CONTROLLED DIFFUSION PROCESSES WITE SUCCESSIVE REWARDS

By

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1. Introduction

Controlled Markov Processes have been studied by many authors. As for the reward function appeared in the literatures concerning the optimization of the processes, the functions of additive type and of average type have been treated for a long time in both of the discrete time and the continuous time cases. In the discrete time case Furukawa and Iwamoto [8] introduced the notion of a recursive reward function, giving a generic name to a broad class of reward functions. They studied the nature of optimal policies and the relation between an optimal policy and the optimality equation for the controlled Markov process associated with the recursive reward function. Groenewegen [9] generalized a certain type of the recursive reward function in the sense of Furukawa and Iwamoto to the continuous time case.

In this paper we shall be concerned with the optimal control of continuous time diffusion processes. We introduce the notion of a successive reward function which is an extension of the recursive function to a nonstationarity. Since the successive function can be expressed by a transition of some types of nonlinear semigroups, the nonlinear semigroup theory is applicable to the analysis of controlled diffusion processes associated with the successive reward functions.

The purpose of this paper is to study the controlled diffusion processes associated with the successive reward functions from the standpoints of the nonlinear semigroup theory and of the nonlinear evolution equation theory. In Section 2 we shall give a formalution of controlled diffusion processes. In Section 3 we shall give necessary and sufficient conditions for an optimal control. In Section 4 we shall discuss Bellman semigroups and optimal value functions.

In the last section we shall give several examples.

2. Notations and controlled processes

In this section we shall give some basic notations which will be used throughout this paper. Let $R^+ = [0, \infty)$. Let R^d be the d -dimensional Euclidean space. Ω denotes the set of all continuous mappings $\omega: R^+ \rightarrow R^d$. A mapping $x(t)$ is given by $x(t, \omega) = \omega(t)$ for $t \in R^+$ and $\omega \in \Omega$. For each $t \in R^+$, $\mathcal{F}_t^0(\mathcal{F}^0)$ denotes the smallest σ -algebra generated by $\{x(s): s \in R^+, s \leq t\}$ (resp. $\{\omega(s): s \in R^+\}$). Let P be a probability measure on $(\Omega, \mathcal{F}^0)$ and let $\mathcal{F}_t(\mathcal{F})$ be the completion of $\mathcal{F}_t^0$ (resp. $\mathcal{F}^0$) with respect to P . Γ is a convex subset of d_1 -dimensional Euclidean space, the action space. U denotes the set of all bounded continuous mappings $u(t, x)$ from $R^+ \times R^d$ to Γ , such that there exist first and second-order derivatives of the mapping u with respect to $x \in R$ and the derivatives are bounded and continuous in $(t, x) \in R^+ \times R^d$, and such that there exists a constant $K_1 > 0$ satisfying that $|u(t, x) - u(s, y)|_{d_1} \leq K_1\{|t - s| + |x - y|_d\}$ for all $x, y \in R^d$ and $t, s \in R^d$, where $|\cdot|_d$ denotes the Euclidean norm in R^d . For each $(\gamma, x) \in \Gamma \times R^d$, $\sigma(\gamma, x)$ and $b(\gamma, x)$ denote a d -dimensional matrix and a d -dimensional vector respectively. Suppose that $\sigma(\gamma, x)$ and $b(\gamma, x)$ satisfy (i), (ii), (iii) and (iv):

- (i) $\sigma(\gamma, x)$ and $b(\gamma, x)$ are bounded and continuous in $\Gamma \times R^d$.
- (ii) $\sigma(\gamma, x)$ and $b(\gamma, x)$ have bounded continuous first and second-order derivatives with respect to $(\gamma, x) \in \Gamma \times R^d$.
- (iii) There exists a constant $\delta > 0$ such that $(\sigma(\gamma, x)\xi, \xi)_d \geq \delta|\xi|_d^2$ for all $(\gamma, x) \in \Gamma \times R^d$ and $\xi \in R^d$, where $(\cdot, \cdot)_d$ denotes the Euclidean inner product in R^d .
- (iv) There exists a constant $K_2 > 0$ such that

$$\|\sigma(\gamma, x) - \sigma(\gamma', x')\|_d + |b(\gamma, x) - b(\gamma', x')|_d \leq K_2\{|\gamma - \gamma'|_{d_1} + |x - x'|_d\}$$

for all $\gamma, \gamma' \in \Gamma$ and $x, x' \in R^d$, where $\|\cdot\|_d$ denotes the operator norm.

U is called the family of admissible controls. We shall fix arbitrary $u \in U$ for the rest of this section. $(w(t), \mathcal{F}_t, P)$ is a fixed d -dimensional Wiener process. By Corollary 5.1.2 in Strook and Varadhan [18], for any $(s, x) \in R^+ \times R^d$ we have a unique solution $(X_t^u)_{t \geq s}$ of

$$(2.1) \quad \begin{cases} X_t^u = x + \int_s^t \sigma(u(r, X_r^u), X_r^u) dw(r) \\ \quad + \int_s^t b(u(r, X_r^u), X_r^u) dr \\ \text{for each } t \geq s. \end{cases} \quad P\text{-a.s.}$$

Then $((t, X_t^u), \mathcal{F}_t, P)$ is a Feller continuous diffusion process with state space $R^+ \times R^d$ from Theorems 5.3.1 and 6.3.4 in [18]. Now we introduce the differential generator G_t^u of a process $X^u = (X_t^u, \mathcal{F}_t, P)$ and its domain D . D denotes the set of all continuous functions k on R^d such that k vanishes at infinity and such that k is twice continuously differentiable and its first and second-order derivatives vanish at infinity. Let

$$G_t^u k(x) = \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(u(t, x), x) \frac{\partial^2}{\partial x_i \partial x_j} k(x) + \sum_{i=1}^d b_i(u(t, x), x) \frac{\partial}{\partial x_i} k(x)$$

for $k \in D$ and $x \in R^d$, where $a(\gamma, x) = \sigma(\gamma, x)\sigma^*(\gamma, x)$ and $\sigma^*(\gamma, x)$ is the transpose of $\sigma(\gamma, x)$ for $(\gamma, x) \in \Gamma \times R^d$.

$E^{t,x}[\cdot]$ denotes the expectation on pathes starting at $(t, x) \in R^+ \times R^d$ and $E^x[\cdot]$ denotes $E^{0,x}[\cdot]$ for $x \in R^d$. Then we have $E^x \left[\int_0^t G_r^u k(X_r^u) dr \right] = E^x [k(X_t^u)] - k(x)$ for all $(t, x) \in R^+ \times R^d$ and $k \in D$ from Theorems 3.2.6 and 6.3.4 in [18]. Let L be the space of all bounded continuous functions on R^d , which vanish at infinity, endowed with the supremum norm $\|\cdot\|$. Define $P_t^u k(x) = E^x[k(X_t^u)]$ for $k \in L$ and $(t, x) \in R^+ \times R^d$. Then we have $P_t^u L \subset L$ for all $t \in R^+$ from Theorem 6.3.4 in [18]. A_0^u denotes the infinitesimal generator of $\{P_t^u\}_{t \geq 0}$ at time 0. We have $A_0^u = G_0^u$ on D from Friedman [7, p. 117]. Now U_M denotes the family of Markov controls such that $U_M = \{u \in U : u(t, x) = u(0, x) \text{ for all } (t, x) \in R^+ \times R^d\}$. Then for $u \in U_M$, G_t^u is independent of t and $\{P_t^u\}$ is a time-homogeneous semigroup on L . Therefore we express G_0^u and A_0^u as G^u and A^u respectively. Moreover we express G^u , A^u and $\{P_t^u\}$ as G^γ , A^γ and $\{P_t^\gamma\}$ respectively, if u is a constant control $u \equiv \gamma$ for some $\gamma \in \Gamma$. Then we have $D \subset D_{A^u} \subset L$, $D \subset D_{A^\gamma} \subset L$ for each $u \in U_M$ and $\gamma \in \Gamma$ and D is dense in L . In this paper we use the following notations. L_0 is the set of bounded measurable functions $k(t, x)$ on $R^+ \times R^d$ such that $k(t, \cdot) \in L$ for each $t \in R^+$. We treat L a subspace of L_0 in the sense that $k' \in L_0$ if we put $k'(t, x) = k(x)$, $(t, x) \in R^+ \times R^d$, for some $k \in L$. In the following sections U_1 denotes the family of all mappings $u: R^d \times R^d \rightarrow \Gamma$ for which there exist a natural number n , a partition $\{0 = t_0 < t_1 < \dots < t_{n+1} < \infty\}$ of R^+ and a finite sequence $\{u_i\}_{0 \leq i \leq n+1} \subset U$ such that

$$u(t, x) = \begin{cases} u_i(t, x) & \text{for } (t, x) \in [t_i, t_{i+1}) \times R^d \text{ and } 0 \leq i \leq n, \\ u_{n+1}(t, x) & \text{for } (t, x) \in [t_{n+1}, \infty) \times R^d. \end{cases}$$

Then we can introduce processes $\{X_t^u\}_{t \geq 0}$ and $\{P_t^u\}_{t \geq 0}$ for $u \in U_1$ from the same reason as the case of $u \in U$. For each $u \in U_1$ and $t \geq s$, $\mathcal{F}_{t,s}^u$ denotes the completion of $\sigma(X_r^u: t \geq r \geq s)$ with respect to P . D_s denotes the space of all bounded smooth functions on R^d vanishing at infinity and L_+ denotes the set of all non-negative, bounded, smooth and Hölder continuous functions on R^d .

3. Successive reward functions

In this section we shall introduce a system of successive reward functions. Let $T=(0, \infty)$. For each $u \in U_1$, $A_{t,s,r}^u(\omega)$ and $M_{t,s,r}^u(\omega)$ are functions of $(t, s, r, \omega) \in R^+ \times T \times R^+ \times \Omega$ and $B_{t,s}$ is a function of $(t, s) \in R^+ \times T$. We assume (c.i)~(c.vi):

(c.i) For $(t, s, r, \omega, u) \in R^+ \times T \times R^+ \times \Omega \times U_1$, $A_{0,s,r}^u(\omega)=0$, $M_{0,s,r}^u(\omega)=1$ and $M_{t,s,r}^u(\omega) \geq 0$.

(c.ii) For each $(t, s, r, u) \in R^+ \times T \times R^+ \times U_1$, $A_{t,s,r}^u(\cdot)$ and $M_{t,s,r}^u(\cdot)$ are bounded and $\mathcal{F}_{t+s,s}^u$ -measurable.

(c.iii) For each $u \in U_1$, $A_{t,s,r}^u$ and $M_{t,s,r}^u$ are continuous a.s. P with respect to $(t, s, r) \in R^+ \times T \times R^+$.

(c.iv) $B_{\cdot,\cdot}$ is bounded and continuous on $R^+ \times T$.

(c.v) For each $(t, s, r, \omega, u) \in R^+ \times T \times R^+ \times \Omega \times U_1$, $A_{t,s,r}^u(\omega)$ and $M_{t,s,r}^u(\omega)$ are independent of $\{u(r', y): (r', y) \in [0, r] \times R^d \text{ or } (r', y) \in [t+r, \infty) \times R^d\}$.

(c.vi) For each $(t, s, r, w, u) \in R^+ \times T \times T \times R^+ \times U_1$ it holds that

$$(3.1) \quad \begin{cases} B_{t,s+r} = B_{t+s,r} B_{t,s}, \\ M_{t+s,r,w}^u = M_{t,s+r,w}^u M_{s,r,t+w}^u & \text{a.s. } P \\ B_{t+s,r} M_{t,s,w}^u + M_{t,s+r,w}^u B_{s,r} & \text{a.s. } P \\ \text{and} \\ B_{t+s,r} A_{t,s,w}^u + A_{t+s,r,w}^u \\ \quad = A_{t,s+r,w}^u + M_{t,s+r,w}^u A_{s,r,t+w}^u & \text{a.s. } P. \end{cases}$$

Next we consider a family $\{C_{t,r}^u\}_{(t,r) \in R^+ \times R^+}$ of functions which satisfy (c. vii)~(c.xi):

(c.vii) For each $(t, r, u) \in T \times R^+ \times U_1$, $C_{t,r}^u$ is an extended valued $\mathcal{F}_{t+r,r}^u$ -measurable function such that $C_{t,r}^u$ is uniformly bounded in $u \in U_1$ and such that $C_{t,r}^u$ is continuous a.s.(P) with respect to $(t, r) \in T \times R^+$.

(c.viii) For each $(t, s, r, u) \in T \times T \times R^+ \times U_1$, it holds that

$$(3.2) \quad \begin{cases} C_{t+s,r}^u = B_{t,s} C_{t,r}^u + A_{t,s,r}^u + M_{t,s,r}^u C_{s,t+r}^u \\ \text{a.s. } P \end{cases}$$

(c.ix) $s\text{-}\lim_{t \downarrow 0} E^{r,x}[B_{t,s} C_{t,r}^u] = 0$ for each $(s, r, u) \in T \times R^+ \times U_1$.

(c.x) For each $(t, s, u) \in T \times T \times U_1$, we define a control $u'(r, \cdot) = u(r-t, \cdot)$ for $r \geq t$. Then it holds that $C_{s,t}^{u'} = C_{s,0}^u$ a.s.(P).

(c.xi) Let $(t, r, x) \in T \times R^+ \times R^d$ be fixed. If there exist $\{u^n\}_n \subset U_1$ and $u \in U_1$ such that

$E^{r,x}[\sup_{r \leq r' \leq t+r} |X_{r'}^{u^n} - X_{r'}^u|^2] \rightarrow 0$ as $n \rightarrow \infty$, then we can extract a subsequence $\{u^{n'}\}_{n'}$, such that $C_{t,r}^{u^{n'}} \rightarrow C_{t,r}^u$ a.s. $P^{r,x}$ as $n' \rightarrow \infty$.

Then we will call $\{C_{t,r}^u\}$ a family of successive functions and the integrand $f_t^u(x) = E^x[C_{t,0}^u]$, $x \in R^d$, its value function. We shall treat the family $\{C_{t,r}^u\}$ in Section 3 and 4. Examples of the family of successive functions will be given in Section 5. Define $f_t(x) = \sup_{u \in U} f_t^u(x)$ for each $(t, x) \in T \times R^d$.

DEFINITION 3.1. For each fixed $t \in T$, a control $u^* \in U$ is called optimal at time t if $f_t^{u^*}(x) = f_t(x)$ for all $x \in R^d$.

We define the operator $S_{t+s,s}^u$ induced by $\{C_{t,r}^u\}$ as follows:

$S_{t+s,s}^u k(r, x) = E^{r,x}[B_{t,s} C_{t,r}^u + A_{t,s,r}^u + M_{t,s,r}^u k(t+r, X_{t+r}^u)]$ and $S_{s,s}^u k(r, \cdot) = s\text{-}\lim_{t \downarrow 0} S_{t+s,s}^u k(r, \cdot)$ for $(t, s, r, x, u) \in T \times T \times R^+ \times R^d \times U_1$ and $k \in L_0$.

LEMMA 3.1. We have (i) and (ii):

(i) $\{S_{t,s}^u\}_{t \geq s}$ is a nonautonomous semigroup and $S_{s,s}^u = I$ for each $(x, u) \in T \times U_M$, where I is the identity operator.

(ii) For $(t, s, u, u') \in T \times T \times U \times U$ ($t \geq s$), we put

$$\bar{u}(r, y) = \begin{cases} u(r, y) & \text{if } (r, y) \in [0, t] \times R^d, \\ u'(r-t, y) & \text{if } (r, y) \in (t, \infty) \times R^d. \end{cases}$$

Then we have $S_{t+s,s}^u f_s^{u'}(0, x) = f_{t+s}^{\bar{u}}(x)$ for all $x \in R^d$.

PROOF. We can easily check (i) and (ii) by virtue of (c.i)~(c.x) and the Markov property.

Let $(s, u) \in T \times U_M$ be fixed. We shall define the generator $\mathcal{A}_s^u$ of $\{S_{t+s,s}^u\}_{t \geq 0}$ and its domain $D_{\mathcal{A}_s^u}$ as follows. $D_{\mathcal{A}_s^u}$ express the set of all elements k of L for which $[S_{t+s,s}^u k(0, \cdot) - k(\cdot)]/t$ converges strongly as $t \downarrow 0$ and the limit belongs to L .

Put $\mathcal{A}_s^u k(\cdot) = s\text{-}\lim_{t \downarrow 0} [S_{t+s,s}^u k(0, \cdot) - k(\cdot)]/t$ for $k \in D_{\mathcal{A}_s^u}$.

We assume the following assumptions in this paper.

ASSUMPTIONS. (I) $D_{\mathcal{A}_s^u} = D_{A^u}$ for each $(s, u) \in T \times U_M$.

(II) $R(I - \lambda \mathcal{A}_s^u) = L$ for each $(s, u) \in T \times U_M$ and $\lambda > 0$, where $R(I - \lambda \mathcal{A}_s^u)$ denotes the range of $I - \lambda \mathcal{A}_s^u$.

(III) Let $u \in U$ and $k \in L$ be fixed. Then it holds that $S_{t+s,s}^u k(0, \cdot) \in L$ for all $(t, s) \in R^+ \times T$.

(IV) For each $u \in U$, f_t^u has a derivative with respect to t on T and the derivative is strongly continuous and of bounded variation on each finite closed subinterval of T .

(V) For each $(s, u) \in T \times U_M$, $\mathcal{A}_s^u$ is dispersive, that is, for each $k_1, k_2 \in D_{A^u}$ it holds that

$$\| [k_1 - k_2 - \lambda(\mathcal{A}_s^u k_1 - \mathcal{A}_s^u k_2)]^+ \| \geq \| (k_1 - k_2)^+ \| \quad \text{for all } \lambda > 0,$$

where $k^+ = \max \{k, 0\}$ for $k \in L$.

(VI) For each $u \in U_M$, there exist continuous mappings $h_i^u (i=0, 1, \dots, n): T \rightarrow L$, which are of bounded variation on each finite closed subinterval of T , an increasing function $h_{n+1}^u: R^+ \rightarrow R^+$ and a constant K' such that

$$\begin{aligned} \|\mathcal{A}_t^u k - \mathcal{A}_s^u k\| &\leq \|h_0^u(t) - h_0^u(s)\| h_{n+1}^u(\|k\|) (1 + \|\mathcal{A}_s^u k\|) \\ &\quad + K' \sum_{i=1}^n \|h_i^u(t) - h_i^u(s)\| \end{aligned}$$

for all $(t, s) \in T \times T$ and all $k \in D_{A^u}$.

(VII) For each $(u, u') \in U_M \times U$ if $f_s^{u'} \in D$ for all $s \in T$, then $\mathcal{A}_s^u f_s^{u'}$ has the bounded variation of s on each finite closed subinterval of T .

We give the following lemma by the help of Crandall and Pazy [2].

LEMMA 3.2. Suppose Assumptions (I), (II), (IV), (V) and (VI) be satisfied. Let $\tau, \sigma \in T (\tau < \sigma)$ and $u \in U_M$ be arbitrary but fixed. Then we have

$$S_{t,\tau}^u k(0, \cdot) = \lim_{n \rightarrow \infty} \prod_{i=1}^n J_{(t-\tau)/n}^u \left(\tau + i \left(\frac{t-\tau}{n} \right) \right) k$$

for $k \in D_{A^u}$ and $t \in (\tau, \sigma]$, where $J_\mu^u(s) = (I - \mu \mathcal{A}_s^u)^{-1}$ for $\mu > 0$ and $s \in T$. Moreover if $k \in L$, then the following Cauchy problem (3.3) has a unique solution $\Phi(t) = S_{t,\tau}^u k(0, \cdot)$ ($t \in [\tau, \sigma]$):

$$(3.3) \quad \begin{cases} \frac{d}{dt} \Phi(t) = \mathcal{A}_t^u \Phi(t) & \text{for all } t \in (\tau, \sigma) \\ \text{and } \Phi(\tau) = k. \end{cases}$$

PROOF. We shall use slight modifications of Theorems 2.1, 3.1 and 3.2 in Crandall and Pazy [2]. Consider the norm $\|k\|_1 = \|k^+\| + \|k^-\|$ for $k \in L$, where $k^- = -\min\{k, 0\}$ for $k \in L$. Then from Assumption (V) we have

$$\|k_1 - k_2 - \lambda(\mathcal{A}_s^u k_1 - \mathcal{A}_s^u k_2)\|_1 \geq \|k_1 - k_2\|_1$$

for $u \in U_M$, $k_1, k_2 \in D_{A^u}$, $s \in T$ and $\lambda > 0$. Therefore the condition (A.1) in Crandall and Pazy [2] is satisfied for the Banach norm $\|\cdot\|_1$ which is equivalent to $\|\cdot\|$, since $\|k\| \leq \|k\|_1 \leq 2\|k\|$ for $k \in L$. Therefore $-\mathcal{A}_s^u$ is m-accretive in the sense of Evans [6] from Assumptions (I) and (II). It is well-known that an m-accretive operator is maximal and closed. Consequently from Assumptions (I)~(VI) it is sufficient to check the condition (C.2) in Crandall and Pazy [2]. However, as is pointed out in Remark 7.1 of Evans [6], the condition (C.2) can be slightly weakened. This fact will be clarified below. By Assumption (VI), in a line similar to the proof of Lemma 3.2 in Crandall and Pazy [2], we can derive

$$(3.4) \quad \begin{cases} \|J_\lambda^u(t)k - J_\lambda^u(s)k\|_1 \leq \lambda \|h_0^u(t) - h_0^u(s)\| h_{n+2}^u(\|k\|_1) \\ \quad \cdot (1 + |\mathcal{A}_s^u k|_1) + 2\lambda K' \sum_{i=1}^n \|h_i^u(t) - h_i^u(s)\|_1 \\ \leq \lambda \left\{ \sum_{i=0}^n \|h_i^u(t) - h_i^u(s)\|_1 \right\} h_{n+2}^u(\|k\|_1) (1 + |\mathcal{A}_s^u k|_1) \end{cases}$$

for $k \in L$, $\lambda > 0$ and $t \in T$, where $h_{n+2}^u(r) = 4 \max\{h_{n+1}^u(\bar{K}(r+1)), K'\}$ for a suitable constant $\bar{K}$ and $|\mathcal{A}_s^u k|_1 = \lim_{\lambda \downarrow 0} \|\lambda^{-1}(I - J_\lambda^u(s))k\|_1$. Here we used the inequalities $\|\cdot\| \leq \|\cdot\|_1 \leq 2\|\cdot\|$. Comparing (3.4) with the condition (C.2) in Crandall and Pazy [2], it will be seen that we have only to consider $\sum_{i=0}^n \|h_i^u(t) - h_i^u(s)\|_1$ instead of $\|h_1^u(t) - h_1^u(s)\|$ in [2]. In the proofs of Theorems 2.1, 3.1, 3.2 and Lemma 3.2 in [2], the condition (C.2) was used only for estimating the continuity and the variation of $J_\lambda^u(\cdot)$. Since h_i^u ($i=0, 1, 2, \dots, n$) are continuous and of bounded variation, the proof of this lemma follows the same line as a series of proofs of Theorem 2.1, 3.1, 3.2 and Lemma 3.2 in [2]. The details are omitted. At last Lemma 3.1 (i) gives us the solution of (3.3) by $\Phi(t) = S_{t,\tau}^u k(0, \cdot)$ ($t \in [\tau, \sigma]$).

The following lemma is based on the idea of Sato [17] and Konishi [11].

LEMMA 3.3. Suppose Assumptions (I), (II) and (V) be satisfied. Let $k_1, k_2 \in L$ ($k_1 \geq k_2$), $k_3 \in L$ ($k_3 \geq 0$) and $u \in U_M$ be arbitrary but fixed. Then we have

$$(I - \lambda(\mathcal{A}_s^u + k_3))^{-1}k_1 \geq (I - \lambda\mathcal{A}_s^u)^{-1}k_2$$

for each $s \in T$ and $\lambda > 0$.

PROOF. Let $k_1, k_2 \in L$ ($k_1 \geq k_2$), $k_3 \in L$ ($k_3 \geq 0$), $u \in U_M$, $s \in T$ and $\lambda > 0$ be fixed. We put $k'_1 = (I - \lambda(\mathcal{A}_s^u + k_3))^{-1}k_1$ and $k'_2 = (I - \lambda\mathcal{A}_s^u)^{-1}k_2$. Then we have $k_1 + \lambda k_3 = k'_1 - \lambda\mathcal{A}_s^u k'_1$ and $k'_1 - \lambda\mathcal{A}_s^u k'_1 \geq k_2 = k'_2 - \lambda\mathcal{A}_s^u k'_2$. From Assumption (V) we obtain $k'_1 \geq k'_2$. This completes the proof.

Now we shall study properties of controlled processes. Let $(t, v, w) \in T \times U \times U$ be fixed. Define $\bar{u} \in U_1$ by

$$(3.5) \quad \bar{u}(s, y) = \begin{cases} v(s, y) & \text{if } (s, y) \in [0, t] \times R^d, \\ w(s-t, y) & \text{if } (s, y) \in (t, \infty) \times R^d, \end{cases}$$

and for each n define u^n by

$$(3.6) \quad u^n(s, y) = \begin{cases} v(s, y) & \text{if } (s, y) \in [0, t] \times R^n, \\ \Phi_n(s)v(t, y) + (1 - \Phi_n(s))w(1/n, y) & \text{if } (s, y) \in (t, t+1/n] \times R^d, \\ w(s-t, y) & \text{if } (s, y) \in (t+1/n, \infty) \times R^d, \end{cases}$$

where $\Phi_n(s) = -ns + nt + 1$ for each natural number n and $s \in [t, t+1/n]$. Then it holds that $u^n \in U$ and $\{u^n\}_n$ converges pointwise to $\bar{u}$. And we have the following lemma.

LEMMA 3.4. It holds that

$$E^{r,x} \left[\sup_{r \leq s \leq t_1} |X_s^{u^n} - X_s^{\bar{u}}|_d^2 \right] \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

for each $t_1 > t > r \geq 0$ and $x \in R^d$.

PROOF. Let $(r, x) \in [0, t) \times R^d$ be fixed. Then we have

$$\begin{aligned} & E^{r,x} [|X_{t+1/n}^{u^n} - X_{t+1/n}^{\bar{u}}|_d^2] \\ & \leq E^{r,x} \left[\left| \int_t^{t+1/n} \{ \sigma(u^n(s, X_s^{u^n}), X_s^{u^n}) - \sigma(\bar{u}(s, X_s^{\bar{u}}), X_s^{\bar{u}}) \} dw(r) \right|_d^2 \right] \end{aligned}$$

$$+ E^{t,x} \left[\left| \int_t^{t+1/n} \{b(u^n(s, X_s^{u^n}), X_s^{u^n}) - b(\bar{u}(s, X_s^{\bar{u}}), X_s^{\bar{u}})\} dr \right|^2 \right]$$

$\leq (1/n + 1/n^2)K$ for some constant K . By applying Corollary 5.1.2 in Strook and Varadhan [18] to the above inequality, the lemma is proved.

THEOREM 3.1. *Suppose Assumptions (I)~(VI) be satisfied. Let $\tau, \sigma \in T$ ($\tau < \sigma$) be fixed. If a control $u^* \in U$ satisfies that $f_s^{u^*} \in D$ for all $s \in (\tau, \sigma)$ and u^* is optimal at every time $s \in [\tau, \sigma]$, then it holds that*

$$(3.7) \quad \sup_{u \in U_M} \mathcal{A}_s^u f_s^{u^*} \leq \frac{d}{ds} f_s^{u^*} \text{ for all } s \in (\tau, \sigma).$$

PROOF. Let $u^* \in U$ be optimal at every time $s \in [\tau, \sigma]$, and suppose that $f_s^{u^*} \in D$ for all $s \in (\tau, \sigma)$. Let $(t, x, u) \in T \times R^d \times U_M$ be arbitrary but fixed. Define u^n and $\bar{u}$ by (3.5) and (3.6) in which v, w and r are replaced by u, u^* and 0 respectively. Now, since $u^n \in U$ for each n , we have (3.8):

$$(3.8) \quad f_{t+s}^{u^*}(x) \geq f_{t+s}^{u^n}(x) \quad \text{for all } n.$$

By tending n to infinity in (3.8), from (c.vii), (c.viii), (c.xi) and Lemma 3.4, we have

$$\begin{aligned} f_{t+s}^{u^*}(x) &\geq f_{t+s}^{\bar{u}}(x) = E^x[C_{t+s,0}^{\bar{u}}] \\ &= E^x[B_{t,s}C_{t,0}^{\bar{u}} + A_{t,s,0}^{\bar{u}} + M_{t,s,0}^{\bar{u}}C_{s,t}^{\bar{u}}]. \end{aligned}$$

Using (c.ii) and the Markov property we have

$$f_{t+s}^{u^*}(x) \geq E^x[B_{t,s}C_{t,0}^{\bar{u}} + A_{t,s,0}^{\bar{u}} + M_{t,s,0}^{\bar{u}}E^{t,X_t^{\bar{u}}}[C_{s,t}^{\bar{u}}]].$$

From (c.v), (c.vii), (c.x) and (3.5) we have

$$\begin{aligned} f_{t+s}^{u^*}(x) &\geq E^x[B_{t,s}C_{t,0}^u + A_{t,s,0}^u + M_{t,s,0}^u E^{0,X_t^u}[C_{s,0}^{u^*}]] \\ &= S_{t+s,s}^u f_s^{u^*}(0, x) \end{aligned}$$

for $(t, x, u, s) \in T \times R^d \times U_M \times (\tau, \sigma)$. From Assumption (IV) and that $f_s^{u^*} \in D$ for all $s \in (\tau, \sigma)$, we obtain $\frac{d}{ds} f_s^{u^*} \geq \mathcal{A}_s^u f_s^{u^*}$ for all $s \in (\tau, \sigma)$, which yields (3.7).

Next we examine the sufficiency of (3.7) for the optimality of u^* .

THEOREM 3.2. *Suppose Assumptions (I)~(VII) be satisfied. Let $\tau, \sigma \in T$ ($\tau < \sigma$) be fixed. If a control u^* is optimal at time τ and satisfies $f_s^{u^*} \in D$ for all $s \in (\tau, \sigma)$, and if (3.7) holds, then u^* is optimal at every time $s \in [\tau, \sigma]$.*

PROOF. Let $u^* \in U$ be optimal at time τ and let u^* satisfy $f_s^{u^*} \in D$ for all $s \in (\tau, \sigma)$ and (3.7). From (3.7) we have

$\mathcal{A}_s^u f_s^{u^*} \leq \frac{d}{ds} f_s^{u^*}$ for all $(s, u) \in (\tau, \sigma) \times U_M$. We define the function $k_s^u \in L$ and the operator $\mathcal{B}_s^u$ on D_{A^u} by

$$(3.9) \quad \begin{cases} k_s^u = \frac{d}{ds} f_s^{u^*} - \mathcal{A}_s^u f_s^{u^*} & \text{and} \\ \mathcal{B}_s^u k = \mathcal{A}_s^u k + k_s^u & \text{for } (s, u) \in T \times U_M \text{ and } k \in D_{A^u}. \end{cases}$$

Now for $(t, s, u) \in T \times T \times U_M$ we have

$$(3.10) \quad \begin{cases} \|\mathcal{A}_t^u f_t^{u^*} - \mathcal{A}_s^u f_s^{u^*}\| \\ \leq \|\mathcal{A}_t^u f_t^{u^*} - \mathcal{A}_s^u f_t^{u^*}\| + \|\mathcal{A}_s^u f_t^{u^*} - \mathcal{A}_s^u f_s^{u^*}\|. \end{cases}$$

Hence $\mathcal{A}_s^u$ is a closed operator as we have seen in the proof of Lemma 3.2. Hence from Assumptions (IV) and (VI), the closedness of $\mathcal{A}_s^u$ and (3.10), it follows that k_s is strongly continuous in $s \in T$ and of bounded variation on each finite closed subinterval of T . Now we have (3.11) $\frac{d}{ds} f_s^{u^*} = \mathcal{B}_s^u f_s^{u^*}$ for all $s \in (\tau, \sigma)$. From Assumptions (I)~(VII) for $\mathcal{A}_s^u$, $\mathcal{B}_s^u$ satisfies (I), (II), (V) and (VI). By applying Lemma 3.2 to (3.11), we obtain

$$(3.12) \quad f_s^{u^*} = \lim_{n \rightarrow \infty} \prod_{i=1}^n \tilde{J}_{(s-t)/n}^u \left(t + i \left(\frac{s-t}{n} \right) \right) f_t^{u^*}$$

for each $\tau \leq t < s \leq \sigma$, where $\tilde{J}_\mu^u(t) = (I - \mu \mathcal{B}_t^u)^{-1}$ for $\mu > 0$ and $t \in (\tau, \sigma]$.

Now we consider

$$(3.13) \quad S_{s,t}^u f_t^{u^*} = \lim_{n \rightarrow \infty} \prod_{i=1}^n J_{(s-t)/n}^u \left(t + i \left(\frac{s-t}{n} \right) \right) f_t^{u^*}$$

for each $\tau \leq t \leq s \leq \sigma$, where J^u is defined in Lemma 3.2. Since $k_s^u \geq 0$, from Lemma 3.3 we have

$$(3.14) \quad \begin{cases} \prod_{i=1}^n \tilde{J}_{(s-t)/n}^u \left(t + i \left(\frac{s-t}{n} \right) \right) f_t^{u^*} \\ \geq \prod_{i=1}^n J_{(s-t)/n}^u \left(t + i \left(\frac{s-t}{n} \right) \right) f_t^{u^*} \end{cases}$$

for each $\tau \leq t < s \leq \sigma$. Therefore from (3.12), (3.13) and (3.14) we obtain

$$(3.15) \quad f_s^{u^*} \geq S_{s,t}^u f_t^{u^*} \quad \text{for } \tau \leq t < s \leq \sigma \text{ and } u \in U_M.$$

Let $(s, \bar{u}) \in (\tau, \sigma) \times U$ be arbitrary but fixed. For each natural number n , we consider a subdivision $I = \{\tau = t_0 < t_1 < t_2 < \dots < t_n = \sigma\}$ of $[\tau, \sigma]$ such that $t_i = \tau + (\sigma - \tau)i/n$ for $0 \leq i \leq n$. For each n , there exists i_0 such that $t_{i_0} \leq s < t_{i_0+1}$. Define controls $u^i \in U_M$ by

$$u^i(x) = \begin{cases} \bar{u}(0, x) & \text{for } i = i_0 + 1 \text{ and } x \in R^d, \\ \bar{u}(s - t_{i_0} + (i_0 - i)\Delta_n, x) & \text{for } 1 \leq i \leq i_0 \text{ and } x \in R^d, \end{cases}$$

where $\Delta_n = (\sigma - \tau)/n$.

And define a mapping $u_n(t, x)$ by

$$u_n(t, x) = \begin{cases} u^{i_0+1}(t, x) & \text{for } (t, x) \in [0, s - t_{i_0}) \times R^d, \\ u^{i_0+1-i}(t, x) & \text{for } (t, x) \in [\tau_i, \tau_{i+1}) \times R^d \text{ and } 1 \leq i \leq i_0, \\ \bar{u}(t, x) & \text{for } (t, x) \in [s - \tau, \infty) \times R^d, \end{cases}$$

where $\tau_i = s - t_{i_0} + (i - 1)\Delta_n$ for $0 \leq i \leq i_0$.

Then from (3.15) we have

$$f^{u*}_{t_1} \geq S^{u^1}_{t_1, \tau} f^{u*}_{\tau}.$$

Moreover by operating $S^{u^i}_{t_i, t_{i-1}}$ ($2 \leq i \leq i_0$) and using (3.15) inductively we have

$$f^{u*}_{t_{i_0}} \geq S^{u^{i_0}}_{t_{i_0}, t_{i_0-1}} S^{u^{i_0-1}}_{t_{i_0-1}, t_{i_0-2}} \dots S^{u^1}_{t_1, \tau} f^{u*}_{\tau}.$$

And by operating $S^{u^{i_0+1}}_{s, t_{i_0}}$ and using (3.15) we have

$$(3.16) \quad f^{u*}_s \geq S^{u^{i_0+1}}_{s, t_{i_0}} S^{u^{i_0}}_{t_{i_0}, t_{i_0-1}} \dots S^{u^1}_{t_1, \tau} f^{u*}_{\tau}.$$

From (3.16), Lemma 3.1, the definition of u_n and the optimality of u^* at time τ we obtain $f^{u*}_s \geq f^{u_n}_s$ for all n . On the other hand we have $|u_n(t, x) - \bar{u}(t, x)|_{d_1} \leq K_0 \Delta_n$ for all n and $(t, x) \in R^+ \times R^d$, where K_0 is the Lipschitz constant for $\bar{u}$. Since $\bar{u} \in U$, there exists a constant K such that

$$(3.17) \quad \begin{cases} |u_n(r, X^{u_n}_r) - \bar{u}(r, X^{\bar{u}}_r)|_{d_1} \\ \leq |u_n(r, X^{u_n}_r) - \bar{u}(r, X^{u_n}_r)|_{d_1} + |\bar{u}(r, X^{u_n}_r) - \bar{u}(r, X^{\bar{u}}_r)|_{d_1} \\ \leq K \{\Delta_n + |X^{u_n}_r - X^{\bar{u}}_r|_d\} \text{ for all } n \text{ and } r \in R^+. \end{cases}$$

Moreover, by using Lipschitz conditions, as a result of standard arguments we have

$$E^x[|X^{u_n}_t - X^{\bar{u}}_t|_d^2] \leq K' \left\{ \Delta_n + E^x \left[\int_0^t |X^{u_n}_r - X^{\bar{u}}_r|_d^2 dr \right] \right\}$$

for all $(t, x) \in [0, \eta] \times R^d$ and n , where η is an arbitrary but fixed constant and K' is a constant which is independent of n and (t, x) . By Gronwall's inequality we obtain

$$(3.18) \quad E^x[|X_t^{u_n} - X_t^{\bar{u}}|_d^2] \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

for all $(t, x) \in R^+ \times R^d$.

In addition we have

$$\begin{aligned} & E^x \left[\sup_{0 \leq t \leq s} |X_t^{u_n} - X_t^{\bar{u}}|_d^2 \right] \\ & \leq 2E^x \left[\sup_{0 \leq t \leq s} \left| \int_0^t \{ \sigma(u_n(r, X_r^{u_n}), X_r^{u_n}) - \sigma(\bar{u}(r, X_r^{\bar{u}}), X_r^{\bar{u}}) \} dw(r) \right|_d^2 \right] \\ & + 2E^x \left[\sup_{0 \leq t \leq s} \left| \int_0^t \{ b(u_n(r, X_r^{u_n}), X_r^{u_n}) - b(\bar{u}(r, X_r^{\bar{u}}), X_r^{\bar{u}}) \} dr \right|_d^2 \right]. \end{aligned}$$

By applying Theorem 3.6 in Friedman [7 p. 701] and (3.17) we obtain

$$(3.19) \quad \begin{cases} E^x \left[\sup_{0 \leq t \leq s} |X_t^{u_n} - X_t^{\bar{u}}|_d^2 \right] \\ \leq 10K'' E^x \left[\int_0^s \{ \Delta_n^2 + |X_r^{u_n} - X_r^{\bar{u}}|_d^2 \} dr \right] \\ = 10sK'' \Delta_n^2 + 10K'' E^x \left[\int_0^s |X_r^{u_n} - X_r^{\bar{u}}|_d^2 dr \right] \end{cases}$$

for all $x \in R^d$ and n , where K'' is a constant independent of n and s .

From (3.18) and (3.19) for all $x \in R^d$ we have

$$E^x \left[\sup_{0 \leq t \leq s} |X_t^{u_n} - X_t^{\bar{u}}|_d^2 \right] \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore from (c.xi), (c.vii) and this fact, there exists a subsequence $\{n'\}$ such that

$$f_s^{u_{n'}}(x) \longrightarrow f_s^{\bar{u}}(x) \text{ as } n' \rightarrow \infty \quad \text{for all } x \in R^d.$$

Consequently we obtain $f_s^{u^*} \geq f_s^{\bar{u}}$ for all $s \in (\tau, \sigma)$ and $\bar{u} \in U$. Since $f_s^{u^*}$ is a continuous function from Assumption (IV) we obtain $f_s^{u^*} \geq f_s$ for all $s \in [\tau, \sigma]$. This completes the proof.

4. Bellman semigroups and their generators

In this section we shall investigate, according to Pliska [15], Bellman semigroups for the generators which have been defined in Section 3. It is obvious that

$\mathcal{A}_s^u$ becomes to be semi-linear by the definition of semigroup $\{S_{t,s}^u\}_{t \geq s}$. Therefore in this section we shall treat the case where $\mathcal{A}_s^u$ is expressed in the form:

$$(4.1) \quad \mathcal{A}_s^u k = A_0^u k - g_s k + k_s$$

for $(s, u) \in T \times U_M$ and $k \in D$, where $k_s \in L$ for each $s \in T$, g_s is a nonnegative bounded continuous on R^d for each $s \in T$, and both g_s and k_s are strongly continuous function of s and of bounded variation on each finite closed subinterval of T .

We have the following lemma.

LEMMA 4.1. *The generator $\mathcal{A}_s^u$ expressed by (4.1) has the following properties:*

- (i) *Assumptions (I), (II), (V) and (VI) are satisfied.*
- (ii) *If Assumption (IV) is satisfied, then Assumption (VII) is satisfied.*

PROOF. (i) Assumptions (I) and (VI) are clearly satisfied. As for Assumption (II) we consider the operator $\bar{A}^u k = A^u k - g_s k$, $k \in D_{A^u}$, with killing g_s . Then, corresponding to each fixed $(s, u) \in T \times U$, we can construct a time-homogeneous contraction semigroup. Therefore from Theorem 1.4 in Dynkin [5] we have $L = R(I - \lambda \bar{A}^u)$ for all $\lambda > 0$. Consequently Assumption (II) is satisfied. Moreover $\bar{A}^u$ is dispersive (s) in the sense of Sato [17] from Theorem 1 in [17]. The dispersiveness (s) implies the dispersiveness in the sense of Assumption (V) by virtue of (1.3) in [17].

(ii) Let $(u, u') \in U_M \times U$ be fixed and $f_s^{u'} \in D$ for all $s \in T$. For each finite closed subinterval T' of T we have

$$(4.2) \quad \begin{cases} \|\mathcal{A}_t^u f_t^{u'} - \mathcal{A}_s^u f_s^{u'}\| \\ \leq \|\mathcal{A}_t^u f_t^{u'} - \mathcal{A}_s^u f_t^{u'}\| + \|\mathcal{A}_s^u f_s^{u'} - \mathcal{A}_s^u f_t^{u'}\| \\ \leq \|(g_t - g_s)f_t^{u'} + (k_t - k_s)\| + \|A^u(f_t^{u'} - f_s^{u'}) + g_s(f_t^{u'} - f_s^{u'})\| \\ \leq K\{\|g_t - g_s\| + \|k_t - k_s\| + \|f_t^{u'} - f_s^{u'}\|\}, \end{cases}$$

where $K = \sup_{t \in T'} \{\|A^u\| + \|g_t\| + \|f_t^{u'}\| + 1\}$ and $\|A^u\|$ denotes the operator norm of A^u . Hence we have $K < \infty$, since A^u is a continuous operator and g_t and $f_t^{u'}$ are continuous in $t \in T'$. By the assumptions on g_t and k_t , Assumption (IV) and (4.2), Assumption (VII) turns out to be satisfied.

Now we define the operator $\mathcal{A}_s$ by

$$(4.3) \quad \mathcal{A}_s k(x) = \sup_{u \in U_M} \mathcal{A}_s^u k(x) \quad \text{for } (s, x) \in T \times R^d \quad \text{and } k \in D.$$

From Friedman [7, p. 117] we have

$$\mathcal{A}_s k(x) = \sup_{u \in U_M} [G^u k(x) - g_s(x)k(x) + k_s(x)].$$

Here G^u has been defined in Section 2. From the definition of G^u it is trivial that

$$(4.4) \quad \mathcal{A}_s k(x) = \sup_{\gamma \in \Gamma} \mathcal{A}_s^\gamma k(x) \quad \text{for } (s, x) \in T \times R^d \quad \text{and } k \in D.$$

LEMMA 4.2. *If the action space is compact, then $-\mathcal{A}_s$ is accretive in the sense of Pliska [15].*

PROOF. We have $\mathcal{A}_s k \in L$ for $s \in T$ and $k \in D$ since Γ is compact. Let $s \in T$ be fixed. We consider the operator $\bar{A}^\gamma k = A^\gamma k - g_s k$ for $k \in D$ and $\gamma \in \Gamma$, then we can construct the corresponding time-homogeneous semigroup in the same manner as Lemma 4.1. We apply Proposition 3.4(a) and Theorem 3.5 in [15], noting that in our case the state space need not be compact by Sato [17, §6.2]. Therefore $-\mathcal{A}_s$ is accretive.

PROPOSITION 4.1. *Suppose Assumption (III), (IV) be satisfied. Suppose Γ be compact. Let $\tau, \sigma \in T$ ($\tau < \sigma$) be fixed. If a control $u^* \in U$ satisfies $f_s^{u^*} \in D$ for all $s \in (\tau, \sigma)$ and u^* is optimal at every time $s \in [\tau, \sigma]$, then it holds that*

$$(4.5) \quad \mathcal{A}_s f_s^{u^*} = \frac{d}{ds} f_s^{u^*} \quad \text{for all } s \in (\tau, \sigma).$$

PROOF. For $t \in T$ define the control u_t^* by

$$u_t^*(s, y) = u^*(t + s, y) \quad \text{for } (s, y) \in R^+ \times R^d.$$

Then from Lemma 3.1 and the optimality of u^* we have

$$f_{t+s}^{u^*}(x) = S_{t+s, s}^{u^*} f_s^{u^*}(0, x) \leq S_{t+s, s}^{u^*} f_s^{u^*}(0, x)$$

for $(t, s, x) \in R^+ \times [\tau, \sigma] \times R^d$. Therefore we obtain $\frac{d}{ds} f_s^{u^*} \leq \mathcal{A}_s^{u^*} f_s^{u^*}$ for all $s \in (\tau, \sigma)$. On the other hand we have

$$\mathcal{A}_s^{u^*} f_s^{u^*} = G_0^{u^*} f_s^{u^*} - g_s f_s^{u^*} + k_s \leq \sup_{u \in U_M} \mathcal{A}_s^u f_s^{u^*}$$

for all $s \in (\tau, \sigma)$. These inequalities and (3.7) imply (4.5).

Next we introduce the Assumption (VIII) which is due to Pliska [15]. Let E be a fixed d -dimensional bounded domain with the smooth boundary ∂E . In the rest of this section we treat the case where controlled processes X^u have the state space E and the absorbing boundary ∂E . In this case we take $\tilde{L} = \{k \in L: \text{Supp}(k) \subset E \text{ and } k=0 \text{ on } \partial E\}$ and $\tilde{D} = \{k \in D: k, \frac{\partial}{\partial x_i} k, \frac{\partial^2}{\partial x_i \partial x_j} k \in \tilde{L} \text{ (} 1 \leq i, j \leq d \text{)}\}$ instead of L and D respectively. Then Theorem 3.1, 3.2, Lemmas 4.1, 4.2 and Proposition 4.1 hold in the same line as the case where state space is R^d .

ASSUMPTION (VIII). Let $s \in T$ be fixed. For each $g \in \tilde{L}$ and $\lambda > 0$ there exists an element $k \in \tilde{D}$ such that

$$\sup_{\gamma \in \Gamma} A^\gamma k - g_s k - \lambda k = g.$$

THEOREM 4.1. Suppose Assumptions (III), (IV) and (VIII) be satisfied. Suppose Γ be compact. Then the limit

$$S_{t,s}k = \lim_{n \rightarrow \infty} \prod_{i=1}^n J_{(t-s)/n} \left(s + i \left(\frac{t-s}{n} \right) \right) k$$

exists for $k \in \tilde{L}$ and $t, s \in T$ ($t \geq s$), where $J_\mu(s) = (I - \mu \mathcal{A}_s)^{-1}$ for $\mu > 0$ and $s \in T$. And $\{S_{t,s}\}_{t \geq s}$ is a nonautonomous semigroup on L which satisfies (i), (ii) and (iii):

(i) $\|S_{t,s}k_1 - S_{t,s}k_2\| \leq \|k_1 - k_2\|$ for $t, s \in T$ ($t \geq s$) and $k_1, k_2 \in L$.

(ii) $S_{t,s}k$ is strongly continuous in $\{(t, s) \in T \times T: t \geq s\}$ for $k \in L$.

(iii) Let $\tau, \sigma \in T$ ($\tau < \sigma$) be fixed. If there exists a solution of Cauchy problem

$$(4.6) \quad \begin{cases} \frac{d}{dt} \Phi(t) = \mathcal{A}_t \Phi(t) & \text{for all } t \in (\tau, \sigma) \\ \text{and } \Phi(\tau) = k \in \tilde{L}, \end{cases}$$

then $\Phi(t) = S_{t,\tau}k$ for all $t \in [\tau, \sigma]$.

PROOF. This theorem is an application of Crandall and Pazy [2]. It can be proved by the same manner as in Lemma 3.2 that the condition (C.2) in [2] is satisfied in our case. It is obvious that the conditions (A.1), (A.2) and (A.3) in [2] are satisfied, since $-\mathcal{A}_s$ is m -accretive.

5. Examples

First we define common notations throughout this section. Let $u \in U_1$ be fixed. We call $m^u = \{m_{t,r}^u\}_{(t,r) \in R^+ \times R^+}$ a multiplicative functional (abbreviated as M.F.) of $\psi \in L_+$, if

$$m_{t,r}^u = \exp\left(-\int_r^{t+r} \psi(X_s^u) ds\right) \quad \text{for } (t, r) \in R^+ \times R^+.$$

And we call $a^u = \{a_{t,r}^u\}_{(t,r) \in R^+ \times R^+}$ and additive functional (abbreviated as A.F.) of (m^u, ϕ) for $\phi \in L$ if

$$a_{t,r}^u = \int_0^t m_{s,r}^u \phi(X_{s+r}^u) ds \quad \text{for } (t, r) \in R^+ \times R^+.$$

Define $Q_t^u k(r, x) = E^{r,x}[m_{t,r}^u k(X_{t+r}^u)]$ for $(r, x) \in R^+ \times R^d$ and $k \in L$. Then the generator of $\{Q_t^u\}_{t \geq 0}$ at $t=r=0$ is $A_0^u - \psi I$ if $u \in U$. We can classify examples in two cases:

Case 1. Let m^u be a M.F. of $\psi \in L_+$ and A^u be an A.F. of (m^u, ϕ) ($\phi \in L$). Let α be a nonnegative function on R^+ which is twice continuously differentiable on $(0, \infty)$ and satisfies $\alpha(t) > 0$ and $\alpha'(t) > 0$ and $\alpha'(t) \geq 0$ for all $t \in (0, \infty)$. We take $B_{s,t} = 0$, $A_{t,s,r}^u = \alpha(t+s)^{-1} a_{t,r}^u$ and $M_{t,s,r}^u = \alpha(s)\alpha(t+s)^{-1} m_{t,r}^u$ for each $(t, s, r) \in R^+ \times T \times R^+$. Then we can easily check that this case satisfies conditions (c.i)~(c.vi) and (c.ix). We have $S_{t+s,s}^u k(r, x) = \alpha(t+s)^{-1} \left\{ \int_0^t Q_{r'}^u \phi(r, x) dr' + \alpha(s) Q_t^u k(r, x) \right\}$ for $(r, x, t, s) \in R^+ \times R^d \times T \times T$ and $k \in L$. Then Assumption (III) is satisfied. Now for each $u \in U$ we have the generator $\mathcal{A}_s^u$ in the form of (4.1) with $g_s = \psi + \alpha'(s)\alpha(s)^{-2}$ and $k_s = \alpha(s)^{-1}\phi$. Therefore Lemma 4.1 holds. Consequently it is sufficient to check conditions (c.vii), (c.viii), (c.x), (c.xi) and Assumption (IV).

Case 2. Take u , m^u , a^u and α in the same manner as Case 1. We take $B_{s,t} = \alpha(s)\alpha(t+s)^{-1}$, $A_{t,s,r}^u = v\alpha(t+s)^{-1} a_{t,r}^u$ and $M_{t,s,r}^u = \alpha(s)\alpha(t+s)^{-1} m_{t,r}^u$ for each $(t, s, r) \in R^+ \times T \times R^+$, where v is a fixed constant. This case also satisfies (c.i)~(c.vi). We have $S_{t+s,s}^u k(r, x) = \alpha(t+s)^{-1} \left\{ \alpha(t) E^{r,x}[C_{t,r}^u] + v\alpha \int_0^t Q_{r'}^u \phi(r, x) dr' + \alpha(s) Q_t^u k(r, x) \right\}$. If $f_s^u \in L$ for $(s, u) \in T \times U$, then Assumption (III) is also satisfied. And if $\lim_{t \downarrow 0} \alpha(t) f_t^u = 0$ and the strongly right derivative $\frac{d^+}{dt} \alpha(t) f_t^u|_{t=0}$ at $t=0$ is independent of u and belongs to L , then the generators $\mathcal{A}_s^u$ are expressed in the form of (4.1) with $g_s = \psi + \alpha'(s)\alpha(s)^{-2}$ and $k_s = \frac{d^+}{dt} \alpha(t) f_t^u|_{t=0} + v\alpha(s)^{-1}\phi$.

Therefore Lemma 4.1 holds and the condition (c.ix) is satisfied. Consequently if we have (5.1):

$$(5.1) \quad \begin{cases} f_s^u \in L, \lim_{t \downarrow 0} \alpha(t) f_t^u = 0 \quad \text{and} \quad \frac{d^+}{dt} \alpha(t) f_t^u|_{t=0} \\ \text{is independent of } u \text{ and belongs to } L \text{ for all} \\ (s, u) \in T \times U. \end{cases}$$

then the facts to be checked are same as those in Case 1.

REMARK 5.1 (i) As for the discussion about Assumption (VIII) we may refer to Pliska [15].

(ii) We note that if $c_{t,r}^u$ is expressed in a finite linear combination of $a_{t,r}^u$, $m_{t,r}^u$ and $k(X_{t,r}^u)$ for $(t, r, u) \in R^+ \times R^+ \times U_1$ and $k \in L$, then the conditions (c.vii), (c.x) and (c.xi) are satisfied. Examples 1 ~ 8, 13 and 14 are contained in this case.

(iii) In each example we can easily check the condition (c.viii) by using explicit solutions. Assumption (IV) is investigated in each example. We also investigate, in each example, the condition (5.2):

$$(5.2) \quad f_t^u \in D \quad \text{for all } (t, u) \in T \times U.$$

This condition was assumed in Theorems 3.1, 3.2 and Proposition 4.1.

LEMMA 5.1. Let $u \in U$, $\Phi \in L_+$, $\phi \in D$ and $k \in D_s$ be fixed. We define the function $R_t^u k(x)$ by

$$R_t^u k(x) = \int_0^t Q_r^u \phi(0, x) dr + Q_t^u k(0, x)$$

for $(t, x) \in R^+ \times R^d$. Then $R_t^u k$ has properties (i) ~ (iii):

- (i) $R_t^u k \in D$ for all $t \in R^+$
- (ii) $\frac{d}{dt} R_t^u k(\cdot) = Q_t^u \phi(0, \cdot) + Q_t^u G_t^u k(0, \cdot) \in L$ for all $t \in T$,
- (iii) $\frac{d}{dt} R_t^u k$ is a strongly continuous function on $t \in T$ which is of bounded variation on each finite closed subinterval of T .

PROOF. (i) follows from Theorem 2.9.10 in Krylov [12]. (ii) is trivial. (iii) We have $\frac{d}{dt} Q_t^u \phi(0, \cdot) = Q_t^u G_t^u \phi(0, \cdot)$. Therefore $Q_t^u \phi(0, \cdot)$ is a strongly continuous function of $t \in T$ which is of bounded variation on each finite closed subinterval of T . And we have

$$\begin{aligned} & \|Q_t^u G_t^u k(0, \cdot) - Q_s^u G_s^u k(0, \cdot)\| \\ & \leq \|Q_t^u G_t^u k(0, \cdot) - Q_s^u G_t^u k(0, \cdot)\| + \|Q_s^u G_t^u k(0, \cdot) - Q_s^u G_s^u k(0, \cdot)\| \end{aligned}$$

for $t, s \in T$. Since $G_t^u k \in D$ for each $t \in T$, $Q_s^u G_t^u k(0, \cdot)$ is strongly continuous of $s \in T$ and of bounded variation on each finite closed subinterval of T for each fixed $t \in T$. On the other hand we have $\|Q_s^u G_t^u k(0, \cdot) - Q_s^u G_s^u k(0, \cdot)\| \leq \|G_t^u k - G_s^u k\|$ for $t, s \in T$. $G^u k$ is a strongly continuous function of $s \in T$ and of bounded variation on each finite closed subinterval of T from Lipschitz conditions of controls $u \in U$ and coefficients of G_t^u . This completes the proof.

EXAMPLE 1. Let m^u be a M.F. of $\psi \in L_+$. Let $g \in D_s$ be fixed. We consider

$$(5.2) \quad C_{t+s,r}^u = m_{t,r}^u C_{s,t+r}^u \quad \text{a. s. } P.$$

Then $C_{t,r}^u = m_{t,r}^u g(X_{t+r}^u)$ is a solution of (5.2). And $S_{t+s,s}^u k = Q_t^u k$ for all $(t, s) \in R^+ \times T$ and $k \in L$. This example implies reward systems with discounted rate ψ . This example is contained in Case 1. Now we have $f_t^u = Q_t^u g(0, \cdot) \in L$. From Remark 5.1(ii) and Lemma 5.1, conditions (c.i)~(c.xi), Assumptions (I)~(VII) and (5.2) are satisfied.

EXAMPLE 2. Let m^u be a M.F. of $\psi \in L_+$. Let a^u be an A.F. of (m^u, ϕ) . ($\phi \in D$). Let $g \in D_s$ be fixed. We treat

$$(5.3) \quad C_{t+s,r}^u = a_{t,r}^u + m_{t,r}^u C_{s,t+r}^u \quad \text{a. s. } P.$$

Then $C_{t,r}^u = a_{t,r}^u + m_{t,r}^u g(X_{t+r}^u)$ is a solution of (5.3) and $S_{t+s,s}^u k = \int_0^t Q_r^u \phi dr + Q_t^u k$ for $k \in L$. The systems with discounted rate ψ are well-known in the theory of controlled Markov processes. For example, Pliska [15], Nisio [14], Krylov [12] and etc. have treated this case. This example is contained in Case 1. Now we have $f_t^u = \int_0^t Q_r^u \phi(0, \cdot) dr + Q_t^u g(0, \cdot) \in L$. From Remark 5.1 (ii) and Lemma 5.1, conditions (c.i)~(c.xi), Assumptions (I)~(VII) and (5.2) are satisfied.

EXAMPLE 3. Let m^u be a M.F. of $\psi \in L_+$. Let $g \in D_s$ be fixed.

$$(5.4) \quad C_{t+s,r}^u = \frac{s}{t+s} m_{t,r}^u C_{s,t+r}^u \quad \text{a. s. } P.$$

Then $C_{t,r}^u = t^{-1} m_{t,r}^u g(X_{t+r}^u)$ is a solution of (5.4) for $(t, r) \in T \times R^+$. This example implies reward systems with discounted rate ψ per unit time. This example is contained in Case 1. Now we have $f_t^u = t^{-1} Q_t^u g(0, \cdot) \in L$. Conditions (c.i)~

(c.xi) and Assumptions (I)~(VII) are satisfied and we have $f_s^u \in D$ for all $s \in T$ similarly.

EXAMPLE 4. Let m^u be a M.F. of $\psi \in L_+$. Let $g \in D_s$ be fixed.

$$(5.5) \quad C_{t+s,r}^u = \frac{tC_{t,r}^u + sm_{t,r}^u C_{s,t+r}^u}{t+s} \quad \text{a. s. } P.$$

Then $C_{t,r}^u = t^{-1} \int_0^t m_{s,r}^u g(X_{s+r}^u) ds$, $(t, r) \in T \times R^+$, is a solution of (5.5). And $S_{t+s,s}^u k(0, \cdot) = t(t+s)^{-1} f_t^u + s(t+s)^{-1} Q_t^u k(0, \cdot)$ for $(t, s) \in T \times T$ and $k \in L$. This example implies total reward systems with average return criteria per unit time. This case has been studied by Kushner [13], Kakumanu [10], Doshi [4] and others, in optimisation problems with asymptotic rewards at infinite time. Ross [16] studied an optimal stopping problem in this case. However we treat optimal control problems in this paper. This example is contained in Case 2. Now we put $f_t^u = t^{-1} \int_0^t Q_r^u g(0, \cdot) dr \in L$ for $(t, u) \in T \times U$. Since $\alpha(t) f_t^u = \int_0^t Q_r^u g(0, \cdot) dr \rightarrow 0$ as $t \rightarrow 0$ and $\frac{d^+}{dt} \alpha(t) f_t^u|_{t=0} = g \in L$, (5.1) is satisfied. From Remark 5.1(ii) and Lemma 5.1, conditions (c.i)~(c.vii), Assumptions (I)~(VII) and (5.2) are satisfied.

EXAMPLE 5. Let $\lambda > 0$ be fixed. Let a^u be an A.F. of $(\{e^{-\lambda t}\}, \phi)$ ($\phi \in D$). Let $g \in D_s$ be fixed.

$$(5.6) \quad C_{t+s,r}^u = \frac{a_{t,r}^u + (e^{-\lambda t} - e^{-\lambda(t+s)}) C_{s,t+r}^u}{1 - e^{-\lambda(t+s)}} \quad \text{a. s. } P.$$

Then $C_{t,r}^u = (1 - e^{-\lambda t})^{-1} (a_{t,r}^u + e^{-\lambda t} g(X_{t+r}^u))$ for $(t, r) \in T \times R_+$ is a solution of (5.6). And $S_{t+s,s}^u k(r, x) = (1 - e^{-\lambda(t+s)})^{-1} \cdot \left\{ \int_0^t e^{-\lambda r'} E^{r,x} [\phi(X_{r'+r}^u)] dr' + (e^{-\lambda t} - e^{-\lambda(t+s)}) E^{r,x} [k(X_{t+r}^u)] \right\}$ for $(r, x) \in R^+ \times R^d$, $(t, s) \in T \times T$, $k \in L$ and $u \in U_1$. This example implies a total reward system with discounted return in gambling theory. Relating to this example, we may refer to Ross [16] in which stopping problems were treated. This example is contained Case 1. Now we have $f_t^u = (1 - e^{-\lambda t})^{-1} \left\{ \int_0^t e^{-\lambda r} P_r^u \phi dr + e^{-\lambda t} P_t^u g \right\} \in L$ for $t \in T$ and $u \in U$. From Remark 5.1(ii) and Lemma 5.1, conditions (c.i)~(c.xi), Assumptions (I)~(VII) and (5.2) are satisfied.

EXAMPLE 6. Let $\lambda > 0$ and $g \in D_s$ be fixed.

$$(5.7) \quad C_{t+s,r}^u = \frac{(1-e^{-\lambda t})C_{t,r}^u + (e^{-\lambda t} - e^{-(t+s)})C_{s,t+r}^u}{1-e^{-\lambda(t+s)}} \quad \text{a. s. } P.$$

Then $C_{t,r}^u = (1-e^{-\lambda t})^{-1} \int_0^t e^{-\lambda s} g(X_{s+r}^u) ds$ for $(t, r) \in T \times R^+$ is a solution of (5.7). This example bears a resemblance to Example 5 but is contained in Case 2. Now we have $f_t^u = (1-e^{-\lambda t})^{-1} \int_0^t e^{-\lambda s} P_s^u g ds \in L$ for $t \in T$ and $u \in U$. Since $\alpha(t)f_t^u = \int_0^t e^{-\lambda s} P_s^u g ds \rightarrow 0$ as $t \rightarrow 0$ and $\frac{d^+}{dt} \alpha(t)f_t^u|_{t=0} = g \in L$, (5.1) is satisfied. Moreover from Remark 5.1(ii) and Lemma 5.1, conditions (c.i)~(c.xi), Assumptions (I)~(VII) and (5.2)

EXAMPLE 7. Let $\lambda > 0$ be fixed. Let m^u be a M.F. of $\psi \in L_+$. Let a^u be an A.F. of (m^u, ϕ) ($\phi \in D$), Let $g \in D_s$ be fixed. We treat

$$(5.8) \quad C_{t+s,r}^u = \frac{a_{t,r}^u + \lambda s m_{t,r}^u C_{s,t+r}^u}{\lambda(t+s)} \quad \text{a. s. } P.$$

Then $C_{t,r}^u = (\lambda t)^{-1} \{a_{t,r}^u + m_{t,r}^u g(X_{t+r}^u)\}$ for $(t, r) \in T \times R^+$ is a solution of (5.8). This example represents a total reward system with average return criterion in gambling theory. Ross [16] has studied an optimal stopping problem of this case with $\lambda = 1$. This example is contained in Case 1. From Remark 5.1(ii) and Lemma 5.1 conditions (c.i)~(c.xi), Assumptions (I)~(VII) and (5.2) are satisfied.

EXAMPLE 8. Let $\lambda > 0$ and $g \in D_s$ be fixed.

$$(5.9) \quad \begin{cases} C_{t+s,r}^u = e^{-\lambda s} C_{t,r}^u + e^{-\lambda t} C_{s,t+r}^u \\ \text{and } C_{0,r}^u = 0 \end{cases} \quad \text{a. s. } P.$$

Then $C_{t,r}^u = e^{-\lambda t} \int_0^t g(X_{s+r}^u) ds$ for $(t, r) \in R^+ \times R^+$ is a solution of (5.9). This example implies a discounted total reward system and is contained in Case 2. Now we have $f_t^u = e^{-\lambda t} \int_0^t P_s^u g ds \in L$. Since $\alpha(t)f_t^u = \int_0^t P_s^u g ds \rightarrow 0$ as $t \rightarrow 0$ and $\frac{d^+}{dt} \alpha(t)f_t^u = g \in L$, (5.1) is satisfied. From Remark 5.1(ii) and Lemma 5.1 conditions (c.i)~(c.xi), Assumptions (I)~(VII) and (5.2) are satisfied.

EXAMPLE 9. Let m^u be a M.F. of $\psi \in L_+$. Let a^u be an A.F. of (m^u, g) for fixed $g \in D_s$.

$$(5.10) \quad C_{t+s,r}^u = C_{t,r}^u + s a_{t,r}^u + m_{t,r}^u C_{s,t+r}^u \quad \text{a. s. } P.$$

Then $C_{t,r}^u = \int_0^t a_{s,r}^u ds$ for $(t, r) \in R^+ \times R^+$ is a solution of (5.10). Now we have $f_t^u = \int_0^t \int_0^r Q_s^u g ds dr \in L$. This example is contained in Case 2. We can easily check conditions (c.vii), (c.x), (c.xi) and (5.2), by using Remark 5.1(ii) and Lemma 5.1. Since $\alpha(t)f_t^u = \int_0^t \int_0^r Q_s^u g ds dr \rightarrow 0$ as $t \rightarrow 0$ and $\frac{d}{dt} \alpha(t)f_t^u = 0$, (5.1) is also satisfied. Consequently conditions (c.i)~(c.xi), Assumptions (I)~(VII) and (5.2) are satisfied.

EXAMPLE 10. Let m^u be a M.F. of $\psi \in L_+$. Let a^u be an A.F. of (m^u, g) ($g \in D$).

$$(5.11) \quad C_{t+s,r}^u = \frac{tC_{t,r}^u + sa_{t,r}^u + sm_{t,r}^u C_{s,t+r}^u}{t+s} \quad \text{a. s. } P.$$

Then $C_{t,r}^u = \frac{1}{t} \int_0^t a_{s,r}^u ds$ for $(t, r) \in T \times R^+$ is a solution of (5.11). We can check conditions and assumptions in a manner similar to Example 9.

EXAMPLE 11. Let $\lambda > 0$ be fixed. Take m^u and a^u in the same manner as Example 10.

$$(5.12) \quad C_{t+s,r}^u = e^{-\lambda s} C_{t,r}^u + s e^{-\lambda(t+s)} a_{t,r}^u + e^{-\lambda t} m_{t,r}^u C_{s,t+r}^u \quad \text{a. s. } P.$$

Then $C_{t,r}^u = e^{-\lambda t} \int_0^t a_{s,r}^u ds$ for $(t, r) \in R^+ \times R^+$ is a solution of (5.12). We can check conditions and assumptions in a manner similar to Example 9.

EXAMPLE 12. Let $\lambda > 0$ be fixed. Take m^u and a^u in the same manner as Example 10.

$$(5.13) \quad C_{t+s,r}^u = \frac{(1 - e^{-\lambda t}) C_{t,r}^u + s a_{t,r}^u + (1 - e^{-\lambda s}) m_{t,r}^u C_{s,t+r}^u}{1 - e^{-\lambda(t+s)}} \quad \text{a. s. } P.$$

Then $C_{t,r}^u = (1 - e^{-\lambda t})^{-1} \int_0^t a_{s,r}^u ds$ for $(t, r) \in T \times R^+$ is a solution of (5.13). We can check conditions and assumptions in a manner similar to Example 9.

EXAMPLE 13. Let m^u be a M.F. of $\psi \in L_+$. Let $\lambda > 0$ and $g \in D_s$ be fixed. We consider

$$(5.14) \quad C_{t+s,r}^u = \frac{1 - e^{-\lambda s}}{1 - e^{-\lambda(t+s)}} m_{t,r}^u C_{s,t+r}^u \quad \text{a. s. } P.$$

Then $C_{t,r}^u = (1 - e^{-\lambda t})^{-1} m_{t,r}^u g(X_{t+r}^u)$ for $(t, r) \in T \times R^+$ is a solution of (5.14).

This example resembles Example 5.

EXAMPLE 14. Take a function α in the same manner as Case 1. Let m^u be a M.F. of $\psi \in L_+$. Let $g \in D_s$ be fixed. We consider

$$(5.15) \quad C_{t+s,r}^u = \frac{\alpha(s)}{\alpha(t+s)} m_{t,r}^u C_{s,t+r}^u \quad \text{a. s. } P.$$

Then $C_{t,r}^u = \alpha(t)^{-1} m_{t,r}^u g(X_{t+r}^u)$ for $(t, r) \in T \times R^+$ is a solution of (5.15). This example is a generalization of Example 13.

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