A METHOD FOR COMPUTING SOME BIFUR- CATIONS OF PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS INVOLVING PARAMETERS

By

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§1. Introduction

The present paper is concerned with some bifurcations of periodic solutions of ordinary differential equations involving parameters. Let

$$(1.1) \quad \frac{dx}{dt} = X(x, B, t)$$

be a given real nonlinear periodic system, where x and $X(x, B, t)$ are real n -dimensional vectors, and B is a parameter. Assume that $X(x, B, t)$ is periodic in t of period 2π and is twice continuously differentiable with respect to (x, B) in the region $\Delta \times R$, and that $X(x, B, t)$ and its first and second partial derivatives with respect to (x, B) are all continuous on $\Delta \times R$. Here Δ is a given region of the (x, B) -space and R is the real line.

Let $\hat{x}(t)$ be a 2π -periodic solution of (1.1) at $B = \hat{B}$, and $\hat{\Phi}(t)$ be the fundamental matrix of the homogeneous system

$$(1.2) \quad \frac{dh}{dt} = X_x(\hat{x}(t), \hat{B}, t)h$$

such that $\hat{\Phi}(0) = E_n$ ($n \times n$ unit matrix), where $X_x(x, B, t)$ is the Jacobian matrix of $X(x, B, t)$ with respect to x . We call $\hat{B}$ a "bifurcation point" if the matrix $\hat{\Phi}(2\pi)$ has an eigenvalue $e^{i\hat{\theta}} = \cos \hat{\theta} + i \sin \hat{\theta}$ and the condition $\text{rank}[E_n - \hat{\Phi}(2\pi)] = n$ is satisfied and for an arbitrary B ($0 < |B - \hat{B}| < \delta$) $\hat{\Phi}(2\pi)$ has no eigenvalue whose absolute value is equal to one, where i denotes the imaginary unit and $\hat{\theta}$ is a real number not equal to 0 and π , and δ is a small positive number, and $\Phi(t)$ is the

fundamental matrix of the first variation equation of (1.1) (at a given B ($0 < |B - \hat{B}| < \delta$)) with respect to $x = x(t)$ such that $\Phi(0) = E_n$. Here $x(t)$ is a 2π -periodic solution of (1.1) at a given B ($0 < |B - \hat{B}| < \delta$). We study the bifurcation point $\hat{B}$ and the 2π -periodic solution $\hat{x}(t)$ of (1.1) right at $B = \hat{B}$, and we propose a method for computing $(\hat{x}(t), \hat{B})$ with high accuracy. We also consider some bifurcations of periodic solutions of a real n -dimensional autonomous system

$$(1.3) \quad \frac{dx}{d\tau} = X(x, B).$$

Our method mentioned above can be also applied to such bifurcations of periodic solutions of (1.3).

H. Kawakami [11] proposed a method different from ours. His method is based on the use of the characteristic equation of the matrix $\hat{\Phi}(2\pi)$. For details of the method, see Remark in Section 2. However, it seems that our method is simpler and easier to handle than his method from a theoretical viewpoint.

In Section 2, we discuss bifurcations of 2π -periodic solutions of (1.1). Section 3 is devoted to the study of bifurcations of periodic solutions of (1.3). In Section 4, we present some examples to illustrate our theory and method.

§ 2. The Case of Periodic Differential Systems

In this section, we consider the above-mentioned bifurcation point $\hat{B}$ and the 2π -periodic solution $\hat{x}(t)$ of (1.1) right at $B = \hat{B}$. As is seen from Section 1, there exists a complex n -dimensional vector $\hat{H}$ ($\neq 0$) such that

$$(2.1) \quad \hat{\Phi}(2\pi)\hat{H} = e^{i\theta}\hat{H} \quad (\text{or } [e^{i\theta}E_n - \hat{\Phi}(2\pi)]\hat{H} = 0).$$

When we put $\hat{H}(t) = \hat{\Phi}(t)\hat{H}$ for this vector $\hat{H}$, $\hat{H}(t)$ is a solution of (1.2) and satisfies

$$(2.2) \quad e^{i\theta}\hat{H}(0) - \hat{H}(2\pi) = 0.$$

However, no solution of (1.2) satisfying the condition (2.2) is uniquely determined unless an additional condition is added because, when $\hat{H}(t)$ is a solution, $\alpha\hat{H}(t)$ is also a solution for an arbitrary constant α . Therefore we need to give some additional conditions. In order to simplify the following argument, we assume that

$$(2.3) \quad \text{rank } [e^{i\theta}E_n - \hat{\Phi}(2\pi)] = \text{rank } \hat{\Sigma}_0 = n - 1,$$

where $\hat{\Xi}_0$ is an $n \times (n-1)$ matrix obtained from $e^{i\theta}E_n - \hat{\Phi}(2\pi)$ by deleting the first column vector and the ranks of the complex matrices $e^{i\theta}E_n - \hat{\Phi}(2\pi)$ and $\hat{\Xi}_0$ are considered on the complex number field. Then the above-mentioned $\hat{H}(t)$ is uniquely determined if we add the following condition

$$(2.4) \quad H_1(0) - 1 = 0,$$

where $H(0) = (H_1(0), H_2(0), \dots, H_n(0))^T$. Here $(\dots)^T$ denotes the transposed vector of a vector $(\dots)$. For this uniquely determined $\hat{H}(t)$, $\hat{H} = \hat{H}(0)$ can be written in the form

$$(2.5) \quad \hat{H} = \hat{h} + i\hat{k},$$

where $\hat{h}$ and $\hat{k}$ are real vectors. When we put

$$(2.6) \quad \hat{h}(t) = \hat{\Phi}(t)\hat{h} \quad \text{and} \quad \hat{k}(t) = \hat{\Phi}(t)\hat{k},$$

$\hat{h}(t)$ and $\hat{k}(t)$ are solutions of (1.2) and satisfy the conditions

$$(2.7) \quad \begin{cases} \hat{h}(0) \cos \hat{\theta} - \hat{k}(0) \sin \hat{\theta} - \hat{h}(2\pi) = 0, \\ \hat{h}(0) \sin \hat{\theta} + \hat{k}(0) \cos \hat{\theta} - \hat{k}(2\pi) = 0 \end{cases}$$

and

$$(2.8) \quad \begin{cases} \hat{h}_1(0) - 1 = 0, \\ \hat{k}_1(0) = 0, \end{cases}$$

where $\hat{h}(0) = (\hat{h}_1(0), \dots, \hat{h}_n(0))^T$ and $\hat{k}(0) = (\hat{k}_1(0), \dots, \hat{k}_n(0))^T$. Therefore, in order to obtain the bifurcation point $\hat{B}$ and the 2π -periodic solution $\hat{x}(t)$ of (1.1) right at $B = \hat{B}$, we consider the following system

$$(2.9) \quad F(\mathbf{x}) = \begin{pmatrix} \varphi(0, \mathbf{x}) - \varphi(2\pi, \mathbf{x}) \\ \varphi_1(0, \mathbf{x}) \cos \theta - \varphi_2(0, \mathbf{x}) \sin \theta - \varphi_1(2\pi, \mathbf{x}) \\ \varphi_1(0, \mathbf{x}) \sin \theta + \varphi_2(0, \mathbf{x}) \cos \theta - \varphi_2(2\pi, \mathbf{x}) \\ h_1 - 1 \\ k_1 \end{pmatrix}$$

$$= \begin{pmatrix} x(0) - \varphi(2\pi, \mathbf{x}) \\ h(0) \cos \theta - k(0) \sin \theta - \varphi_1(2\pi, \mathbf{x}) \\ h(0) \sin \theta + k(0) \cos \theta - \varphi_2(2\pi, \mathbf{x}) \\ h_1 - 1 \\ k_1 \end{pmatrix} = 0,$$

where $\mathbf{x} = (x(0), h(0), k(0), B, \theta)^T$, $x(0) = (x_1, \dots, x_n)^T$, $h(0) = (h_1, \dots, h_n)^T$, $k(0) = (k_1, \dots, k_n)^T$, and $(\varphi(t, \mathbf{x}), \varphi_1(t, \mathbf{x}), \varphi_2(t, \mathbf{x}))^T$ is a solution of

$$(2.10) \quad \begin{cases} \frac{dx}{dt} = X(x, B, t), \\ \frac{dh}{dt} = X_x(x, B, t)h, \\ \frac{dk}{dt} = X_x(x, B, t)k \end{cases}$$

satisfying the initial condition $(\varphi(0, \mathbf{x}), \varphi_1(0, \mathbf{x}), \varphi_2(0, \mathbf{x}))^T = (x(0), h(0), k(0))^T$. As is seen from the above argument, the system (2.9) has a solution $\hat{\mathbf{x}} = (\hat{x}(0), \hat{h}(0), \hat{k}(0), \hat{B}, \hat{\theta})^T$, where $\hat{h}(0) = \hat{h}$ and $\hat{k}(0) = \hat{k}$. From the assumption, the function $F(\mathbf{x})$ defined by the equality (2.9) is continuously differentiable with respect to $\mathbf{x}$ and we denote by $F'(\mathbf{x})$ the Jacobian matrix of $F(\mathbf{x})$ with respect to $\mathbf{x}$. Then, for the solution $\hat{\mathbf{x}}$, we have

$$(2.11) \quad \det F'(\hat{\mathbf{x}}) = \begin{vmatrix} E_n - \hat{\Phi}(2\pi) & 0 & 0 & -\hat{\xi}_1(2\pi) & 0 \\ -\hat{\Phi}_2(2\pi) & \cos \hat{\theta} E_n - \hat{\Phi}(2\pi) & -\sin \hat{\theta} E_n & -\hat{\xi}_2(2\pi) & \hat{\eta}_2 \\ -\hat{\Phi}_3(2\pi) & \sin \hat{\theta} E_n & \cos \hat{\theta} E_n - \hat{\Phi}(2\pi) & -\hat{\xi}_3(2\pi) & \hat{\eta}_3 \\ 0 \dots 0 & 1 \dots 0 & 0 \dots 0 & 0 & 0 \\ 0 \dots 0 & 0 \dots 0 & 1 \dots 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} E_n - \hat{\Phi}(2\pi) & 0 & 0 & 0 & 0 & 0 & 0 \\ -\hat{\Phi}_2(2\pi) & 0 & \hat{A}_0 & 0 & -\hat{C}_0 & \hat{\eta}_2 & \hat{\eta}_2 \\ -\hat{\Phi}_3(2\pi) & 0 & \hat{C}_0 & 0 & \hat{A}_0 & \hat{\eta}_3 & \hat{\eta}_3 \\ 0 \dots 0 & 1 & 0 \dots 0 & 0 & 0 \dots 0 & 0 & 0 \\ 0 \dots 0 & 0 & 0 \dots 0 & 1 & 0 \dots 0 & 0 & 0 \end{vmatrix},$$

where

$\hat{\Phi}_2(t)$ ($n \times n$ matrix) is a solution of

$$(2.12) \quad \frac{dz}{dt} = X_x(\hat{x}(t), \hat{B}, t)z + \{X_{xx}(\hat{x}(t), \hat{B}, t)\hat{h}(t)\}\hat{\Phi}(t)$$

such that $\hat{\Phi}_2(0)=0$ ($n \times n$ zero matrix), where $X_{xx}(x, B, t)$ denotes the second derivative of $X(x, B, t)$ with respect to x ;

$\hat{\Phi}_3(t)$ ($n \times n$ matrix) is a solution of

$$(2.13) \quad \frac{dw}{dt} = X_x(\hat{x}(t), \hat{B}, t)w + \{X_{xx}(\hat{x}(t), \hat{B}, t)\hat{k}(t)\}\hat{\Phi}(t)$$

such that $\hat{\Phi}_3(0)=0$ ($n \times n$ zero matrix);

$(\hat{\xi}_1(t), \hat{\xi}_2(t), \hat{\xi}_3(t))^T$ (3n-dimensional vector) is a solution of

$$(2.14) \quad \begin{cases} \frac{d\hat{\xi}_1}{dt} = X_x(\hat{x}(t), \hat{B}, t)\hat{\xi}_1 + X_B(\hat{x}(t), \hat{B}, t), \\ \frac{d\hat{\xi}_2}{dt} = X_x(\hat{x}(t), \hat{B}, t)\hat{\xi}_2 + \{X_{xx}(\hat{x}(t), \hat{B}, t)\hat{\xi}_1 + X_{xB}(\hat{x}(t), \hat{B}, t)\hat{h}(t), \\ \frac{d\hat{\xi}_3}{dt} = X_x(\hat{x}(t), \hat{B}, t)\hat{\xi}_3 + \{X_{xx}(\hat{x}(t), \hat{B}, t)\hat{\xi}_1 + X_{xB}(\hat{x}(t), \hat{B}, t)\hat{k}(t) \end{cases}$$

such that $(\hat{\xi}_1(0), \hat{\xi}_2(0), \hat{\xi}_3(0))^T = (0, 0, 0)^T$, where $X_B(x, B, t)$ and $X_{xB}(x, B, t)$ are the partial derivatives of $X(x, B, t)$ and $X_x(x, B, t)$ with respect to B , respectively;

$$(2.15) \quad \begin{cases} \hat{\eta}_2 = -\hat{h}(0) \sin \hat{\theta} - \hat{k}(0) \cos \hat{\theta}, \\ \hat{\eta}_3 = \hat{h}(0) \cos \hat{\theta} - \hat{k}(0) \sin \hat{\theta}; \end{cases}$$

$\hat{A}_0$ and $\hat{C}_0$ are the $n \times (n-1)$ matrices obtained from $\cos \hat{\theta} E_n - \hat{\Phi}(2\pi)$ and $\sin \hat{\theta} E_n$ by deleting the first column vectors, respectively;

$$(2.16) \quad \begin{cases} \hat{\gamma}_2 = -\hat{\xi}_2(2\pi) - \hat{\Phi}_2(2\pi)\hat{\mu}, \\ \hat{\gamma}_3 = -\hat{\xi}_3(2\pi) - \hat{\Phi}_3(2\pi)\hat{\mu}, \end{cases}$$

where $\hat{\mu}$ is a solution of the equation

$$(2.17) \quad [E_n - \hat{\Phi}(2\pi)]\hat{\mu} = \hat{\xi}_1(2\pi) \quad (\text{where } \mu = (\mu_1, \dots, \mu_n)^T).$$

From (2.11), we readily get the following theorem.

THEOREM 1.

The matrix $F'(\hat{x})$ is non-singular if and only if

$$(2.18) \quad \text{rank} \begin{pmatrix} \hat{A}_0 & -\hat{C}_0 & \hat{\gamma}_2 & \hat{\eta}_2 \\ \hat{C}_0 & \hat{A}_0 & \hat{\gamma}_3 & \hat{\eta}_3 \end{pmatrix} = 2n.$$

Thus, if the condition (2.18) is satisfied, then, by applying the Newton method to the system (2.9), we can compute the solution $\hat{x} = (\hat{x}(0), \hat{h}(0), \hat{k}(0), \hat{B}, \hat{\theta})^T$ with high accuracy. Consequently we can also obtain a highly accurate approximation to the bifurcation point $\hat{B}$.

Next, we consider the case where a bifurcation point degenerates to a singular point with a higher singularity, that is, $\hat{\Phi}(2\pi)$ has a double eigenvalue one and the condition

$$(2.19) \quad \text{rank} [E_n - \hat{\Phi}(2\pi)] = n - 1$$

is satisfied. In this case, we suppose that the dimension of the parameter B is two, that is, $B = (B_1, B_2)^T$.

Let $\hat{x}(t)$ be a 2π -periodic solution of (1.1) at $B = \hat{B} = (\hat{B}_1, \hat{B}_2)^T$ and $\hat{\Phi}(t)$ be the fundamental matrix of (1.2) (at $x = \hat{x}(t)$ and $B = \hat{B} = (\hat{B}_1, \hat{B}_2)^T$) such that $\hat{\Phi}(0) = E_n$. As is mentioned in the above, we assume that

$$(2.20) \quad J(1) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad (a \neq 0),$$

where $J(1)$ denotes the Jordan block corresponding to the eigenvalue one of the matrix $\hat{\Phi}(2\pi)$. Then, due to (2.20), there exist real n -dimensional vectors $\hat{p}$ and $\hat{q}$ such that

$$(2.21) \quad \begin{cases} \hat{\Phi}(2\pi)\hat{p} = \hat{p}, \\ \hat{\Phi}(2\pi)\hat{q} = \hat{q} + a\hat{p}. \end{cases}$$

When we put

$$(2.22) \quad \begin{cases} \hat{p}(t) = \hat{\Phi}(t)\hat{p}, \\ \hat{q}(t) = \hat{\Phi}(t)\hat{q} - \frac{a}{2\pi} t \hat{\Phi}(t)\hat{p}, \end{cases}$$

$(\hat{p}(t), \hat{q}(t))^T$ is a 2π -periodic solution of

$$(2.23) \quad \begin{cases} \frac{dp}{dt} = X_x(\hat{x}(t), \hat{B}, t)p, \\ \frac{dq}{dt} = X_x(\hat{x}(t), \hat{B}, t)q - \frac{a}{2\pi}p. \end{cases}$$

However, no 2π -periodic solution of (2.23) is uniquely determined unless appropriate additional conditions are added because, when $(\hat{p}(t), \hat{q}(t))^T$ is a 2π -periodic solution, $(\alpha\hat{p}(t), \alpha\hat{q}(t) + \beta\hat{p}(t))^T$ is also a 2π -periodic solution for arbitrary constants α and β . Therefore we need to give some additional conditions.

In order to simplify the following argument, we assume that

$$(2.24) \quad \text{rank}[E_n - \hat{\Phi}(2\pi)] = \text{rank} \hat{D}_1(2\pi) = n-1,$$

where $\hat{D}_1(t)$ is the $n \times (n-1)$ matrix obtained from $E_n - \hat{\Phi}(t)$ by deleting the first column vector. If the conditions

$$(2.25) \quad \begin{cases} p_1(0) - 1 = 0, \\ q_1(0) = 0 \end{cases}$$

are given, then the 2π -periodic solution $(\hat{p}(t), \hat{q}(t))^T$ is uniquely determined, where $p(0) = (p_1(0), \dots, p_n(0))^T$ and $q(0) = (q_1(0), \dots, q_n(0))^T$. Therefore, in this case, we consider the following system

$$(2.26) \quad G(y) = \begin{pmatrix} \phi(0, y) - \phi(2\pi, y) \\ \phi_1(0, y) - \phi_1(2\pi, y) \\ \phi_2(0, y) - \phi_2(2\pi, y) \\ p_1 - 1 \\ q_1 \end{pmatrix} = \begin{pmatrix} x(0) - \phi(2\pi, y) \\ p(0) - \phi_1(2\pi, y) \\ q(0) - \phi_2(2\pi, y) \\ p_1 - 1 \\ q_1 \end{pmatrix} = 0,$$

where $y = (x(0), p(0), q(0), B)^T$, $x(0) = (x_1, \dots, x_n)^T$, $p(0) = (p_1, \dots, p_n)^T$, $q(0) = (q_1, \dots, q_n)^T$, $B = (B_1, B_2)^T$, and $(\phi(t, y), \phi_1(t, y), \phi_2(t, y))^T$ is a solution of

$$(2.27) \quad \begin{cases} \frac{dx}{dt} = X(x, B, t), \\ \frac{dp}{dt} = X_x(x, B, t)p, \\ \frac{dq}{dt} = X_x(x, B, t)q - \frac{a}{2\pi}p \end{cases}$$

satisfying the initial condition $(\phi(0, y), \phi_1(0, y), \phi_2(0, y))^T = (x(0), p(0), q(0))^T$. As is seen from the above argument, the system (2.26) has a solution $\hat{y} = (\hat{x}(0), \hat{p}(0), \hat{q}(0), \hat{B})^T$. Denoting by $G'(y)$ the Jacobian matrix of $G(y)$ with respect to y , for the solution $\hat{y}$, we have

$$(2.28) \quad \det G'(\hat{y})$$

$$= \begin{vmatrix} E_n - \hat{\Phi}(2\pi) & 0 & 0 & -\hat{\xi}_{11}(2\pi) & -\hat{\xi}_{12}(2\pi) \\ -\hat{\Psi}_2(2\pi) & E_n - \hat{\Phi}(2\pi) & 0 & -\hat{\xi}_{21}(2\pi) & -\hat{\xi}_{22}(2\pi) \\ -\hat{\Psi}_3(2\pi) & a\hat{\Phi}(2\pi) & E_n - \hat{\Phi}(2\pi) & -\hat{\xi}_{31}(2\pi) & -\hat{\xi}_{32}(2\pi) \\ 0 \cdots 0 & 10 \cdots 0 & 00 \cdots 0 & 0 & 0 \\ 00 \cdots 0 & 00 \cdots 0 & 10 \cdots 0 & 0 & 0 \end{vmatrix},$$

where

$(\hat{\Psi}_2(t), \hat{\Psi}_3(t))^T$ ($2n \times n$ matrix) is a solution of

$$(2.29) \quad \begin{cases} \frac{dz_1}{dt} = X_x(\hat{x}(t), \hat{B}, t)z_1 + \{X_{xx}(\hat{x}(t), \hat{B}, t)\hat{p}(t)\}\hat{\Phi}(t), \\ \frac{dz_2}{dt} = X_x(\hat{x}(t), \hat{B}, t)z_2 + \{X_{xx}(\hat{x}(t), \hat{B}, t)\hat{q}(t)\}\hat{\Phi}(t) - \frac{a}{2\pi}z_1 \end{cases}$$

such that $(\hat{\Psi}_2(0), \hat{\Psi}_3(0))^T = (0, 0)^T$ (0 is the $n \times n$ zero matrix);

each $(\hat{\xi}_{1j}(t), \hat{\xi}_{2j}(t), \hat{\xi}_{3j}(t))^T$ ($3n$ -dimensional vector) is a solution of

$$(2.30) \quad \begin{cases} \frac{dv_1}{dt} = X_x(\hat{x}(t), \hat{B}, t)v_1 + X_{B_j}(\hat{x}(t), \hat{B}, t), \\ \frac{dv_2}{dt} = X_x(\hat{x}(t), \hat{B}, t)v_2 + \{X_{xx}(\hat{x}(t), \hat{B}, t)v_1 + X_{xB_j}(\hat{x}(t), \hat{B}, t)\}\hat{p}(t), \\ \frac{dv_3}{dt} = X_x(\hat{x}(t), \hat{B}, t)v_3 + \{X_{xx}(\hat{x}(t), \hat{B}, t)v_1 + X_{xB_j}(\hat{x}(t), \hat{B}, t)\}\hat{q}(t) - \frac{a}{2\pi}v_2 \end{cases}$$

such that $(\hat{\xi}_{1j}(0), \hat{\xi}_{2j}(0), \hat{\xi}_{3j}(0))^T = (0, 0, 0)^T$.

Due to (2.28), we readily get the following theorem.

THEOREM 2.

If the condition $\text{rank}(\hat{D}_1(2\pi), \hat{\xi}_{11}(2\pi)) = n$ is satisfied, then the matrix $G'(\hat{y})$ is non-singular if and only if

$$(2.31) \quad \text{rank} \begin{pmatrix} \hat{D}_1(2\pi) & 0 & (\hat{\#}) & (\hat{@}) \\ a\hat{D}_0(2\pi) & \hat{D}_1(2\pi) & (\hat{\#\#}) & (\hat{@@}) \end{pmatrix} = 2n,$$

where

$\hat{D}_0(2\pi)$ denotes the $n \times (n-1)$ matrix obtained from the fundamental matrix $\hat{\Phi}(2\pi)$ by deleting the first column vector;

$$(2.32) \quad (\hat{\#}) = \hat{\Psi}_2(2\pi)\hat{p}(0), \quad (\hat{\#\#}) = \hat{\Psi}_3(2\pi)\hat{p}(0);$$

$$(2.33) \quad \begin{cases} (\hat{@}) = -\hat{\xi}_{22}(2\pi) + [\hat{D}_2(2\pi), \hat{\xi}_{21}(2\pi)]\hat{h}, \\ (\hat{@@}) = -\hat{\xi}_{32}(2\pi) + [\hat{D}_3(2\pi), \hat{\xi}_{31}(2\pi)]\hat{h}, \end{cases}$$

where $\hat{D}_2(2\pi)$ and $\hat{D}_3(2\pi)$ are the $n \times (n-1)$ matrices obtained from $-\hat{\Psi}_2(2\pi)$ and $-\hat{\Psi}_3(2\pi)$ by deleting the first column vectors, respectively, and $\hat{h}$ is a solution of the equation

$$(2.34) \quad [\hat{D}_1(2\pi), \hat{\xi}_{11}(2\pi)]\eta = \hat{\xi}_{12}(2\pi) \quad (\eta = (\eta_1, \dots, \eta_n)^T).$$

Here each $[\hat{D}_j(2\pi), \hat{\xi}_{j1}(2\pi)]$ denotes the $n \times n$ matrix consisting of the $n \times (n-1)$ matrix $\hat{D}_j(2\pi)$ and the n -dimensional vector $\hat{\xi}_{j1}(2\pi)$.

REMARK.

H. Kawakami [11] proposed a method for computing bifurcation points. He considered the following system

$$(2.35) \quad W(z) = \begin{pmatrix} \varphi(0, z) - \varphi(2\pi, z) \\ g(z) \end{pmatrix} = 0$$

instead of the system (2.9), where $z = (x(0), B)^T$, $x(0) = (x_1, \dots, x_n)^T$, and $\varphi(t, z)$ is a solution of (1.1) such that $\varphi(0, z) = x(0)$, and the equation $g(z) = 0$ is the one obtained by eliminating θ from a couple of equations

$$(2.36) \quad \begin{cases} g_1(z, \theta) = \text{Re} \{ \det(e^{i\theta} E_n - \Phi(2\pi)) \} = 0, \\ g_2(z, \theta) = \text{Im} \{ \det(e^{i\theta} E_n - \Phi(2\pi)) \} = 0. \end{cases}$$

Here $\text{Re} \{ \cdot \}$ denotes the real part of a complex number $\{ \cdot \}$ and $\text{Im} \{ \cdot \}$ denotes the imaginary part of $\{ \cdot \}$. When we compare the system (2.35) with the system (2.9), our method seems to be simpler than his method.

§3. The Case of Autonomous Differential Systems

In this section, we consider some bifurcations of ω -periodic solutions (ω is unknown) of a real nonlinear autonomous system

$$(3.1) \quad \frac{dx}{d\tau} = X(x, B),$$

where x and $X(x, B)$ are real n -dimensional vectors, and B is a parameter, and $X(x, B)$ is defined in some region Ω of the (x, B) -space and is twice continuously differentiable with respect to (x, B) in Ω . We transform τ to t by $\tau = \frac{\omega}{2\pi}t$, then the system (3.1) becomes

$$(3.2) \quad \frac{dx}{dt} = \frac{\omega}{2\pi} X(x, B).$$

Then the problem of computing some bifurcation points of ω -periodic solutions of (3.1) is reduced to the one of computing some bifurcation points of 2π -periodic solutions of (3.2). However, as is well-known, no 2π -periodic solution of (3.2) is uniquely determined by the periodic boundary condition alone because, when $x(t)$ is a solution of (3.2), $x(t+\alpha)$ is also a solution of (3.2) for an arbitrary constant α . So we need to add one more condition. We consider how to choose an additional condition. Let $\hat{x}(t)$ be a 2π -periodic solution of (3.2) at $\omega = \hat{\omega}$ and $B = \hat{B}$ and $\hat{\Phi}_1(t)$ be the fundamental matrix of the homogeneous system

$$(3.3) \quad \frac{dh'}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})h'$$

satisfying the initial condition $\hat{\Phi}_1(0) = E_n$, where $X_x(x, B)$ denotes the Jacobian matrix of $X(x, B)$ with respect to x . We assume that

$$(3.4) \quad \hat{\Phi}_1(2\pi) \text{ has a simple eigenvalue one.}$$

As is well-known, $X(\hat{x}(t), \hat{B})$ is a 2π -periodic solution of (3.3). Therefore the n -dimensional vector $\hat{c} = X(\hat{x}(0), \hat{B})$ ($\neq 0$) is an eigenvector corresponding to the eigenvalue one, that is,

$$(3.5) \quad \hat{\Phi}_1(2\pi)\hat{c} = \hat{c} \quad (\text{or } [E_n - \hat{\Phi}_1(2\pi)]\hat{c} = 0).$$

Since the eigenvalue one of $\hat{\Phi}_1(2\pi)$ is simple, by (3.5), we have

$$(3.6) \quad \text{rank } [E_n - \hat{\Phi}_1(2\pi)] = n - 1 < n = \text{rank } [E_n - \hat{\Phi}_1(2\pi), \hat{c}].$$

On the other hand, it follows from (3.5) that there exists a positive integer j ($1 \leq j \leq n$) such that

$$(3.7) \quad \hat{c}_j \neq 0$$

and

$$(3.8) \quad \text{rank} [E_n - \hat{\Phi}_1(2\pi)] = \text{rank } \hat{D}_{-j} = n-1,$$

where $\hat{c} = (\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n)^T$ and $\hat{D}_{-j}$ is the $n \times (n-1)$ matrix obtained from $E_n - \hat{\Phi}_1(2\pi)$ by deleting the j -th column vector. Then, by (3.6) and (3.8), we have

$$(3.9) \quad \text{rank} \begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) & \hat{c} \\ 0 \dots 0 \underset{\substack{\uparrow \\ j}}{1} 0 \dots 0 & 0 \end{pmatrix} = n+1,$$

where $\uparrow$ indicates the location of the j -th column vector of the $(n+1) \times (n+1)$ matrix $\begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) & \hat{c} \\ 0 \dots 0 \underset{\substack{\uparrow \\ j}}{1} 0 \dots 0 & 0 \end{pmatrix}$.

The condition (3.9) implies that we may employ the condition

$$(3.10) \quad x_j(0) - d = 0$$

as an additional condition, where $x(0) = (x_1(0), \dots, x_n(0))^T$ and d is a constant number. Then the 2π -periodic solution $\hat{x}(t)$ of (3.2) at $\omega = \hat{\omega}$ and $B = \hat{B}$ satisfying the condition (3.10) is uniquely determined.

Now we assume that

$$(3.11) \quad \hat{\Phi}_1(2\pi) \text{ has a simple eigenvalue } e^{i\hat{\sigma}},$$

where $\hat{\sigma}$ is a real number and is neither 0 nor π . We call $\hat{B}$ a "bifurcation point" of periodic solutions of (3.2) satisfying (3.10) if the conditions (3.4) and (3.11) are satisfied and for an arbitrary B ($0 < |B - \hat{B}| < \delta$) one of the eigenvalues of $\Phi_1(2\pi)$ is one and the absolute values of the remaining $(n-1)$ eigenvalues are all different from one, where δ is a small positive number and $\Phi_1(2\pi)$ is the fundamental matrix of the first variation equation of (3.2) (at a given B ($0 < |B - \hat{B}| < \delta$)) with respect to $x = x(t)$ such that $\Phi_1(0) = E_n$. Here $x(t)$ is a 2π -periodic solution of (3.2) at a given B ($0 < |B - \hat{B}| < \delta$) and also satisfies (3.10). We propose a method for computing the bifurcation point $\hat{B}$ and the 2π -periodic solution $\hat{x}(t)$ (of (3.2) at $\omega = \hat{\omega}$ and $B = \hat{B}$ satisfying (3.10)) with high accuracy.

From (3.11) it follows that there exists a complex n -dimensional vector $\hat{K} (\neq 0)$ such that

$$(3.12) \quad \hat{\Phi}_1(2\pi)\hat{K} = e^{i\hat{\sigma}}\hat{K}.$$

When we put $\hat{K}(t) = \hat{\Phi}_1(t)\hat{K}$, $\hat{K}(t)$ is a solution of (3.3) and satisfies

$$(3.13) \quad e^{i\hat{\sigma}}\hat{K}(0) - \hat{K}(2\pi) = 0.$$

However, no solution of (3.3) satisfying the condition (3.13) is uniquely determined unless an additional condition is added because, when $\hat{K}(t)$ is a solution, $\alpha\hat{K}(t)$ is also a solution for an arbitrary constant α . Therefore we need to give some additional condition.

In order to simplify the following argument, we assume that

$$(3.14) \quad \text{rank} [e^{i\hat{\sigma}}E_n - \hat{\Phi}_1(2\pi)] = \text{rank } \hat{\Lambda}_0 = n-1,$$

where $\hat{\Lambda}_0$ is the $n \times (n-1)$ matrix obtained from $e^{i\hat{\sigma}}E_n - \hat{\Phi}_1(2\pi)$ by deleting the first column vector, and we consider in (3.14) the ranks of the complex matrices $e^{i\hat{\sigma}}E_n - \hat{\Phi}_1(2\pi)$ and $\hat{\Lambda}_0$ on the complex number field. Then we employ the following condition

$$(3.15) \quad K_1(0) - 1 = 0$$

as an additional condition, where $K(0) = (K_1(0), \dots, K_n(0))^T$. Then the solution $\hat{K}(t)$ of (3.3) satisfying (3.13) and (3.15) is uniquely determined. For such a uniquely determined $\hat{K}(t)$, the complex vector $\hat{K} = \hat{K}(0)$ can be written in the form

$$(3.16) \quad \hat{K} = \hat{h}' + i\hat{k}',$$

where $\hat{h}'$ and $\hat{k}'$ are real vectors. When we put

$$(3.17) \quad \hat{h}'(t) = \hat{\Phi}_1(t)\hat{h}' \quad \text{and} \quad \hat{k}'(t) = \hat{\Phi}_1(t)\hat{k}',$$

$\hat{h}'(t)$ and $\hat{k}'(t)$ are solutions of (3.3) and satisfy the conditions

$$(3.18) \quad \begin{cases} \hat{h}'(0) \cos \hat{\sigma} - \hat{k}'(0) \sin \hat{\sigma} - \hat{h}'(2\pi) = 0, \\ \hat{h}'(0) \sin \hat{\sigma} + \hat{k}'(0) \cos \hat{\sigma} - \hat{k}'(2\pi) = 0 \end{cases}$$

and

$$(3.19) \quad \begin{cases} \hat{h}'_1(0) - 1 = 0, \\ \hat{k}'_1(0) = 0, \end{cases}$$

where $\hat{h}'(0) = (\hat{h}'_1(0), \dots, \hat{h}'_n(0))^T$ and $\hat{k}'(0) = (\hat{k}'_1(0), \dots, \hat{k}'_n(0))^T$. Therefore, in order to obtain the bifurcation point $\hat{B}$ and the 2π -periodic solution $\hat{x}(t)$ of (3.2) at $\omega = \hat{\omega}$ and $B = \hat{B}$ satisfying (3.10), we consider the following system

$$(3.20) \quad M(v) = \begin{pmatrix} \psi(0, v) - \psi(2\pi, v) \\ x_j - d \\ \psi_1(0, v) \cos \sigma - \psi_2(0, v) \sin \sigma - \psi_1(2\pi, v) \\ \psi_1(0, v) \sin \sigma + \psi_2(0, v) \cos \sigma - \psi_2(2\pi, v) \\ h'_1 - 1 \\ k'_1 \end{pmatrix} = 0,$$

where $v = (u(0), h'(0), k'(0), B, \sigma)^T$, $u(0) = (x(0), \omega)^T$, $x(0) = (x_1, \dots, x_n)^T$, $h'(0) = (h'_1, \dots, h'_n)^T$, $k'(0) = (k'_1, \dots, k'_n)^T$, and $(\psi(t, v), \psi_1(t, v), \psi_2(t, v))^T$ is a solution of

$$(3.21) \quad \begin{cases} \frac{dx}{dt} = \frac{\omega}{2\pi} X(x, B), \\ \frac{dh'}{dt} = \frac{\omega}{2\pi} X_x(x, B)h', \\ \frac{dk'}{dt} = \frac{\omega}{2\pi} X_x(x, B)k' \end{cases}$$

such that $(\psi(0, v), \psi_1(0, v), \psi_2(0, v))^T = (x(0), h'(0), k'(0))^T$. As is seen from the above argument, the system (3.20) has a solution $\hat{v} = (\hat{u}(0), \hat{h}'(0), \hat{k}'(0), \hat{B}, \hat{\sigma})^T$, where $\hat{h}'(0) = \hat{h}'$ and $\hat{k}'(0) = \hat{k}'$. From the assumption, the function $M(v)$ defined by the equality (3.20) is continuously differentiable with respect to v and we denote by $M'(v)$ the Jacobian matrix of $M(v)$ with respect to v . Then, for the solution $\hat{v}$, we have

$$(3.22) \quad \det M'(\hat{v})$$

$$= \begin{vmatrix} \Gamma & 0 & 0 & -\begin{pmatrix} \hat{\xi}'_1(2\pi) \\ 0 \end{pmatrix} & 0 \\ -\hat{\Phi}'_2(2\pi) & -\hat{v}'_2(2\pi) & \cos \hat{\sigma} E_n - \hat{\Phi}'_1(2\pi) & -\sin \hat{\sigma} E_n & -\hat{\xi}'_2(2\pi) & \hat{\eta}'_2 \\ -\hat{\Phi}'_3(2\pi) & -\hat{v}'_3(2\pi) & \sin \hat{\sigma} E_n & \cos \hat{\sigma} E_n - \hat{\Phi}'_1(2\pi) & -\hat{\xi}'_3(2\pi) & \hat{\eta}'_3 \\ 00 \dots 0 & 0 & 10 \dots 0 & 00 \dots 0 & 0 & 0 \\ 00 \dots 0 & 0 & 00 \dots 0 & 10 \dots 0 & 0 & 0 \end{vmatrix},$$

where

$$(3.23) \quad \Gamma = \begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) & -\hat{c} \\ 0 \cdots 0 & 10 \cdots 0 \\ \uparrow \\ j \end{pmatrix} \quad ((n+1) \times (n+1) \text{ matrix});$$

$\hat{\Phi}'_2(t)$ ($n \times n$ matrix) is a solution of

$$(3.24) \quad \frac{dy}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})y + \left\{ \frac{\hat{\omega}}{2\pi} X_{xx}(\hat{x}(t), \hat{B})\hat{h}'(t) \right\} \hat{\Phi}_1(t)$$

such that $\hat{\Phi}'_2(0) = 0$ ($n \times n$ zero matrix), where $X_{xx}(x, B)$ denotes the second derivative of $X(x, B)$ with respect to x ;

$\hat{\Phi}'_3(t)$ ($n \times n$ matrix) is a solution of

$$(3.25) \quad \frac{dz}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})z + \left\{ \frac{\hat{\omega}}{2\pi} X_{xx}(\hat{x}(t), \hat{B})\hat{k}'(t) \right\} \hat{\Phi}_1(t)$$

such that $\hat{\Phi}'_3(0) = 0$ ($n \times n$ zero matrix);

$(\hat{v}_1(t), \hat{v}_2(t), \hat{v}_3(t))^T$ ($3n$ -dimensional vector) is a solution of

$$(3.26) \quad \begin{cases} \frac{dv_1}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})v_1 + \frac{1}{2\pi} X(\hat{x}(t), \hat{B}), \\ \frac{dv_2}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})v_2 + \left\{ \frac{\hat{\omega}}{2\pi} X_{xx}(\hat{x}(t), \hat{B})v_1 + \frac{1}{2\pi} X_x(\hat{x}(t), \hat{B}) \right\} \hat{h}'(t), \\ \frac{dv_3}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})v_3 + \left\{ \frac{\hat{\omega}}{2\pi} X_{xx}(\hat{x}(t), \hat{B})v_1 + \frac{1}{2\pi} X_x(\hat{x}(t), \hat{B}) \right\} \hat{k}'(t) \end{cases}$$

such that $(\hat{v}_1(0), \hat{v}_2(0), \hat{v}_3(0))^T = (0, 0, 0)^T$. In fact, $\hat{v}_1(t) = \frac{t}{2\pi} \hat{\Phi}_1(t) \hat{c}$ ($\hat{c} = X(\hat{x}(0), \hat{B})$);

$(\hat{\xi}'_1(t), \hat{\xi}'_2(t), \hat{\xi}'_3(t))^T$ ($3n$ -dimensional vector) is a solution of

$$(3.27) \quad \begin{cases} \frac{d\mu_1}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})\mu_1 + \frac{\hat{\omega}}{2\pi} X_B(\hat{x}(t), \hat{B}), \\ \frac{d\mu_2}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})\mu_2 + \frac{\hat{\omega}}{2\pi} \{ X_{xx}(\hat{x}(t), \hat{B})\mu_1 + X_{xB}(\hat{x}(t), \hat{B}) \} \hat{h}'(t), \\ \frac{d\mu_3}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})\mu_3 + \frac{\hat{\omega}}{2\pi} \{ X_{xx}(\hat{x}(t), \hat{B})\mu_1 + X_{xB}(\hat{x}(t), \hat{B}) \} \hat{k}'(t) \end{cases}$$

such that $(\hat{\xi}'_1(0), \hat{\xi}'_2(0), \hat{\xi}'_3(0))^T = (0, 0, 0)^T$, where $X_B(x, B)$ and $X_{xB}(x, B)$ are the

partial derivatives of $X(x, B)$ and $X_x(x, B)$ with respect to B , respectively;

$$(3.28) \quad \begin{cases} \hat{h}_2 = -\hat{h}'(0) \sin \hat{\sigma} - \hat{k}'(0) \cos \hat{\sigma}, \\ \hat{h}_3 = \hat{h}'(0) \cos \hat{\sigma} - \hat{k}'(0) \sin \hat{\sigma}. \end{cases}$$

From (3.22) we readily get the following theorem.

THEOREM 3.

The matrix $M'(\hat{\theta})$ is non-singular if and only if

$$(3.29) \quad \text{rank} \begin{pmatrix} \hat{F}_0 & -\hat{G}_0 & \delta_2 & \hat{h}'_2 \\ \hat{G}_0 & \hat{F}_0 & \delta_3 & \hat{h}'_3 \end{pmatrix} = 2n,$$

where

$\hat{F}_0$ and $\hat{G}_0$ are the $n \times (n-1)$ matrices obtained from $\cos \hat{\sigma} E_n - \hat{\Phi}_1(2\pi)$ and $\sin \hat{\sigma} E_n$ by deleting the first column vectors, respectively;

$$(3.30) \quad \begin{cases} \delta_2 = -\hat{\xi}'_2(2\pi) - [\hat{\Phi}'_2(2\pi), \hat{v}_2(2\pi)]\hat{\lambda}, \\ \delta_3 = -\hat{\xi}'_3(2\pi) - [\hat{\Phi}'_3(2\pi), \hat{v}_3(2\pi)]\hat{\lambda}, \end{cases}$$

where $\hat{\lambda}$ ($(n+1)$ -dimensional vector) is a solution of the equation

$$(3.31) \quad \Gamma \hat{\lambda} = \begin{pmatrix} \hat{\xi}'_1(2\pi) \\ 0 \end{pmatrix}.$$

Here each $[\hat{\Phi}'_j(2\pi), \hat{v}_j(2\pi)]$ denotes the $n \times (n+1)$ matrix consisting of the $n \times n$ matrix $\hat{\Phi}'_j(2\pi)$ and the n -dimensional vector $\hat{v}_j(2\pi)$.

Thus, if the condition (3.29) is satisfied, then we can obtain an approximation to the solution $\hat{\theta}$ of (3.20) as accurately as we desire by applying the Newton method to the system (3.20). Therefore we can also obtain a desired approximation to the bifurcation point $\hat{B}$.

Next, we study the case where a bifurcation point degenerates to a singular point with a higher singularity. In this case, we suppose that the dimension of the parameter B is two, that is, $B = (B_1, B_2)^T$.

Let $\hat{x}(t)$ be a 2π -periodic solution of (3.2) at $\omega = \hat{\omega}$ and $B = \hat{B} = (\hat{B}_1, \hat{B}_2)^T$ and $\hat{\Phi}_1(t)$ be the fundamental matrix of (3.3) such that $\hat{\Phi}_1(0) = E_n$. Then we classify the problems into the following two cases:

$$(3.32) \quad (i) \quad J(1) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \quad (a \neq 0)$$

$$(3.33) \quad (ii) \quad J(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \quad (a \neq 0)$$

where $J(1)$ denotes the Jordan block corresponding to the eigenvalue one of the matrix $\hat{\Phi}_1(2\pi)$.

First, we consider the case (i). Due to (3.32), there exist real n -dimensional vectors $\hat{r}$ and $\hat{s}$ such that

$$(3.34) \quad \begin{cases} \hat{\Phi}_1(2\pi)\hat{c} = \hat{c}, \\ \hat{\Phi}_1(2\pi)\hat{r} = \hat{r} + a\hat{c}, \\ \hat{\Phi}_1(2\pi)\hat{s} = \hat{s} + a\hat{r} \end{cases}$$

because $\hat{c} = X(\hat{x}(0), \hat{B})$ is an eigenvector corresponding to the eigenvalue one of $\hat{\Phi}_1(2\pi)$. From the first of (3.34) it follows that there exists a positive integer j ($1 \leq j \leq n$) such that

$$(3.35) \quad \hat{c}_j \neq 0$$

and

$$(3.36) \quad \text{rank} [E_n - \hat{\Phi}_1(2\pi)] = \text{rank } \hat{D}_{-j} = n - 1,$$

where $\hat{c} = (\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n)^T$ and $\hat{D}_{-j}$ is the $n \times (n-1)$ matrix obtained from $E_n - \hat{\Phi}_1(2\pi)$ by deleting the j -th column vector. Then it follows from (3.34) and (3.36) that

$$(3.37) \quad \text{rank} \begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) & \hat{c} \\ 0 \cdots 0 \underset{\uparrow j}{1} 0 \cdots 0 & 0 \end{pmatrix} = \text{rank} \begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) \\ 0 \cdots 0 \underset{\uparrow j}{1} 0 \cdots 0 \end{pmatrix} = n.$$

This tells us that, in this case, we may employ the condition

$$(3.38) \quad x_j(0) - d_1 = 0$$

as an additional condition, where $x(0)=(x_1(0), \dots, x_n(0))^T$ and d_1 is a constant number. Then, without loss of generality, we can assume that the above-mentioned 2π -periodic solution $\hat{x}(t)$ of (3.2) at $\omega=\hat{\omega}$ and $B=\hat{B}$ satisfies the condition (3.38).

On the other hand, when we put

$$(3.39) \quad \begin{cases} \hat{r}(t) = \hat{\Phi}_1(t)\hat{r} - \frac{a}{2\pi}t\hat{\Phi}_1(t)\hat{c}, \\ \hat{s}(t) = \hat{\Phi}_1(t)\hat{s} - \frac{a}{2\pi}t\hat{\Phi}_1(t)\hat{r} + \left(\frac{a}{2\pi}\right)^2 \frac{t^2}{2}\hat{\Phi}_1(t)\hat{c} + \frac{a^2}{4\pi}t\hat{\Phi}_1(t)\hat{c}, \end{cases}$$

$(\hat{r}(t), \hat{s}(t))^T$ is a 2π -periodic solution of

$$(3.40) \quad \begin{cases} \frac{dr}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})r - \frac{a}{2\pi} X(\hat{x}(t), \hat{B}), \\ \frac{ds}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})s - \frac{a}{2\pi}r + \frac{a^2}{4\pi} X(\hat{x}(t), \hat{B}). \end{cases}$$

However, no 2π -periodic solution of (3.40) is uniquely determined unless appropriate additional conditions are added because, when $(\hat{r}(t), \hat{s}(t))^T$ is a 2π -periodic solution, $(\hat{r}(t) + \alpha\hat{\Phi}_1(t)\hat{c}, \hat{s}(t) + \alpha\hat{r}(t) + \beta\hat{\Phi}_1(t)\hat{c})^T$ is also a 2π -periodic solution for arbitrary constants α and β . Therefore, from (3.36) and (3.37), we add the conditions

$$(3.41) \quad \begin{cases} r_j(0) = 0, \\ s_j(0) = 0 \end{cases}$$

as additional conditions, where $r(0)=(r_1(0), \dots, r_n(0))^T$ and $s(0)=(s_1(0), \dots, s_n(0))^T$. Then the 2π -periodic solution $(\hat{r}(t), \hat{s}(t))^T$ of (3.40) satisfying (3.41) is uniquely determined. This tells us that, in order to obtain the singular point $\hat{B}=(\hat{B}_1, \hat{B}_2)^T$, we may consider the following system

$$(3.42) \quad P(w) = \begin{pmatrix} \psi(0, w) - \psi(2\pi, w) \\ x_j - d_1 \\ \zeta_1(0, w) - \zeta_1(2\pi, w) \\ \zeta_2(0, w) - \zeta_2(2\pi, w) \\ r_j \\ s_j \end{pmatrix} = 0,$$

where $\mathbf{w} = (u(0), r(0), s(0), B)^T$, $u(0) = (x(0), \omega)^T$, $x(0) = (x_1, \dots, x_n)^T$, $r(0) = (r_1, \dots, r_n)^T$, $s(0) = (s_1, \dots, s_n)^T$, $B = (B_1, B_2)^T$, and $(\psi(t, \mathbf{w}), \zeta_1(t, \mathbf{w}), \zeta_2(t, \mathbf{w}))^T$ is a solution of

$$(3.43) \quad \begin{cases} \frac{dx}{dt} = \frac{\omega}{2\pi} X(x, B), \\ \frac{dr}{dt} = \frac{\omega}{2\pi} X_x(x, B)r - \frac{a}{2\pi} X(x, B), \\ \frac{ds}{dt} = \frac{\omega}{2\pi} X_x(x, B)s - \frac{a}{2\pi} r + \frac{a^2}{4\pi} X(x, B) \end{cases}$$

satisfying the initial condition $(\psi(0, \mathbf{w}), \zeta_1(0, \mathbf{w}), \zeta_2(0, \mathbf{w}))^T = (x(0), r(0), s(0))^T$. As is seen from the above argument, the system (3.42) has a solution $\hat{\mathbf{w}} = (\hat{u}(0), \hat{r}(0), \hat{s}(0), \hat{B})^T$, where $(\hat{r}(0), \hat{s}(0))^T$ is of course the value of the above-mentioned 2π -periodic solution $(\hat{r}(t), \hat{s}(t))^T$ at $t=0$, which satisfies (3.41).

Secondly, we consider the case (ii). In addition to the condition (3.33), we assume that

$$(3.44) \quad \text{rank} [E_n - \hat{\Phi}_1(2\pi), \hat{c}] = n - 1.$$

Due to (3.33) and (3.44), there exist real n -dimensional vectors $\hat{y}$ and $\hat{z}$ such that $\hat{c}$ and $\hat{y}$ are linearly independent and

$$(3.45) \quad \begin{cases} \hat{\Phi}_1(2\pi)\hat{c} = \hat{c}, \\ \hat{\Phi}_1(2\pi)\hat{y} = \hat{y}, \\ \hat{\Phi}_1(2\pi)\hat{z} = \hat{z} + a\hat{y}, \end{cases}$$

where $\hat{c} = X(\hat{x}(0), \hat{B})$. Since $\hat{c}$ and $\hat{y}$ are eigenvectors corresponding to the eigenvalue one and are linearly independent, there exist positive integers l and m ($1 \leq l, m \leq n$, $l \neq m$) such that

$$(3.46) \quad \text{rank} \begin{pmatrix} \hat{c}_l & \hat{y}_l \\ \hat{c}_m & \hat{y}_m \end{pmatrix} = 2$$

and

$$(3.47) \quad \text{rank} [E_n - \hat{\Phi}_1(2\pi)] = \text{rank } \hat{D}_{-l, -m} = n - 2,$$

where $\hat{c} = (\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n)^T$, $\hat{y} = (\hat{y}_1, \hat{y}_2, \dots, \hat{y}_n)^T$ and $\hat{D}_{-l, -m}$ is the $n \times (n-2)$ matrix

obtained from $E_n - \hat{\Phi}_1(2\pi)$ by deleting the l -th column vector and the m -th column vector. By (3.44) and (3.47), we see that

$$(3.48) \quad \text{rank} \begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) & \hat{c} \\ 0 \dots 0 \underset{\substack{\uparrow \\ l}}{1} 0 \dots 0 & 0 \end{pmatrix} = \text{rank } \hat{V}_{-m} = n$$

and

$$(3.49) \quad \text{rank} \begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) & \hat{c} \\ 0 \dots 0 \underset{\substack{\uparrow \\ m}}{1} 0 \dots 0 & 0 \end{pmatrix} = \text{rank } \hat{W}_{-l} = n,$$

where $\hat{V}_{-m}$ is the $(n+1) \times n$ matrix obtained from $\begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) & \hat{c} \\ 0 \dots 0 \underset{\substack{\uparrow \\ l}}{1} 0 \dots 0 & 0 \end{pmatrix}$ by deleting the m -th column vector and $\hat{W}_{-l}$ is the $(n+1) \times n$ matrix obtained from $\begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) & \hat{c} \\ 0 \dots 0 \underset{\substack{\uparrow \\ m}}{1} 0 \dots 0 & 0 \end{pmatrix}$ by deleting the l -th column vector.

Therefore, in this case, we employ either of the following

$$(3.50) \quad \begin{cases} x_l(0) - d_2 = 0 \end{cases}$$

$$(3.51) \quad \begin{cases} x_m(0) - d_3 = 0 \end{cases}$$

as an additional condition, where $x(0) = (x_1(0), \dots, x_n(0))^T$ and d_2 and d_3 are constant numbers. For the sake of simplicity, we employ the condition (3.50). Then, without loss of generality, we can assume that the above-mentioned 2π -periodic solution $\hat{x}(t)$ of (3.2) at $\omega = \hat{\omega}$ and $B = \hat{B}$ satisfies the condition (3.50). Due to (3.48), the equation

$$(3.52) \quad \begin{cases} \hat{A}\tilde{p} = 0, \\ p_m - 1 = 0 \end{cases} \quad \left(\hat{A} = \begin{pmatrix} E_n - \hat{\Phi}_1(2\pi) & -\hat{c} \\ 0 \dots 0 \underset{\substack{\uparrow \\ l}}{1} 0 \dots 0 & 0 \end{pmatrix} \right)$$

has only one solution $\hat{\tilde{p}} = (\hat{p}, \hat{p}_{n+1})^T$, where $\tilde{p} = (p, p_{n+1})^T$, $p = (p_1, \dots, p_n)^T$. For

the solution $\hat{p}=(\hat{p}, \hat{p}_{n+1})^T$, the assumption (3.44) implies $\hat{p}_{n+1}=0$. Therefore $\hat{p}$ becomes a solution of the equation

$$(3.53) \quad [E_n - \hat{\Phi}_1(2\pi)]q = 0 \quad (\text{where } q = (q_1, \dots, q_n)^T).$$

Due to (3.45), this solution $\hat{p}$ can be written in the form

$$(3.54) \quad \hat{p} = \hat{\mu}\hat{c} + \hat{\gamma}\hat{y}$$

because $\hat{c}$ and $\hat{y}$ are solutions of (3.53) and are linearly independent, where $\hat{\mu}$ and $\hat{\gamma}$ are constant numbers. From (3.45) and (3.54) we see that

$$(3.55) \quad \hat{\Phi}_1(2\pi)(\hat{\gamma}\hat{z}) = \hat{\gamma}\hat{z} + a\hat{\gamma}\hat{y} = \hat{\gamma}\hat{z} + a(\hat{p} - \hat{\mu}\hat{c}).$$

When we put

$$(3.56) \quad \begin{cases} \hat{y}(t) = \hat{\Phi}_1(t)\hat{p}, \\ \hat{z}(t) = \hat{\Phi}_1(t)(\hat{\gamma}\hat{z}) - \frac{a}{2\pi}t\hat{\Phi}_1(t)(\hat{p} - \hat{\mu}\hat{c}), \end{cases}$$

by (3.52) and (3.55), $(\hat{y}(t), \hat{z}(t))^T$ is a 2π -periodic solution of

$$(3.57) \quad \begin{cases} \frac{dy}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})y + \frac{\hat{p}_{n+1}}{2\pi} X(\hat{x}(t), \hat{B}) = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})y, \\ \frac{dz}{dt} = \frac{\hat{\omega}}{2\pi} X_x(\hat{x}(t), \hat{B})z - \frac{a}{2\pi}(y - \hat{\mu}X(\hat{x}(t), \hat{B})). \end{cases}$$

By (3.48) we see that the 2π -periodic solution $(\hat{y}(t), \hat{z}(t))^T$ of (3.57) satisfying the conditions

$$(3.58) \quad \begin{cases} y_l(0) = 0, & y_m(0) - 1 = 0, \\ z_l(0) = 0, & z_m(0) = 0 \end{cases}$$

is uniquely determined, where $y(0) = (y_1(0), \dots, y_n(0))^T$ and $z(0) = (z_1(0), \dots, z_n(0))^T$. Therefore, in order to obtain the singular point $\hat{B} = (\hat{B}_1, \hat{B}_2)^T$, we may consider the following system

$$(3.59) \quad Q(\mathbf{w}') = \begin{pmatrix} \left(\begin{array}{c} \psi(0, \mathbf{w}') - \psi(2\pi, \mathbf{w}') \\ x_l - d_2 \end{array} \right) \\ \left(\begin{array}{c} \lambda_1(0, \mathbf{w}') - \lambda_1(2\pi, \mathbf{w}') \\ y_l \end{array} \right) \\ \left(\begin{array}{c} \lambda_2(0, \mathbf{w}') - \lambda_2(2\pi, \mathbf{w}') \\ z_l \\ y_m - 1 \\ z_m \end{array} \right) \end{pmatrix} = 0,$$

where $\mathbf{w}' = (u(0), y(0), v, z(0), \mu, B)^T$, $u(0) = (x(0), \omega)^T$, $x(0) = (x_1, \dots, x_n)^T$, $y(0) = (y_1, \dots, y_n)^T$, $z(0) = (z_1, \dots, z_n)^T$, $B = (B_1, B_2)^T$, and $(\psi(t, \mathbf{w}'), \lambda_1(t, \mathbf{w}'), \lambda_2(t, \mathbf{w}'))^T$ is a solution of

$$(3.60) \quad \begin{cases} \frac{dx}{dt} = \frac{\omega}{2\pi} X(x, B), \\ \frac{dy}{dt} = \frac{\omega}{2\pi} X_x(x, B)y + \frac{v}{2\pi} X(x, B), \\ \frac{dz}{dt} = \frac{\omega}{2\pi} X_x(x, B)z - \frac{a}{2\pi}(y - \mu X(x, B)) \end{cases}$$

satisfying the initial condition $(\psi(0, \mathbf{w}'), \lambda_1(0, \mathbf{w}'), \lambda_2(0, \mathbf{w}'))^T = (x(0), y(0), z(0))^T$. As is seen from the above argument, the system (3.59) has a solution $\hat{\mathbf{w}}' = (\hat{u}(0), \hat{y}(0), 0, \hat{z}(0), \hat{\mu}, \hat{B})^T$, where $(\hat{y}(0), \hat{z}(0))^T$ is of course the value of the above-mentioned 2π -periodic solution $(\hat{y}(t), \hat{z}(t))^T$ at $t=0$, which satisfies (3.58).

§4. Numerical Examples

In this section, to illustrate our theory and method mentioned in Sections 2 and 3, we give some examples.

First, we compute a bifurcation point of periodic solutions of nonlinear periodic systems.

Example 1 ([11]).

We consider van der Pol's equation with an external forcing term

$$(4.1) \quad \frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = B \cdot \cos vt \quad (\mu=0.2).$$

We put $v=0.9$ and employ B as a parameter and compute a bifurcation point. The results of numerical computations are as follows:

$$(4.2) \quad \begin{cases} \hat{x}_1(0)=1.25704, & \hat{x}_2(0)=-0.55890, & \hat{B}=0.29563, \\ \hat{\theta}=0.59920 & (\cos \hat{\theta}=0.82579, \sin \hat{\theta}=0.56398), \end{cases}$$

where $x_1(t)=x(t)$ and $x_2(t)=\frac{dx}{dt}(t)$.

Secondly, we compute a singular point to which a bifurcation point degenerates.

Example 2.

Now, in (4.1), we take v and B as parameters. Then, as the values of v and B vary, a bifurcation point degenerates to a singular point with a higher singularity, as is mentioned in the latter half of Section 2. We compute such a singular point. The results of numerical computations are as follows:

$$(4.3) \quad \begin{cases} \hat{x}_1(0)=0.99960, & \hat{x}_2(0)=-0.90135, \\ \hat{v}=0.94934, & \hat{B}=0.19239 \end{cases}$$

and

$$(4.4) \quad \begin{cases} \hat{x}_1(0)=-0.99968, & \hat{x}_2(0)=-1.10124, \\ \hat{v}=1.04940, & \hat{B}=0.20741. \end{cases}$$

Lastly, we compute a bifurcation point of periodic solutions of nonlinear autonomous systems.

Example 3.

We consider an ω -periodic solution of an autonomous system

$$(4.5) \quad \begin{cases} \frac{dx}{d\tau} = y, \\ \frac{dy}{d\tau} = \varepsilon(1 - \gamma z^2 - \eta y^2)y - x, \\ \frac{dz}{d\tau} = az - dz^3 + bx + c, \end{cases}$$

where $\varepsilon=0.4$, $a=0.04$, $d=0.01$, $b=0.4$, $c=-0.04$, $\gamma=1.0$, and η is a parameter. In this example, we adopt the condition

$$(4.6) \quad y(0)=0$$

as an additional condition corresponding to (3.10) in Section 3. We compute a bifurcation point and the results of numerical computations are as follows:

$$(4.7) \quad \begin{cases} \hat{x}(0)=0.45085, & \hat{y}(0)=0, & \hat{z}(0)=0.97025, \\ \hat{\omega}=6.29435, & \hat{\eta}=0.27152, & \hat{\sigma}=0.55230 \\ (\cos \hat{\sigma}=0.85132, & \sin \hat{\sigma}=0.52465). \end{cases}$$

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