p^\rho k-1} f(i) = \sum_{j=0}^{p^\rho k-1} f(n+j)$. Because f is uniformly continuous on $\mathbb{Z}_p$ by compactness of $\mathbb{Z}_p$, this sum tends to 0 uniformly with respect to n when ρ tends to infinity. Therefore we can obtain $F \in \mathcal{C}(\mathbb{Z}_p, \mathbb{C}_p)$ by setting $F(x) = \lim_{\substack{n \rightarrow x \\ n \in \mathbb{N}}} F(n)$.

By definition the function F satisfies $\Delta F(x) = f(x)$ and $F(0) = 0$, and it is clear that such function F is unique for a given f . By a theorem of Mahler we have $f(x) = \sum_{k=0}^{\infty} \Delta^k f(0) \binom{x}{k}$ on $\mathbb{Z}_p$, where $\Delta^k f(0) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(j)$. Then it is known that $f \in \mathcal{UD}(\mathbb{Z}_p, \mathbb{C}_p)$ is equivalent to $k|\Delta^k f(0)| \rightarrow 0$ ($k \rightarrow \infty$) [8]. From $\Delta F(n) = f(n)$ we see $\Delta^k F(0) = \Delta^{k-1} f(0)$ ($k \geq 1$). Hence $k|\Delta^k F(0)| = (k-1) \cdot |\Delta^{k-1} f(0)| + |\Delta^{k-1} f(0)|$ and this tends to 0 when k tends to infinity. Thus we have $F \in \mathcal{UD}(\mathbb{Z}_p, \mathbb{C}_p)$ if $f \in \mathcal{UD}(\mathbb{Z}_p, \mathbb{C}_p)$.

Similarly the assertions are valid for any function in $\mathcal{C}(\mathbb{Z}_p^*, \mathbb{C}_p)$, $\mathcal{UD}(\mathbb{Z}_p^*, \mathbb{C}_p)$.

Now, for $e \geq 0$, $1 \leq v < p^{e+1}$, $(v, p) = 1$ we have $\delta_v^{(e)}(x)$, the characteristic function of the set $v + p^{e+1}\mathbb{Z}_p$. This function $\delta_v^{(e)}(x)$ belongs to $\mathcal{UD}(\mathbb{Z}_p, \mathbb{C}_p)$. The logarithmic function $\log x$ also belongs to $\mathcal{UD}(\mathbb{Z}_p^*, \mathbb{C}_p)$. Hence we obtain by Lemma 2 the log gamma function and the partial log gamma function as follows:

$$(3) \quad \log \Gamma_p(x) = \lim_{\substack{n \rightarrow x \\ n \in \mathbb{N}}} \sum_{b=1}^{n-1} \log b,$$

$$(4) \quad \log \Gamma_p(x, v, p^{e+1}) = \lim_{\substack{n \rightarrow x \\ n \in \mathbb{N}}} \sum_{b=1}^{n-1} \delta_v^{(e)}(b) \log b.$$

Then these functions belong to $\mathcal{UD}(\mathbb{Z}_p, \mathbb{C}_p)$.

From the definition we see immediately

$$(5) \quad \log \Gamma_p(x) = \sum_{v=1}^{p^{e+1}} \log \Gamma_p(x, v, p^{e+1}),$$

$$(6) \quad \log \Gamma_p(n, v, p^{e+1}) = 0 \quad \text{for } n \in \mathbb{Z}, 0 \leq n \leq v.$$

$$(7) \quad \log \Gamma_p(x+1, v, p^{e+1}) = \log \Gamma_p(x, v, p^{e+1}) + \delta_v^{(e)}(x) \log x$$

for any $x \in \mathbb{Z}_p$.

$$(8) \quad \log \Gamma_p(x, -v, p^{e+1}) = -\log \Gamma_p(1-x, v, p^{e+1}).$$

We set $N_v(x) = \lim_{\substack{n \rightarrow x \\ n \in \mathbb{N}}} \sum_{i=0}^{n-1} \delta_v^{(e)}(i)$. Then we have

PROPOSITION 1. For any $c > 1$, $c \equiv 1 \pmod{p^{e+1}}$ and any $x \in \mathbb{Z}_p$ we have

$$(9) \quad \sum_{i=0}^{c-1} \log \Gamma_p\left(\frac{x+i}{c}, v, p^{e+1}\right) \\ = \log \Gamma_p(x, v, p^{e+1}) + \sum_{i=0}^{c-1} \log \Gamma_p\left(\frac{i}{c}, v, p^{e+1}\right) - N_v(x) \log c.$$

PROOF. We see with natural numbers n

$$\begin{aligned} \log \Gamma_p\left(\frac{x+i}{c}, v, p^{e+1}\right) &= \lim_{n \rightarrow \frac{x}{c}} \log \Gamma_p\left(\frac{i}{c} + n, v, p^{e+1}\right) \\ &= \log \Gamma_p\left(\frac{i}{c}, v, p^{e+1}\right) + \lim_{n \rightarrow \frac{x}{c}} \sum_{j=0}^{n-1} \delta_v^{(e)}\left(\frac{i}{c} + j\right) \log\left(\frac{i}{c} + j\right) \\ &= \log \Gamma_p\left(\frac{i}{c}, v, p^{e+1}\right) + \lim_{n \rightarrow \frac{x}{c}} \sum_{j=0}^{n-1} \delta_v^{(e)}(i + cj) \log(i + cj) \\ &\quad - \lim_{n \rightarrow \frac{x}{c}} \sum_{j=0}^{n-1} \delta_v^{(e)}(i + cj) \log c. \end{aligned}$$

Hence we have

$$\begin{aligned} \sum_{i=0}^{c-1} \log \Gamma_p\left(\frac{x+i}{c}, v, p^{e+1}\right) \\ = \sum_{i=0}^{c-1} \log \Gamma_p\left(\frac{i}{c}, v, p^{e+1}\right) + \lim_{n \rightarrow x} \sum_{k=0}^{n-1} \delta_v^{(e)}(k) \log k - N_v(x) \log c. \end{aligned}$$

Now, we know the Diamond function $D_p(x)$ for any $x \in \mathbb{C}_p$, $|x| > 1$ defined by

$$D_p(x) = \frac{1}{2} - x + \lim_{\rho \rightarrow \infty} \frac{1}{p^\rho} \sum_{k=0}^{p^\rho-1} (x+k) \log(x+k).$$

This limit exists because $(x+a) \log(x+a) \in \mathcal{UD}(\mathbb{Z}_p, \mathbb{C}_p)$, and can be denoted by $\int_{\mathbb{Z}_p} (x+a) \log(x+a) d\mu_0(a)$, where $d\mu_0$ means a p -adic measure defined by $\mu_0(a + p^\rho \mathbb{Z}_p) = \frac{1}{p^\rho}$ for any interval $a + p^\rho \mathbb{Z}_p$ in $\mathbb{Z}_p$.

PROPOSITION 2. Notations e, v being the same as above, we have

$$(10) \quad \int_{\mathbf{Z}_p} \log \Gamma_p(a+1, v, p^{e+1}) d\mu_0(a) = -D_p\left(\frac{v}{p^{e+1}}\right) - \frac{v}{p^{e+1}} + \frac{1}{2}.$$

PROOF. The integral on the left-hand side is equal to

$$\begin{aligned} & \lim_{\rho \rightarrow \infty} \frac{1}{p^\rho} \sum_{i=1}^{p^\rho-1} \sum_{b=1}^i \delta_v^{(e)}(b) \log b \\ &= \lim_{\rho \rightarrow \infty} \frac{1}{p^\rho} \sum_{b=1}^{p^\rho-1} \delta_v^{(e)}(b) \log b \sum_{i=b}^{p^\rho-1} 1 \\ &= \lim_{\rho \rightarrow \infty} \frac{1}{p^\rho} \sum_{\substack{b=1 \\ b \equiv v \pmod{p^{e+1}}}}^{p^\rho-1} (p^\rho - b) \log b \\ &= \log \Gamma_p(0, v, p^{e+1}) - \lim_{\rho \rightarrow \infty} \frac{1}{p^{\rho-e-1}} \sum_{k=0}^{p^\rho-e-1-1} \left(\frac{v}{p^{e+1}} + k \right) \log \left(\frac{v}{p^{e+1}} + k \right) \\ &= -D_p\left(\frac{v}{p^{e+1}}\right) - \frac{v}{p^{e+1}} + \frac{1}{2}. \end{aligned}$$

REMARK. Because $\log \Gamma_p(x, v, p^{e+1}) \in \mathcal{UD}(\mathbf{Z}_p, \mathbf{C}_p)$ we see

$$\int_{\mathbf{Z}_p} \log \Gamma_p(a+1, v, p^{e+1}) d\mu_0(a) = \lim_{\rho \rightarrow \infty} \frac{1}{p^\rho} \sum_{i=1}^{p^\rho} \log \Gamma_p\left(\frac{i}{1+p^\rho}, v, p^{e+1}\right).$$

Indeed, in general, for any $f \in \mathcal{UD}(\mathbf{Z}_p, \mathbf{C}_p)$ denote by F_f the Newton quotient $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$, $x \neq y$, of f . Then we see readily

$$\begin{aligned} & \frac{1}{p^\rho} \sum_{i=0}^{p^\rho-1} f\left(\frac{i}{1+p^\rho}\right) - \frac{1}{p^\rho} \sum_{i=0}^{p^\rho-1} f(i) \\ &= -\frac{1}{1+p^\rho} \sum_{i=0}^{p^\rho-1} i F_f(i, i) - \frac{1}{1+p^\rho} \sum_{i=0}^{p^\rho-1} i \left(F_f\left(\frac{i}{1+p^\rho}, i\right) - F_f(i, i) \right). \end{aligned}$$

From the uniform continuity of F_f on $\mathbf{Z}_p \times \mathbf{Z}_p$ and Lemma 1 we obtain

$$\lim_{\rho \rightarrow \infty} \frac{1}{p^\rho} \sum_{i=0}^{p^\rho-1} f\left(\frac{i}{1+p^\rho}\right) = \int_{\mathbf{Z}_p} f(a) d\mu_0(a).$$

We have from Proposition 2 and (5) with $e \geq 1$

$$\sum_{k=0}^{p-1} D_p\left(\frac{v+p^e k}{p^{e+1}}\right) = D_p\left(\frac{v}{p^e}\right).$$

Namely, the function $D_p(x)$ gives us a p -adic measure on $\mathbf{Z}_p^*$, as is well known.

§3. The functions $G_p(s, u)$

In this section we assume $|u| \leq 1$ and $|u-1|=1$ and concern ourselves with the continuous function $G_p(s, u)$ on $\mathbf{Z}_p$ defined in Introduction.

PROPOSITION 3. *The function $G_p(s, u)$ is analytic on $\mathbf{Z}_p$ and $G_p(s, u) = \sum_{k=0}^{\infty} \binom{-s}{k} \int_{\mathbf{Z}_p^*} (\langle a \rangle - 1)^k d\mu_u^*(a)$ holds.*

PROOF. From the assumption of u we see

$$\left| \int_{\mathbf{Z}_p^*} (\langle a \rangle - 1)^k d\mu_u^*(a) \right| \leq \sup_{a \in \mathbf{Z}_p^*} |\langle a \rangle - 1|^k = q^{-k}.$$

and $\Delta^k G_p(-s, u)|_{s=0} = \int_{\mathbf{Z}_p^*} (\langle a \rangle - 1)^k d\mu_u^*(a)$. The equality is nothing but the Mahler expansion of $G_p(-s, u)$. Moreover $\frac{1}{k!} \Delta^k G_p(-s, u)|_{s=0} \rightarrow 0$ ($k \rightarrow \infty$) shows that $G_p(s, u)$ is analytic.

REMARK. $f(s) = \sum_{k=0}^{\infty} a_k \binom{s}{k} \in \mathcal{C}(\mathbf{Z}_p, \mathbf{C}_p)$ with $a_k \in \mathbf{C}_p$, where $a_k \rightarrow 0$ ($k \rightarrow \infty$), is analytic if and only if $\frac{1}{k!} a_k \rightarrow 0$ ($k \rightarrow \infty$).

Indeed, the function $f(s)$ is called analytic if $f(s) = \sum_{k=0}^{\infty} c_k s^k$ holds for any $s \in \mathbf{Z}_p$, with $c_k \in \mathbf{C}_p$, where $c_k \rightarrow 0$ ($k \rightarrow \infty$).

The uniqueness of Mahler expansion gives us

$$a_k = \Delta^k f(0) = \sum_{j=k}^{\infty} c_j \Delta^k 0^j$$

and

$$\left| \frac{1}{k!} a_k \right| \leq \max_{k \leq j} \{ |c_j \mathfrak{S}(j, k)| \},$$

where $\mathfrak{S}(j, k) \in \mathbf{Z}$ means the Stirling number of the second kind. Thus $c_k \rightarrow 0$ ($k \rightarrow \infty$) yields $\frac{1}{k!} a_k \rightarrow 0$ ($k \rightarrow \infty$). Conversely, if this fact holds, then $c_k = \sum_{j=k}^{\infty} \frac{1}{j!} a_j S(j, k) \in \mathbf{C}_p$ with the Stirling number $S(j, k)$ of the first kind exist and $f(s)$ can be expressed as $f(s) = \sum_{k=0}^{\infty} c_k s^k$, with $c_k \rightarrow 0$ ($k \rightarrow \infty$). Now, $G_p(s, u) = \int_{\mathbf{Z}_p^*} \langle a \rangle^{-s} d\mu_u^*(a)$ is uniformly convergent with respect to $s \in \mathbf{Z}_p$, hence

$$G_p'(s, u) = - \int_{\mathbf{Z}_p^*} \langle a \rangle^{-s} \log a d\mu_u^*(a).$$

In general we denote by ζ_f a primitive f -th root of unity. For given $f \in \mathbb{N}$ we decompose $f = f_0 p^e$, $(f_0, p) = 1$, $e \geq 0$. If $f_0 > 1$, then $|\zeta_f| = |1 - \zeta_f| = 1$. We select ζ_f as u . Thus we have

$$\begin{aligned} G'_p(0, \zeta_f) &= \lim_{\rho \rightarrow \infty} \sum_{a=1}^{p^\rho} \frac{\zeta_f^{p^\rho - a}}{\zeta_f^{p^\rho} - 1} \log a \\ &= \lim_{\rho \rightarrow \infty} \sum_{a=1}^{p^\rho} \frac{\zeta_f^a}{\zeta_f^{p^\rho} - 1} \log a. \end{aligned}$$

Here we set $\zeta_f = \zeta_{p^e} \zeta_{f_0}$ and choose $r \in \mathbb{N}$ so that $r = p^m$, $m \geq e + 1$, $p^m \equiv 1 \pmod{f_0}$. Then, by making use of p^r instead of p^ρ in the above, we see

$$G'_p(0, \zeta_f) = \lim_{\rho \rightarrow \infty} \sum_{v=1}^{p^{e+1}} \zeta_{p^e}^v \sum_{w=1}^{f_0} \frac{\zeta_{f_0}^w}{\zeta_{f_0} - 1} \sum_{\substack{1 \leq a \leq p^\rho \\ a \equiv w \pmod{f_0}}} \delta_v^{(e)}(a) \log a.$$

Substituting $\frac{\zeta_{f_0}^w}{\zeta_{f_0} - 1} = \sum_{j=0}^{w-1} \zeta_{f_0}^j + \frac{1}{\zeta_{f_0} - 1}$ and using Lemma 1 we see that the limit is equal to

$$\lim_{\rho \rightarrow \infty} \sum_{j=0}^{f_0-1} \zeta_{f_0}^j \sum_{v=1}^{p^{e+1}} \zeta_{p^e}^{f_0 v} \sum_{w=j+1}^{f_0} \sum_{\substack{a \equiv f_0 v \pmod{p^{e+1}} \\ a \equiv w \pmod{f_0}}} \log a.$$

Next, set $k_\rho = \frac{p^\rho - 1}{f_0} \in \mathbb{N}$ and

$$\begin{aligned} A_p^{(v)}(j) &= \{a \in \mathbb{Z}; 1 \leq a \leq p^\rho - 1, a = w + f_0 t, j + 1 \leq w \leq f_0, \\ &0 \leq t \leq k_\rho - 1, a \equiv f_0 v \pmod{p^{e+1}}\}. \end{aligned}$$

Then we can write

$$G'_p(0, \zeta_f) = \sum_{j=0}^{f_0-1} \zeta_{f_0}^j \sum_{v=1}^{p^{e+1}} \zeta_{p^e}^{f_0 v} \lim_{\rho \rightarrow \infty} \sum_{a \in A_p^{(v)}(j)} \log a.$$

Because for each $a \in A_p^{(v)}(j)$ it holds that $k_\rho a \equiv -(k_\rho(f_0 - w) + t + 1) \pmod{p^\rho}$, set

$$B_p^{(v)}(j) = \{b \in \mathbb{Z}; 1 \leq b \leq k_\rho(f_0 - j), b \equiv v \pmod{p^{e+1}}\}.$$

Then it can be seen that the sets $A_p^{(v)}(j)$ and $B_p^{(v)}(j)$ are corresponding each other bijectively.

Here we see

$$\lim_{\rho \rightarrow \infty} \sum_{a \in A_{\rho}^{(v)}(j)} \log a = \lim_{\rho \rightarrow \infty} \sum_{b \in B_{\rho}^{(v)}(j)} \log b + N_v \left(\frac{j}{f_0} \right) \log f_0,$$

where we have used $N_v \left(\frac{j}{f_0} \right) = \lim_{\rho \rightarrow \infty} \#B_{\rho}^{(v)}(j)$ and $\#B_{\rho}^{(v)}(j) = \sum_{b=1}^{k_{\rho}(f_0-j)} \delta_v^{(e)}(b)$.

LEMMA 3. We have $N_v(x) = - \left[\frac{v-x}{p^{e+1}} \right]$ for any $x \in \mathbb{Z}_p$ with a generalized Gauss symbol defined by

$$\left[\frac{y}{p^{e+1}} \right] = \frac{1}{p^{e+1}} (y - r_{p^{e+1}}(y)), \quad 0 \leq r_{p^{e+1}}(y) \leq p^{e+1} - 1,$$

$$y \equiv r_{p^{e+1}}(y) \pmod{p^{e+1}} \quad \text{for any } y \in \mathbb{Z}_p.$$

PROOF. We write $\zeta = \zeta_{p^{e+1}}$ for the sake of simplicity. First, we see

$$\delta_v^{(e)}(x) = \frac{1}{p^{e+1}} \sum_{i=0}^{p^{e+1}-1} \zeta^{(v-x)i}.$$

Therefore we have

$$\sum_{b=1}^{n-1} \delta_v^{(e)}(b) = \frac{1}{p^{e+1}} (n-1) + \frac{1}{p^{e+1}} \sum_{i=1}^{p^{e+1}-1} \frac{\zeta^{vi} - \zeta^{(v-n+1)i}}{\zeta^i - 1}.$$

Consequently we have

$$N_v(x) = \frac{1}{p^{e+1}} (x-1) - \frac{1}{p^{e+1}} \sum_{i=1}^{p^{e+1}-1} \frac{\zeta^{vi} - \zeta^{(v+1-x)i}}{1 - \zeta^i}.$$

However we see easily

$$\sum_{i=1}^{p^{e+1}-1} \frac{\zeta^{vi}}{1 - \zeta^i} = (Bp^{e+1} + v - 1)^1 - B^1 \quad \text{for } 1 \leq v \leq p^{e+1},$$

where B^1 means the first Bernoulli number, namely $B^1 = -\frac{1}{2}$.

Similarly we have

$$\sum_{i=1}^{p^{e+1}-1} \frac{\zeta^{(v+1-x)i}}{1 - \zeta^i} = (Bp^{e+1} + r_{p^{e+1}}(v-x))^1 - B^1.$$

Therefore we have finally

$$N_v(x) = - \frac{1}{p^{e+1}} ((v-x) - r_{p^{e+1}}(v-x))$$

$$= -\left[\frac{v-x}{p^{e+1}} \right].$$

Especially we have

$$N_v\left(\frac{j}{f_0}\right) = \frac{1}{f_0 p^{e+1}} \left(j + f_0 r_{p^{e+1}} \left(v - \frac{j}{f_0} \right) \right) - \frac{1}{p^{e+1}} v.$$

We notice that, when $0 \leq j \leq f_0 - 1$, $1 \leq v \leq p^{e+1}$, $(v, p) = 1$, then $0 \leq j + f_0 r_{p^{e+1}} \left(v - \frac{j}{f_0} \right) \leq f_0 p^{e+1} - 1$ and

$$j + f_0 r_{p^{e+1}} \left(v - \frac{j}{f_0} \right) \equiv \begin{cases} j & (\text{mod } f_0), \\ f_0 v & (\text{mod } p^{e+1}). \end{cases}$$

Therefore we have

$$\begin{aligned} \sum_{j=0}^{f_0-1} \zeta_{f_0}^j \sum_{v=1}^{p^{e+1}} \zeta_{p^{e+1}}^{f_0 v} N_v\left(\frac{j}{f_0}\right) &= \frac{1}{f p} \sum_{k=0}^{p-1} \zeta_f^k k \\ &= \frac{\zeta_f^p - \zeta_f}{(\zeta_f - 1)(\zeta_f^p - 1)}. \end{aligned}$$

Thus we obtain

THEOREM 1. We have for $f_0 > 1$, $\zeta_f = \zeta_{f_0} \zeta_{p^e}$

$$\begin{aligned} G'_p(0, \zeta_f) &= \sum_{j=0}^{f_0-1} \zeta_{f_0}^j \sum_{v=1}^{p^{e+1}} \zeta_{p^{e+1}}^{f_0 v} \log \Gamma_p\left(\frac{j}{f_0}, v, p^{e+1}\right) \\ &\quad + \frac{\zeta_f^p - \zeta_f}{(1 - \zeta_f)(1 - \zeta_f^p)} \log f_0. \end{aligned}$$

In particular we have

$$G'_p(0, \zeta_{f_0}) = \sum_{j=0}^{f_0-1} \zeta_{f_0}^j \log \Gamma_p\left(\frac{j}{f_0}\right) + \frac{\zeta_{f_0}^p - \zeta_{f_0}}{(1 - \zeta_{f_0})(1 - \zeta_{f_0}^p)} \log f_0.$$

§ 4. A calculation of $L'_p(0, \chi)$

Let χ be a primitive Dirichlet character modulo f not equal to the principal character χ^0 . Take $c \in \mathbb{Z}$ such as $c > 1$, $(c, pf) = 1$.

It is known and can be easily proved that for the generalized Bernoulli number B_χ^m ($m \geq 1$) holds

$$\begin{aligned}
(11) \quad & -(1 - \chi(c)c^m) \frac{B_{\chi}^m}{m} \\
&= \frac{\chi(-1)}{f} \tau(\chi) \sum_{i=1}^f \bar{\chi}(i) \sum_{\lambda=1}^{c-1} \frac{\zeta_f^{-i} \zeta_c^{-\lambda}}{1 - \zeta_f^{-i} \zeta_c^{-\lambda}} H^{m-1}(\zeta_f^{-i} \zeta_c^{-\lambda}) \\
&= \frac{\chi(-1)}{f} \tau(\chi) \sum_{i=1}^f \bar{\chi}(i) \sum_{\lambda=1}^{c-1} \int_{\mathbf{Z}_p} a^{m-1} d\mu_{\zeta_f^{-i} \zeta_c^{-\lambda}}(a),
\end{aligned}$$

where $\bar{\chi}$ means the inverse character of χ and $\tau(\chi) = \sum_{a=1}^f \chi(a) \zeta_f^a$, the normalized Gauss sum.

Moreover we have the following

PROPOSITION 4. *For any $m \geq 1$ we have*

$$\begin{aligned}
& -(1 - \chi(c)c^m)(1 - \chi(p)p^{m-1}) \frac{B_{\chi}^m}{m} \\
&= \frac{\chi(-1)}{f} \tau(\chi) \sum_{i=1}^f \bar{\chi}(i) \sum_{\lambda=1}^{c-1} \int_{\mathbf{Z}_p^*} a^{m-1} d\mu_{\zeta_f^{-i} \zeta_c^{-\lambda}}^*(a).
\end{aligned}$$

PROOF. Since we have $d\mu_u(pa) = d\mu_{up}(a)$ to the parameter u , we see

$$\int_{p\mathbf{Z}_p} a^{m-1} d\mu_u(a) = p^{m-1} \int_{\mathbf{Z}_p} a^{m-1} d\mu_{up}(a).$$

Hence it suffices to show

$$-(1 - \chi(c)c^m)\chi(p) \frac{B_{\chi}^m}{m} = \frac{\chi(-1)}{f} \tau(\chi) \sum_{i=1}^f \bar{\chi}(i) \sum_{\lambda=1}^{c-1} \int_{\mathbf{Z}_p} a^{m-1} d\mu_{\zeta_f^{-i} \zeta_c^{-\lambda}}(a).$$

But this is clear in the case $(p, f) = 1$. In the case $f \equiv 0 \pmod{p}$ we can see readily the value of the right-hand side is also equal to 0 by changing i to i' , $i' \equiv ij \pmod{f}$ with $j \in \mathbf{Z}$, $(j, f) = 1$, $j \equiv 1 \pmod{f'}$, $\chi(j) \neq 1$, $f = pf'$.

PROPOSITION 5. *Let χ be any even Dirichlet character and f the conductor of $\chi\omega^{-1}$. Set $d = p - 1$ for $p > 2$ and $d = 2$ for $p = 2$. Then we have for any $s \in \mathbf{Z}_p$ with $s \equiv 0 \pmod{d}$*

$$(1 - \chi(c)\langle c \rangle^{1-s})L_p(s, \chi) = -\frac{\tau(\chi\omega^{-1})}{f} \sum_{i=1}^f \overline{\chi\omega^{-1}}(i) \sum_{\lambda=1}^{c-1} G_p(s, \zeta_f^{-i} \zeta_c^{-\lambda}).$$

PROOF. In Proposition 4 by replacing χ , m with $\chi\omega^{-1}$, $m + 1$ respectively we have for any $m \geq 0$

$$\begin{aligned}
& -(1 - \chi\omega^{-1}(c)c^{m+1})(1 - \chi\omega^{-1}(p)p^m) \frac{B_{\chi}^{m+1}}{m+1} \\
& = -\frac{\tau(\chi\omega^{-1})}{f} \sum_{i=1}^f \overline{\chi\omega^{-1}}(i) \sum_{\lambda=1}^{c-1} \int_{\mathbf{Z}_p^*} a^m d\mu_{\zeta_f^{-i}\zeta_c^{-\lambda}}^*(a).
\end{aligned}$$

When $m \equiv 0 \pmod{d}$, from the definitions of

$$L_p(s, \chi) = -\frac{1}{1-s} \int_{X^*} \chi(a) \langle a \rangle^{1-s} d\mu_0^*(a) \quad \text{and} \quad G_p(s, u)$$

we have

$$(1 - \chi(c) \langle c \rangle^{1+m}) L_p(-m, \chi) = -\frac{\tau(\chi\omega^{-1})}{f} \sum_{i=1}^f \overline{\chi\omega^{-1}}(i) \sum_{\lambda=1}^{c-1} G_p(-m, \zeta_f^{-i}\zeta_c^{-\lambda}).$$

Herein X^* means the multiplicative group $(\mathbf{Z}/f_0\mathbf{Z})^* \times \mathbf{Z}_p^*$ with f_0 p -prime component of the conductor of χ . Thus we have our assertion, since both functions are continuous.

Now we consider the derivatives of the both functions at $s=0$. Namely we see

$$\begin{aligned}
(12) \quad & \chi(c) \langle c \rangle \log \langle c \rangle L_p(0, \chi) + (1 - \chi(c) \langle c \rangle) L'_p(0, \chi) \\
& = -\frac{\tau(\chi\omega^{-1})}{f} \sum_{i=1}^f \overline{\chi\omega^{-1}}(i) \sum_{\lambda=1}^{c-1} G'_p(0, \zeta_f^{-i}\zeta_c^{-\lambda}).
\end{aligned}$$

Set $f = p^e f_0$, $e \geq 0$, $(f_0, p) = 1$ and $\zeta_f = \zeta_{p^e} \zeta_{f_0}$. Furthermore we take c such as $c \equiv 1 \pmod{p^{e+1}}$ and select $l_{j, f_0 v} \in \mathbf{Z}$ so that $l_{j, f_0 v} \equiv j \pmod{f_0}$ and $l_{j, f_0 v} \equiv f_0 v \pmod{p^{e+1}}$. Then from Theorem 1, the formula (12) and Lemma 3 together with $L_p(0, \chi) = -(1 - \chi\omega^{-1}(p)) B_{\chi\omega^{-1}}^1$ we have

$$\begin{aligned}
(13) \quad & \sum_{j=0}^{f_0-1} \sum_{v=1}^{p^{e+1}}^* \chi\omega^{-1}(l_{j, f_0 v}) N_v \left(\frac{j}{f_0} \right) \\
& = (1 - \chi\omega^{-1}(p)) B_{\chi\omega^{-1}}^1 - \frac{1}{p^{e+1}} \sum_{j=0}^{f_0-1} \sum_{v=1}^{p^{e+1}}^* \chi\omega^{-1}(l_{j, f_0 v}) v
\end{aligned}$$

and

$$\begin{aligned}
(14) \quad & (1 - \chi\omega^{-1}(c)c) L'_p(0, \chi) \\
& = (1 - \chi\omega^{-1}(c)c) \sum_{j=0}^{f_0-1} \sum_{v=1}^{p^{e+1}}^* \chi\omega^{-1}(l_{j, f_0 v}) \log \Gamma_p \left(\frac{j}{f_0}, v, p^{e+1} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{f_0-1} \sum_{v=1}^{p^{e+1}} \chi \omega^{-1}(l_{j, f_0 v}) \sum_{i=1}^{c-1} \log \Gamma_p \left(\frac{i}{c}, v, p^{e+1} \right) \\
& + \frac{1}{p^{e+1}} \sum_{j=0}^{f_0-1} \sum_{v=1}^{p^{e+1}} \chi \omega^{-1}(l_{j, f_0 v}) v \log c \\
& + (1 - \chi \omega^{-1}(c)c)(1 - \chi \omega^{-1}(p)) B_{\chi \omega^{-1}}^1 \log f_0.
\end{aligned}$$

If $f_0 \neq 1$, then $\sum_{j=0}^{f_0-1} \chi \omega^{-1}(l_{j, f_0 v}) = 0$. Thus we obtain the following

THEOREM 2. We have for $f_0 > 1$

$$\begin{aligned}
L'_p(0, \chi) &= \sum_{j=0}^{f_0-1} \sum_{v=1}^{p^{e+1}} \chi \omega^{-1}(l_{j, f_0 v}) \log \Gamma_p \left(\frac{j}{f_0}, v, p^{e+1} \right) \\
&+ (1 - \chi \omega^{-1}(p)) B_{\chi \omega^{-1}}^1 \log f_0.
\end{aligned}$$

In the case $(f, p) = 1$ this formula reduces to the Ferrero-Greenberg [2]

$$L'_p(0, \chi) = \sum_{j=0}^{f-1} \chi \omega^{-1}(j) \log \Gamma_p \left(\frac{j}{f} \right) + (1 - \chi \omega^{-1}(p)) B_{\chi \omega^{-1}}^1 \log f.$$

If $f_0 = 1$, the formula (14) becomes to

$$\begin{aligned}
(1 - \chi \omega^{-1}(c)c) L'_p(0, \chi) &= \sum_{v=1}^{p^{e+1}} \chi \omega^{-1}(v) \sum_{i=1}^{c-1} \log \Gamma_p \left(\frac{i}{c}, v, p^{e+1} \right) \\
&+ \frac{1}{p^{e+1}} \sum_{v=1}^{p^{e+1}} \chi \omega^{-1}(v) v \log c.
\end{aligned}$$

Here, by taking $c = 1 + p^\rho$ and tending ρ to infinity we have from Proposition 2 and Remark to it

$$\begin{aligned}
L'_p(0, \chi) &= \sum_{v=1}^{p^{e+1}} \chi \omega^{-1}(v) D_p \left(\frac{v}{p^{e+1}} \right) \\
&= \sum_{v=1}^{p^e} \chi \omega^{-1}(v) D_p \left(\frac{v}{p^e} \right).
\end{aligned}$$

If we take $c = 1 + fp^\rho$ in the formula (14) and make ρ tend to infinity we have just the Diamond formula in general case [1], namely

$$L'_p(0, \chi) = (1 - \chi \omega^{-1}(p)) B_{\chi \omega^{-1}}^1 \log f_0 + \sum_{a=1}^{pf} \chi \omega^{-1}(a) D_p \left(\frac{a}{pf} \right).$$

Indeed we have only to note that

$$D_p\left(\frac{j+f_0r_{p^{e+1}}\left(v-\frac{j}{f_0}\right)}{p^{e+1}f_0}\right)=D_p\left(\frac{v}{p^{e+1}}\right)+\log\Gamma_p\left(\frac{j}{f_0}, v, p^{e+1}\right)$$

for $0\leq j\leq f_0-1$.

Hence, another way to obtain our Theorem 2 is first to derive Diamond's formula, which can quite readily be obtained directly by the definition of $L_p(s, \chi)$ and next to transform the formula into our formula by making use of properties of Diamond's functions $D_p(x)$. However our way in the above gives us more explicit formula for $L'_p(0, \chi)$, namely it contains the parameter c , which is always very useful in the theory of p -adic L -functions.

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