DENSITY THEOREMS FOR INVARIANT CLOSED GEODESICS OF A COMPACT RIEMANNIAN MANIFOLD WITH STRICTLY NEGATIVE CURVATURE

By

Midori S. GOTO and Kagumi UESU

(Received March 28, 1984)

§0. Introduction

Let M be a compact Riemannian manifold of strictly negative sectional curvature. Then it is well known that there is (up to parametrization) a unique closed geodesic in each class of freely-homotopic closed curves in M and that the fundamental group of M has exponential growth, cf. [5], [8] and [9]. The asymptotic behavior of the number of closed geodesics (of multiplicity one) has been studied by many authors, e.g. Bowen [1], [2], DeGeorge [3], Margulis [7], Sinai [11], etc. Among them in [7], Margulis announced that on M , the number of closed geodesics whose lengths are not longer than t grows like a function e^{dt}/ct as t tends to ∞ , where c and d are constants. Later Parry and Pollicott showed that c and d are equal to the topological entropy of the geodesic flow of the unit tangent bundle of M . On the other hand, on a compact manifold, the number of closed geodesics invariant under isometries has been studied by several authors, e.g. Grove, Tanaka cf. [4].

In this paper we are concerned with the number of closed geodesics and present density theorems for the set of certain closed geodesics in the set of all closed geodesics of a compact Riemannian manifold with strictly negative sectional curvature.

First we obtain the following theorem.

THEOREM 1. *Let M be a compact Riemannian manifold with strictly negative sectional curvature, $\pi: M \rightarrow M_0$ a finite normal Riemannian covering, G the group of covering transformations. Let f be the order of G . Denote by $v(t)$ (resp. $v^i(t)$) the number of closed geodesics of M with minimal periods $\leq t$*

(resp. those of M with minimal periods $\leq t$ which have i -images under the action of G). Then

$$\lim_{t \rightarrow \infty} \frac{v^{z_1}(t) + \cdots + v^{z_{r-1}}(t)}{v(t)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{v^f(t)}{v(t)} = 1,$$

where $1 = z_1 < z_2 < \cdots < z_r = f$ are the totality of the divisors of f .

As an immediate consequence we have the following corollary.

COROLLARY 1. *Let M be a compact Riemannian manifold with strictly negative sectional curvature. Let g be an isometry of M . Let $v^{(g)}(t)$ (resp. $v(t)$) denote the number of closed geodesics of M which are preserved under the action of g with minimal periods $\leq t$ (resp. closed geodesics of M with minimal periods $\leq t$). If the group generated by g acts on M freely, then*

$$\lim_{t \rightarrow \infty} \frac{v^{(g)}(t)}{v(t)} = 0.$$

Using the above result we prove,

THEOREM 2. *Let M be a compact Riemannian manifold with strictly negative sectional curvature, $\pi: M \rightarrow M_0$ a finite normal Riemannian covering and G the group of covering transformations. Let g be an element of G which generates a maximal cyclic subgroup G_1 of G . If G_1 has order $p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}$, where p_i 's are prime numbers with $p_1 < p_2 < \cdots < p_k$ and q_i 's are positive integers, then*

$$\lim_{t \rightarrow \infty} \frac{v_0^{(g)}(t)}{v_0(t)} = \frac{p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}}{\#N(G_1)} \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right),$$

where $N(G_1)$ denotes the normalizer of G_1 in G .

Theorem 2 is inspired by a work of T. Sunada [12]. He brought many notions in algebraic number theory to the theory of closed geodesics and obtained density theorem of Tchebotarev's type for closed geodesics.

§1. Terminology and notation

Throughout this paper manifolds are always supposed to be smooth, connected Riemannian manifolds without boundary, and geodesics are oriented and parametrized by arc-lengths. By a closed geodesic we mean a smoothly closed

geodesic. Two geodesics are considered to be the same if their images and their orientations coincide. For a set S , $\#S$ denotes the cardinality of S .

Let $\pi: M \rightarrow M_0$ be a covering, where M and M_0 are Riemannian manifolds. The covering $\pi: M \rightarrow M_0$ is said to be *normal* if the group of covering transformations acts transitively on each fibre $\pi^{-1}(x_0)$ over $x_0 \in M_0$. We note that this definition is equivalent to the ordinary definition, cf. [13] Corollary 1.8.17. If the covering map π is a local isometry, then it is called a *Riemannian covering*. A Riemannian manifold M' is called an *intermediate manifold* of the Riemannian covering $\pi: M \rightarrow M_0$, if there are Riemannian coverings $\pi'': M \rightarrow M'$ and $\pi': M' \rightarrow M_0$ such that $\pi = \pi' \circ \pi''$. We also consider M and M_0 as intermediate manifolds of π . The covering $\pi': M' \rightarrow M_0$ is called an *intermediate covering*.

Again let $\pi: M \rightarrow M_0$ be a covering. For curves γ_0 and γ in M_0 and M respectively, we say that γ *lies above* γ_0 or that γ is a *lift* of γ_0 if $\pi(\gamma)$ coincides with γ_0 geometrically and π restricted to γ preserves orientation. If a lift γ of a closed curve γ_0 in M_0 is closed, then π restricted to γ is an f -fold covering for some positive integer f . We call this integer f the *degree* of γ and denote it by $\deg_\pi \gamma$. It is easy to see that: If $\pi: M \rightarrow M_0$ is an n -fold covering, then there are at most n closed curves in M lying above a closed curve γ_0 in M_0 , and

$$n = \sum_{\gamma: \text{a lift of } \gamma_0} \deg_\pi \gamma.$$

If the covering π is normal, then the group of covering transformations is transitive on the set of all the lifts of a curve in M_0 . Hence all the lifts of a curve in M_0 are closed or all the lifts of a curve in M_0 are non-closed. Furthermore, if the covering is an n -fold normal covering, then all the degrees of lifts of a closed curve in M_0 are the same. Therefore

$$n = (\text{the number of all the lifts of } \gamma_0) \times (\deg_\pi \gamma),$$

where γ_0 is a closed curve in M_0 and γ is a lift of γ_0 . Thus for a closed curve in M_0 and a lift γ of γ_0 , $\deg_\pi \gamma$ and the number of all the lifts of γ_0 are divisors of n .

Let M be a Riemannian manifold. For an isometry g of M , if g restricted to a geodesic is a simple translation along the geodesic, then we say that g *preserves the geodesic*. We have the following proposition.

PROPOSITION 1.1. *Let M be a Riemannian manifold, g an isometry of M without fixed points and γ a geodesic of M . Then $g(\gamma) = \gamma$ (precisely, for any $t \in \mathbf{R}$ there is some $t' \in \mathbf{R}$ such that $g(\gamma)(t) = \gamma(t')$), if and only if g preserves γ .*

PROOF. Suppose $g(\gamma) = \gamma$. Since g has no fixed point, $g(\gamma)(t) = \gamma(t+a)$ for some constant a , by virtue of the following lemma: If γ_1 and γ_2 are geodesics such that $\gamma_1(\mathbf{R}) = \gamma_2(\mathbf{R})$ and $\|\dot{\gamma}_1(t)\| = \|\dot{\gamma}_2(t)\| = 1$, then either $\gamma_1(t) = \gamma_2(t+a)$ or $\gamma_1(t) = \gamma_2(-t+a)$ for some constant a .

The converse is easy.

Q. E. D.

PROPOSITION 1.2. *Let G be a group of isometries of M which acts on M freely and properly discontinuously. Let γ be a geodesic of M . Then the subgroup K of G consisting of all the elements which preserve γ is a cyclic group.*

PROOF. For an isometry g in K , the restriction of g to γ is a translation. The group of all translations of γ is isomorphic to the group S^1 or the group $\mathbf{R}$ according to whether γ is compact or not. Thus we have a representation φ from K to S^1 or $\mathbf{R}$. Since K acts on M freely and properly discontinuously, φ is injective and $\varphi(K)$ is a discrete subgroup. Therefore $\varphi(K)$ is cyclic and so K is cyclic.

Q. E. D.

The authors are indebted to Professor Sugawara for the proof of Proposition 1.2 which simplified the authors's original one.

§2. Known results

We shall summarize known results about the growth of the number of closed geodesics for later use.

Let M denote a compact n -dimensional Riemannian manifold with strictly negative sectional curvature, let K denote the generic sectional curvature of M and let $v(t)$ denote the number of closed geodesics of M with minimal periods $\leq t$.

THEOREM A (W. Parry and M. Pollicott [10]). *The function $v(t)$ is asymptotic to e^{ht}/ht as t tends to positive infinity, where h is the topological entropy of the geodesic flow of the unit tangent bundle of M .*

THEOREM B (A. Manning [6]). *Let h be the topological entropy for the geodesic flow of the unit tangent bundle of M , and $\tilde{M}$ the universal Riemannian covering manifold of M . Let $V(x, r)$ denote the volume of the ball around x with radius r , where $x \in \tilde{M}$. Then $\lim_{r \rightarrow \infty} (\log V(x, r)/r)$ exists and equal to h , where $x \in \tilde{M}$.*

We call $\lim_{r \rightarrow \infty} (\log V(x, r)/r)$ the exponential growth rate of the volume in $\tilde{M}$ and denote it by λ . Since M has strictly negative sectional curvature, the expo-

nential growth rate λ of the volume in $\tilde{M}$ is positive, cf. Sinai [11]. By Theorem A and Theorem B, we have the following proposition.

PROPOSITION 2.1. *Let M and M_0 be compact Riemannian manifolds with strictly negative sectional curvature. Suppose the both universal Riemannian covering manifolds of M and M_0 are isometric. Then $\lim_{t \rightarrow \infty} \frac{v_0(t)}{v(t)} = 1$, where $v_0(t)$ and $v(t)$ are the cardinalities of closed geodesics of M_0 and M respectively with minimal periods $\leq t$.*

§3. First density theorem

We shall give a density theorem, which would enable us to compute the other density theorem.

From now on M denotes a compact Riemannian manifold with strictly negative sectional curvature, $\pi: M \rightarrow M_0$ denotes a normal Riemannian covering and G denotes the group of covering transformations. For a closed geodesic γ of M , $\tau(\gamma)$ denotes the minimal period of γ . For a positive number t , an integer i and an isometry $g \in G$, let

$$\begin{aligned} A(t) &= \{\gamma: \gamma \text{ is a closed geodesic of } M \text{ with } \tau(\gamma) \leq t\}, \\ A_0(t) &= \{\gamma_0: \gamma_0 \text{ is a closed geodesic of } M_0 \text{ with } \tau(\gamma_0) \leq t\}, \\ A^i(t) &= \{\gamma \in A(t): \#G(\gamma) = i\}, \\ A_0^i(t) &= \{\gamma_0 \in A_0(t): \text{For some lift } \gamma \text{ of } \gamma_0 \text{ to } M, \#G(\gamma) = i\}, \\ A^{(g)}(t) &= \{\gamma \in A(t): g \text{ preserves } \gamma\}, \end{aligned}$$

and

$$A_0^{(g)}(t) = \{\gamma_0 \in A_0(t): \text{For some lift } \gamma \text{ of } \gamma_0, g(\gamma) = \gamma\},$$

where $G(\gamma) = \{\text{closed geodesic } g(\gamma): g \in G\}$. We denote by $v(t)$, $v_0(t)$, $v^i(t)$, $v_0^i(t)$, $v^{(g)}(t)$ and $v_0^{(g)}(t)$ the cardinalities of $A(t)$, $A_0(t)$, $A^i(t)$, $A_0^i(t)$, $A^{(g)}(t)$ and $A_0^{(g)}(t)$ respectively.

Let f be the order of G . Since the covering is normal, if i is not a divisor of f , $A^i(t)$ and $A_0^i(t)$ are empty for all t . If i is a divisor of f , it follows that $\tau(\gamma)/\tau(\pi\gamma) = f/i$ for $\gamma \in A^i(t)$, so that

$$(3.1) \quad v^i(tf/i) = i v_0^i(t).$$

Let $1 = z_1 < z_2 < \dots < z_k = f$ be all the divisors of f . Then

$$v(t) = \sum_{i=1}^k v^{z_i}(t).$$

LEMMA 3.1. Suppose G is a cyclic group with a generator g . Then $\gamma \in A^i(t)$ if and only if $\tau(\gamma) \leq t$, $g^i(\gamma) = \gamma$ and $g^j(\gamma) \neq \gamma$ for $1 \leq j \leq i-1$. Hence $A^1(t) = A^{(g)}(t)$.

PROOF. Let γ be a geodesic of M such that $g^i(\gamma) = \gamma$ and $g^j(\gamma) \neq \gamma$ for $1 \leq j \leq i-1$. Then $\gamma, g(\gamma), \dots, g^{i-2}(\gamma)$ and $g^{i-1}(\gamma)$ are mutually distinct closed geodesics of M . Moreover, for an integer l , there are integers $j \in \{0, 1, \dots, i-1\}$ and m such that $l = mi + j$. Hence $g^l(\gamma) = g^j g^{mi}(\gamma) = g^j(\gamma)$, so that $G(\gamma) = \{\gamma, g(\gamma), \dots, g^{i-1}(\gamma)\}$. Thus $\gamma \in A^i(t)$. Because $A(t) = \bigcup_{i=1}^k A^{z_i}(t)$ is a disjoint union, it follows that if $\gamma \in A^i(t)$, then $g^i(\gamma) = \gamma$ and $g^j(\gamma) \neq \gamma$ for $0 \leq j \leq i-1$.

Recalling Proposition 1.1, the last statement follows immediately. Q. E. D.

THEOREM 1. Let M be a compact Riemannian manifold with strictly negative sectional curvature, $\pi: M \rightarrow M_0$ a finite normal Riemannian covering, G the group of covering transformations and f the order of G . Then

$$\lim_{t \rightarrow \infty} \frac{v^{z_1}(t) + \dots + v^{z_r-1}(t)}{v(t)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{v^f(t)}{v(t)} = 1,$$

where $1 = z_1 < z_2 < \dots < z_r = f$ are the totality of the divisors of f .

PROOF. Let λ be the exponential growth rate of the volume in the universal Riemannian covering of M and M_0 . For any $\varepsilon > 0$ we have

$$v(t) \geq e^{(\lambda - \varepsilon)t} \quad \text{and} \quad v_0(t) \leq e^{(\lambda + \varepsilon)t}$$

for all sufficiently large t by Theorems A and B. By (3.1)

$$v^z(t) \leq z v_0(zt/f) \leq f v_0\left(\frac{t}{2}\right)$$

for any divisor z of f , $z \neq f$. Hence $\lim_{t \rightarrow \infty} v^z(t)/v(t) = 0$ for such a z . Since $\sum_{i=1}^k v^{z_i}(t) = v(t)$, we get $\lim_{t \rightarrow \infty} \frac{v^f(t)}{v(t)} = 1$. Q. E. D.

Immediately we have,

COROLLARY 1. Let M be a compact Riemannian manifold with strictly negative sectional curvature. Let g be an isometry of M . If the group G generated by g acts on M freely, then

$$\lim_{t \rightarrow \infty} \frac{v^{(\theta)}(t)}{v(t)} = 0.$$

COROLLARY 2. *We retain the same notations as in Theorem 1. Then we have*

$$\lim_{t \rightarrow \infty} \frac{v_0^f(t)}{v_0(t)} = \frac{1}{f}.$$

Furthermore, if f is a prime number, then

$$\lim_{t \rightarrow \infty} \frac{v_0^1(t)}{v_0(t)} = 1 - \frac{1}{f}.$$

PROOF. By Theorems A and 1 and the formula (3.1), it follows

$$\lim_{t \rightarrow \infty} \frac{v_0^f(t)}{v_0(t)} = \frac{1}{f} \lim_{t \rightarrow \infty} \frac{v^f(t)}{v(t)} \cdot \frac{v(t)}{v_0(t)} = \frac{1}{f}.$$

The second statement is easy.

Q. E. D.

§4. Proof of Theorem 2

Again, let M be a compact Riemannian manifold of strictly negative curvature, $\pi: M \rightarrow M_0$ a finite normal Riemannian covering, and G the group of covering transformations. We retain the same notation as in §3. We often consider intermediate coverings $\pi': M' \rightarrow M_0$ of $\pi: M \rightarrow M_0$, for which we construct the same objects such as $A_0^i(t)$, $v_0^i(t)$ and so on for π' . Then we specify the covering as follows $A_0^i[\pi'](t)$, $v_0^i[\pi'](t)$ and so on.

First, we consider the case that G is a cyclic group.

Suppose that G is a cyclic group of order f , generated by an element g , and that there is a non-trivial proper subgroup G_1 of G . Let $M_1 = M/G_1$. Then $\pi': M_1 \rightarrow M_0$ is a normal Riemannian covering. Pick a closed geodesic γ_0 of M_0 and a lift γ of γ_0 to M . If G_1 is generated by g^l , then we have the following lemma.

LEMMA 4.1. $\gamma_0 \in A_0^1[\pi'](t)$ if and only if there is a lift γ of γ_0 such that $g(\gamma) = g^{li}(\gamma)$ for some i , $0 \leq i \leq f/l - 1$.

PROOF. Let $\pi'': M \rightarrow M'$ be the Riemannian covering such that $\pi = \pi' \circ \pi''$. Then the group of covering transformations of π'' is G_1 and the one of π' is naturally identified with G/G_1 . Suppose γ_0 is in $A_0^1[\pi'](t)$. Let γ be a lift of γ_0 to M . Then $\pi''(\gamma)$ is a lift of γ_0 to M' . Since γ_0 is in $A_0^1[\pi'](t)$, $(gG_1)(\pi''\gamma) = \pi''\gamma$. Thus

$g(\gamma) \equiv \gamma \pmod{G_1}$, so that $g(\gamma) = g^{li}(\gamma)$ for some i , $1 \leq i \leq l-1$.

The converse is easy.

Q. E. D.

LEMMA 4.2. Let $\pi: M \rightarrow M_0$ be a normal Riemannian covering and G the group of covering transformations. Suppose that G is a cyclic group of order $p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}$, where p_i 's are prime numbers with $p_1 < p_2 < \cdots < p_k$ and q_i 's are positive integers. Let g be a generator of G , G_1 the subgroup of G generated by $g^{p_1 p_2 \cdots p_k}$, $M_1 = M/G_1$ and $\pi': M_1 \rightarrow M_0$ the natural projection. Then

$$\lim_{t \rightarrow \infty} \frac{v_0^1(t)}{v_0(t)} = \lim_{t \rightarrow \infty} \frac{v_0^1[\pi'](t)}{v_0(t)}.$$

PROOF. Let $\gamma_0 \in A_0^1[\pi'](t)$ and let γ be a lift of γ_0 to M . Then by Lemma 4.1, $g(\gamma) = g^{p_1 p_2 \cdots p_k m}(\gamma)$ for some integer m . Since $p_1 p_2 \cdots p_k m - 1$ and $p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}$ are relatively prime, there are integers s and r such that $(p_1 p_2 \cdots p_k m - 1)s + p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k} r = 1$. Hence $\gamma = (g^{p_1 p_2 \cdots p_k m - 1})^s (\gamma) = g^{1 - p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k} r}(\gamma) = g(\gamma)$. Therefore $A_0^1[\pi'](t) \supset A_0^1(t)$. It is trivial that $A_0^1(t) \subset A_0^1[\pi'](t)$, so that $v_0^1[\pi'](t) = v_0^1(t)$.

Q. E. D.

In view of Lemma 4.2, the general cyclic group cases reduce to the case that G is a cyclic group of order $p_1 p_2 \cdots p_k$ where $p_1, p_2, \dots, p_k$ are prime numbers such that $p_1 < p_2 < \cdots < p_k$. Hence we suppose that G is a group of this type. Let g be a generator of G . For a non-empty subset $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ of $\{1, 2, \dots, k\}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_r$, let $G' = G / \langle g^{p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_r}} \rangle$, $M' = M/G'$ and let $\pi_{\alpha_1 \alpha_2 \cdots \alpha_r}: M' \rightarrow M_0$ be the natural projection, where $\langle g^{p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_r}} \rangle$ is the subgroup of G generated by $g^{p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_r}}$. Then G' acts on M' naturally and $\pi_{\alpha_1 \alpha_2 \cdots \alpha_r}$ is a normal Riemannian covering of order $p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_r}$. We set

$$A_0^1[\pi_{\alpha_1 \alpha_2 \cdots \alpha_r}](t) = \{\gamma_0 \in A_0(t) : \text{For some lift } \gamma_1 \text{ of } \gamma_0 \text{ to } M', \#G'(\gamma_1) = 1\}$$

We claim the following.

LEMMA 4.3. $A_0^1[\pi_{\alpha_1 \alpha_2 \cdots \alpha_r}](t) = \bigcup_{i | p_{\beta_1} p_{\beta_2} \cdots p_{\beta_r}} A_0^1(t)$, where $\{\beta_1, \beta_2, \dots, \beta_{k-r}\}$ is the complementary subset of $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ in $\{1, 2, \dots, k\}$ and $i | s$ implies that i is a divisor of s . Hence $v_0^1[\pi_{\alpha_1 \alpha_2 \cdots \alpha_r}](t) = \sum_{i | p_{\beta_1} p_{\beta_2} \cdots p_{\beta_r}} v_0^1(t)$.

PROOF. If $\gamma_0 \in A_0^1[\pi_{\alpha_1 \alpha_2 \cdots \alpha_r}](t)$, and if γ is a lift of γ_0 to M , then by Lemma 4.1 $g(\gamma) = g^{p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_r} m}(\gamma)$ for some integer m . Hence by Lemma 3.1 γ is in $A_0^1(t)$ for some divisor i of $p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_r} m - 1$. Since $A_0^1(t)$ is empty for non-divisor j of $p_1 p_2 \cdots p_k$, the above integer i must be a divisor of $p_1 p_2 \cdots p_k$. If integers h and

$p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_r}$ are not relatively prime, then h can not be a divisor of $p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_r}m-1$. Therefore the above integer i is a divisor of $p_{\beta_1}p_{\beta_2}\cdots p_{\beta_{k-r}}$, where $\{\beta_1, \beta_2, \dots, \beta_{k-r}\}$ is the complementary subset of $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ in $\{1, 2, \dots, k\}$. Thus $A_0^1[\pi_{\alpha_1\alpha_2\cdots\alpha_r}](t) \subset \bigcup_{i|p_{\beta_1}p_{\beta_2}\cdots p_{\beta_{k-r}}} A_0^i(t)$.

Conversely, let $\gamma_0 \in A_0^i(t)$ for some divisor i of $p_{\beta_1}p_{\beta_2}\cdots p_{\beta_{k-r}}$, where $\{\beta_1, \beta_2, \dots, \beta_{k-r}\}$ is the complementary subset of $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ in $\{1, 2, \dots, k\}$, and let γ be a lift of γ_0 to M . Since i and $p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_r}$ are relatively prime, there are integers l and m such that $i \cdot l + p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_r} \cdot m = 1$. It follows, again by Lemma 3.1, that $g^{1-p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_r}m}(\gamma) = \gamma$, and hence $\gamma_0 \in A_0^1[\pi_{\alpha_1\alpha_2\cdots\alpha_r}](t)$. Q. E. D.

Before going ahead we prepare more notion which makes our argument systematical. Let V be the subspace of the polynomial ring in variables $x_1, x_2, \dots, x_k$ over $\mathbf{R}$ such that every term of each element of V is of degree one or zero with respect to x_i for $1 \leq i \leq k$. Let Φ be a linear function on V defined as follows:

$$\Phi(1) = \lim_{t \rightarrow \infty} \frac{v_0^1(t)}{v_0(t)}$$

and

$$\Phi(x_{i_1}, x_{i_2}, \dots, x_{i_r}) = \lim_{t \rightarrow \infty} \frac{v_0^{p_{i_1}p_{i_2}\cdots p_{i_r}}(t)}{v_0(t)}$$

for $1 \leq i_1 < i_2 < \cdots < i_r \leq k$.

By Corollary 2, we have

$$(4.1) \quad \Phi(x_1x_2\cdots x_k) = \frac{1}{p_1p_2\cdots p_k}.$$

LEMMA 4.4. For a proper subset $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ of $\{1, 2, \dots, k\}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ and its complementary subset $\{\beta_1, \beta_2, \dots, \beta_{k-r}\}$, we have

$$\lim_{t \rightarrow \infty} \frac{v_0^1[\pi_{\alpha_1\alpha_2\cdots\alpha_r}](t)}{v_0(t)} = \Phi((1+x_{\beta_1})(1+x_{\beta_2})\cdots(1+x_{\beta_{k-r}})).$$

PROOF. By the definition of Φ , Theorem A and Lemma 4.3,

$$\begin{aligned} & \lim_{t \rightarrow \infty} v_0^1[\pi_{\alpha_1\alpha_2\cdots\alpha_r}](t)/v_0(t) \\ &= \lim_{t \rightarrow \infty} \sum_{i|p_{\beta_1}p_{\beta_2}\cdots p_{\beta_{k-r}}} v_0^i(t)/v_0(t) \\ &= \Phi((1+x_{\beta_1})(1+x_{\beta_2})\cdots(1+x_{\beta_{k-r}})). \end{aligned}$$

Q. E. D.

LEMMA 4.5. *Let $\pi: M \rightarrow M_0$ be a finite normal Riemannian covering where M and M_0 are compact Riemannian manifolds with strictly negative sectional curvature. Let G be the group of covering transformations. Suppose G is a cyclic group of order $p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}$, where p_i 's are prime numbers such that $p_1 < p_2 < \cdots < p_k$ and q_i 's are positive integers. Then*

$$\lim_{t \rightarrow \infty} \frac{v_0^{(g)}(t)}{v_0(t)} = \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

PROOF. By Lemma 3.1 and Lemma 4.2, it is sufficient to compute $\lim_{t \rightarrow \infty} \frac{v_0^1(t)}{v_0(t)}$ for $q_1 = q_2 = \cdots = q_k = 1$. So we assume that $q_1 = q_2 = \cdots = q_k = 1$. We prove the lemma by induction on k . If $k=1$, then the statement holds as in Corollary 2. Suppose it is true for $r < k$. Then, for a non-empty proper subset $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ of $\{1, 2, \dots, k\}$, we have

$$\lim_{t \rightarrow \infty} \frac{v_0^1[\pi_{\alpha_1 \alpha_2 \dots \alpha_r}](t)}{v_0(t)} = \left(1 - \frac{1}{p_{\alpha_1}}\right) \left(1 - \frac{1}{p_{\alpha_2}}\right) \cdots \left(1 - \frac{1}{p_{\alpha_r}}\right).$$

Hence, by Lemma 4.4, we have

$$(4.2) \quad \Phi((1+x_{p_{\beta_1}})(1+x_{p_{\beta_2}}) \cdots (1+x_{p_{\beta_{k-r}}})) = \left(1 - \frac{1}{p_{\alpha_1}}\right) \left(1 - \frac{1}{p_{\alpha_2}}\right) \cdots \left(1 - \frac{1}{p_{\alpha_r}}\right),$$

where $\{\beta_1, \dots, \beta_{k-r}\}$ is the complementary subset of $\{\alpha_1, \dots, \alpha_r\}$.

The following identity can easily be proved.

$$(4.3) \quad \begin{aligned} & (1+x_1)(1+x_2) \cdots (1+x_k) \\ &= x_1 x_2 \cdots x_k + \sum_{1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq k} (1+x_{i_1})(1+x_{i_2}) \cdots (1+x_{i_{k-1}}) \\ & \quad - \sum_{1 \leq i_1 < i_2 < \cdots < i_{k-2} \leq k} (1+x_{i_1})(1+x_{i_2}) \cdots (1+x_{i_{k-2}}) \\ & \quad + \cdots + (-1)^{k-2} \sum_{i=1}^k (1+x_i) + (-1)^{k-1}. \end{aligned}$$

By the linearity of Φ and the formulas (4.1), (4.2), (4.3), we have

$$\begin{aligned} 1 &= \frac{p_1 p_2 \cdots p_k}{1} + \sum_{i=1}^k \left(1 - \frac{1}{p_i}\right) - \sum_{1 \leq i_1 < i_2 \leq k} \left(1 - \frac{1}{p_{i_1}}\right) \left(1 - \frac{1}{p_{i_2}}\right) \\ & \quad + \cdots + (-1)^{k-2} \sum_{1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq k} \left(1 - \frac{1}{p_{i_1}}\right) \left(1 - \frac{1}{p_{i_2}}\right) \cdots \left(1 - \frac{1}{p_{i_{k-1}}}\right) \\ & \quad + (-1)^{k-1} \Phi(1). \end{aligned}$$

Hence, again using (4.3), we have

$$\Phi(1) = \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

Q. E. D.

We are now in a position to give the proof of Theorem 2. We shall designate a few more notations. Let M be a compact Riemannian manifold with strictly negative sectional curvature, $\pi: M \rightarrow M_0$ a finite normal Riemannian covering and G the group of covering transformations. Let g be an element of G which generates a maximal cyclic subgroup, denoted by G_1 , of G . Let $M_1 = M/G_1$ and $\pi_1: M \rightarrow M_1$ the natural projection. For a positive number t , let

$$A_1^{(g)}(t) = \{\gamma_1 \in A_1(t) : \text{For some lift } \gamma \text{ of } \gamma_1 \text{ to } M, g(\gamma) = \gamma.\},$$

where $A_1(t)$ is the set consisting of all the closed geodesics of M_1 whose minimal periods $\leq t$. Let $v_1(t)$ and $v_1^{(g)}(t)$ denote the cardinalities of $A_1(t)$ and $A_1^{(g)}(t)$ respectively. Since G_1 is a maximal cyclic subgroup, recalling Propositions 1.2, for γ in $A^{(g)}(t)$ there are exactly $(\#G_1)$ -points which are mutually identified under the action of G . Thus the projections of the coverings give rise to the following surjective commutative diagram:

$$\begin{array}{ccc} A^{(g)}((\#G_1)t) & \xrightarrow{\pi_1} & A_1^{(g)}(t) \\ \pi \searrow & & \swarrow \pi' \\ & A_0^{(g)}(t) & \end{array}$$

where π' is the natural projection from M_1 to M_0 . Let γ be a closed geodesic in $A^{(g)}(t)$. By the maximality of the cyclic group G_1 and Proposition 1.2, $gh(\gamma) = h(\gamma)$ if and only if $h^{-1}gh \in G_1$, where $h \in G$. Therefore the map π in the above diagram is $\#N(G_1)/\#G_1$ to 1. On the other hand, π_1 is one to one. Thus π' is $\#N(G_1)/\#G_1$ to 1. It follows that

$$\lim_{t \rightarrow \infty} \frac{v_0^{(g)}(t)}{v_0(t)} = \lim_{t \rightarrow \infty} \frac{v_1^{(g)}(t)}{v_0(t)} \cdot \frac{\#G_1}{\#N(G_1)} = \lim_{t \rightarrow \infty} \frac{v_1(t)}{v_0(t)} \cdot \frac{v_1^{(g)}(t)}{v_1(t)} \cdot \frac{\#G_1}{\#N(G_1)}.$$

Since the covering $\pi_1: M \rightarrow M_1$ is cyclic, we have already computed $\lim_{t \rightarrow \infty} v_1^{(g)}(t)/v_1(t)$ in Lemma 4.4. This completes the proof.

References

- [1] R. BOWEN, *Periodic orbits for hyperbolic flows*, Amer. J. Math., **93** (1971), 1–30.
- [2] R. BOWEN, *The equidistribution of closed geodesics*, Amer. J. Math., **94** (1972), 413–423.
- [3] D. L. DEGEORGE, *Length spectrum for compact locally symmetric spaces of strictly negative curvature*, Ann. Scient. Éc. Norm. Sup., **10** (1977), 133–152.
- [4] K. GROVE and M. TANAKA, *On the number of invariant closed geodesics*, Acta Math. **140** (1978), 33–48.
- [5] W. KLINGENBERG, *Riemannian manifolds with geodesic flow of Anosov type*, Ann. Math. **99** (1974), 1–14.
- [6] A. MANNING, *Topological entropy for geodesic flows*, Ann. of Math., **110** (1979), 567–573.
- [7] G. A. MARGULIS, *Applications of ergodic theory to the investigation of manifolds of negative curvature*, Func. Anal. and its Appl., **3** (1969), 335–336.
- [8] J. MILNOR, *A note on curvature and fundamental group*, J. Diff. Geometry, **2** (1968), 1–7.
- [9] J. PLANTE and W. THURSTON, *Anosov flows and the fundamental group*, Topology, **11** (1972), 147–150.
- [10] W. PARRY and M. POLLICOTT, *An analogue of the prime number theorem for closed orbits on Axiom A flows*, Preprint.
- [11] YA. G. SINAI, *The asymptotic behavior of the number of closed geodesics on a compact manifold of negative curvature*, A.M.S., Transl. (2), **73** (1968), 227–250.
- [12] T. SUNADA, *Tchebotarev's type density theorem for closed geodesics in a compact locally symmetric space of negative curvature*, Preprint.
- [13] J. A. WOLF, *Spaces of constant curvature*, Publish or Perish Inc., 1974.

Midori S. GOTO

Department of Mathematics
Fukuoka Technology University
Shimowajiro, Fukuoka
Japan

Kagumi UESU

Department of Mathematics
Kyushu University 33
Hakozaki, Fukuoka
Japan