ON THE STABLE HUREWICZ IMAGE OF SOME STUNTED PROJECTIVE SPACES, III

To the memory of Professor Shichiro Oka

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Let HP^n (resp. QP^n) ($0 \leq n \leq \infty$) be the $4n$ -dim (resp. $(4n-1)$ -dim) quaternionic projective space (resp. quasi-projective space), and $HP_k^n = HP^n/HP^{k-1}$ (resp. $QP_k^n = QP^n/QP^{k-1}$). We shall use the notation KP^n to denote HP^n or ΣQP^n , where ΣX is the suspension of X ($X = QP^n$), and $KP_k^n = KP^n/KP^{k-1}$. Then the stable order of the attaching map of the top cell of KP_k^n is equal to the order of the cokernel of the stable Hurewicz homomorphism

$$h: \pi_{4n}^S(KP_k^n) \longrightarrow H_{4n}(KP_k^n).$$

For $KP_k^n = HP_1^n = HP^n$, its value is determined by D. M. Segal [8] and McGibbon [6], and for $KP_k^n = QP_1^n = QP^n$ its value is well known (cf. [11], [7]). In [3] and [4], we determined its value for $KP_k^n = HP_2^n$, QP_2^n by using some stable map $\Sigma^8 KP^\infty \rightarrow KP^\infty$ constructed in [7]. In this paper we continue our methods to determine the stable order of the attaching map of the top cell of KP_3^{2n+1} .

In [3; Lemma 9] and [4; Lemma 2.5], we calculated the e -invariants of some stable homotopy classes, which was the important step to prove the key theorem. In this paper we also need such a calculation of e -invariants, and so in §1 we prepare them and the related calculations in KO -theory. In §2 we show the main theorem (Theorem 2.1), and in §3 we give some applications to the stable James numbers.

Since we aim to obtain the 2-primary results in our main theorem and its applications, we identify the spaces and groups with their 2-localized spaces and their 2-components respectively. Also, throughout this paper, we make free use of notations in [3] and [4].

§1. Preliminaries

Let ζ be a canonical quaternionic line bundle over HP^∞ or its restriction over HP^n ($n \geq 1$), and $\zeta = \xi \otimes_H \xi^*$ be the tensor product over the quaternion of ξ and its conjugate bundle ξ^* . Then ζ has a non zero cross section and so can be written as $\zeta = \zeta' \oplus 1$, where 1 denote the trivial 1-dim vector bundle. The $(4n-1)$ -dim quaternionic quasi-projective space QP^n is the Thom space of ζ' over HP^{n-1} , and thus we have $\Sigma QP^n = (HP^{n-1})^\zeta$. Because ζ is KSp -orientable, there exist the Thom isomorphisms

$$V: H^*(HP^{n-1}; Z) \longrightarrow \tilde{H}^{*+3}(QP^n; Z) \quad \text{and}$$

$$U: KO^*(HP^{n-1}) \longrightarrow \widetilde{KO}^{*+3}(QP^n)$$

in the integral cohomology groups and the KO -cohomology groups respectively.

Let $x \in H^4(HP^n; Z)$ and $X = \xi - 1 \in \widetilde{KSp}(HP^n) = \widetilde{KO}^4(HP^n)$ be the respective Euler classes of ξ , and $V = V(1)$ and $U = U(1)$ be the respective Thom classes. Then the unreduced integral cohomology ring (resp. KO -cohomology ring) of HP^n is a truncated polynomial ring with generator x (resp. X) over Z (resp. $\pi_*(KO)$) and the relation $x^{n+1} = 0$ (resp. $X^{n+1} = 0$), and the reduced integral cohomology group (resp. KO -cohomology group) of QP^n is a free $H^*(HP^{n-1}; Z)$ -module (resp. $KO^*(HP^{n-1})$ -module) with one generator V (resp. U). That is,

$$H^*(HP^n; Z) \cong Z[x]/(x^{n+1}), \quad \tilde{H}^*(QP^n; Z) \cong Z\{V, Vx, \dots, Vx^{n-1}\},$$

$$KO_*(HP^n) \cong \pi_*(KO)[X]/(X^{n+1}), \quad \widetilde{KO}^*(QP^n) \cong \pi_*(KO)\{U, UX, \dots, UX^{n-1}\}.$$

Let $ph = ch \circ c: KO^*(\) \rightarrow K^*(\) \rightarrow H^*(\ ; Q)$ be the Pontrjagin character. Then, with respect to the values of ph on the elements of $\widetilde{KO}^*(KP^n)$ ($KP^n = HP^n$ or ΣQP^{n-1}), we have the following lemma, where $N(m, s)$ ($m, s \geq 1$) is the following rational number:

$$N(m, s) = (2/(2m)!) \sum_{j=0}^s (-1)^j \binom{2s}{j} (s-j)^{2m}.$$

LEMMA 1.1. (i) $ph(X^s) = \sum_{m \geq 1} N(m, s) x^m$ for $s \geq 1$.

(ii) $ph(UX^s) = \sum_{m \geq 0} \frac{m+1}{s+1} N(m+1, s+1) V x^m$ for $s \geq 0$.

PROOF. (i) follows from [3; (4.4), (4.5)] and an elementary arithmetic. Following the notations in [1], $sh(\alpha)$ denotes the characteristic class for $SO(2n)$ -bundle α , which corresponds to the element

$$\prod_{1 \leq r \leq n} \frac{e^{\frac{1}{2}x_r} - e^{-\frac{1}{2}x_r}}{x_r}$$

in $H^*(BT; \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_n]$ through $(Bi)^*: H^*(BSO(2n); \mathbb{Q}) \rightarrow H^*(BT; \mathbb{Q})$, where T is the maximal torus in $U(n)$ and Bi is the composition $BT \rightarrow BU(n) \rightarrow BSO(2n)$ of the canonical maps. Then, by [1; Theorem (5.1)], we have $V^{-1}(ph(U)) = sh(\zeta)$ for the preceding bundle ζ over HP^{n-1} . It is easy to see that the total real Pontrjagin class of ζ is $1 + 4x$. Thus using these formulas we have

$$ph(U) = \sum_{m \geq 0} (1/(2m+1)!) Vx^m.$$

By [1; Corollary (5.4)], $ph(UX^s) = ph(U) \cdot ph(X^s)$. But, the coefficient of Vx^m in the expansion of $ph(U)$ is equal to the one of t^{2m+1} in the expansion of $\frac{1}{2} \frac{d}{dt} (e^t + e^{-t})$, and similarly, $N(m, s)$ is equal to the coefficient of t^{2m} in the expansion of $(e^t + e^{-t} - 2)^s$. Therefore the coefficient of Vx^m in the expansion of $ph(UX^s)$ is equal to the one of the t^{2m+1} in the expansion of $\frac{1}{2s+2} \frac{d}{dt} ((e^t + e^{-t} - 2)^{s+1})$, and we have (ii). Q. E. D.

Let W be a finite CW -complex with $\dim W \leq 4k - 2$ ($k \geq 1$). For any element $h \in \pi_{4k-1}^S(W)$, we can consider the cofiber sequence $S^{4k-1} \xrightarrow{h} W \xrightarrow{i} C(h) \xrightarrow{p} S^{4k}$, where $C(h)$ is the mapping cone of h . Then we have the following commutative diagram:

$$\begin{array}{ccccccc} \widetilde{KO}^{4i}(S^{4k}) & \xrightarrow{p^*} & \widetilde{KO}^{4i}(C(h)) & \xrightarrow{i^*} & \widetilde{KO}^{4i}(W) & \longrightarrow & 0 \\ \downarrow ph_{4k} & & \downarrow ph_{4k} & & \downarrow ph_{4k} & & \\ 0 \longrightarrow & H^{4k}(S^{4k}; \mathbb{Q}) & \xrightarrow{p^*} & H^{4k}(C(h); \mathbb{Q}) & \xrightarrow{i^*} & H^{4k}(W; \mathbb{Q}), \end{array}$$

where ph_{4k} denotes the $4k$ -dimensional component of the Pontrjagin character. According to [2] and [11], the e -invariant $e(h)$ of h is a functional Pontrjagin character

$$(p^*)^{-1}(ph_{4k})(i^*)^{-1}: \widetilde{KO}^{4i}(W) \longrightarrow H^{4k}(S^{4k}; \mathbb{Q})/\text{Im } ph_{4k}.$$

Let $a(j) = 1$ if j is even and $= 2$ if j is odd. Then the image of $ph_{4k}: \widetilde{KO}^{4i}(S^{4k}) \rightarrow$

$H^{4k}(S^{4k}; Q) = Q$ is the subgroup $a(k-i)Z$, and thus the range of $e(h)$ is identified with $Q/a(k-i)Z \cong Q/Z$. We refer the fundamental properties of e -invariants to [11; Lemmas 2.1 and 2.2] in which $e(h)$ is denoted by $e_R(h)$. Especially, when $W = S^{4i}$ ($0 \leq i \leq k-1$), $e(h)$ is the Adams e'_R -invariant of $h \in \pi_{4(k-i)-1}^S(S^0)$.

Let $\phi_n: S^{4n-1} \rightarrow KP^{n-1}$ be the attaching map of the top cell of KP^n , where $KP^m = HP^m$ or ΣQP^m ($m = n-1, n$). Then, we have $C(\phi_n) = KP^n$, and so, using Lemma 1.1, we have the following proposition, where the values of e -invariants are in Q/Z .

PROPOSITION 1.2. (i) For $1 \leq s \leq n-1$,

$$e(\phi_n)(X^s) = (1/a(n-s))N(n, s).$$

(ii) For $0 \leq s \leq n-2$.

$$e(\phi_n)(UX^s) = \frac{n}{(s+1)a(n-s+1)} N(n, s+1).$$

where $N(n, l)$ ($l = s, s+1$) are the rational numbers in Lemma 1.1.

REMARK. G. Walker [11] has already given the above calculations by using the KO -groups of complex projective spaces.

Let $\beta_i \in H_{4i}(KP^n)$ ($1 \leq i \leq n$) be the dual of $x^i \in H^{4i}(HP^n)$ if $KP^n = HP^n$ and the dual of $Vx^{i-1} \in H^{4i}(\Sigma QP^n)$ if $KP^n = \Sigma QP^n$. In [3] we denoted β_i by the same letter for the case $KP^n = HP^n$, and in [4] we denoted it by the letter γ_i for the case $KP^n = \Sigma QP^n$.

Let $f: \Sigma^8 KP^\infty \rightarrow KP^\infty$ be the stable map $f(2,1)$ (resp. $g(2,1)$) defined in [7] for the case $KP^\infty = HP^\infty$ (resp. ΣQP^∞). Then, by [7; Theorems 1 and 3],

$$(1.3) \quad f^*(\beta_n) = \begin{cases} ((2n+4)!/(2n)!) \beta_{n+2} & \text{if } KP^\infty = HP^\infty, \\ ((2n+3)!/(2n-1)!) \beta_{n+2} & \text{if } KP^\infty = \Sigma QP^\infty. \end{cases}$$

By the cellular approximation, we have the restriction

$$f: \Sigma^8 KP^n \longrightarrow KP^{n+2} \quad \text{and} \quad f: \Sigma^8 KP_k^n \longrightarrow KP_{k+2}^{n+2} \quad (k \leq n).$$

Then there is the following stable diagram which commutes up to sign:

$$(1.4) \quad \begin{array}{ccccccc} S^{4k+7} & \xrightarrow{\phi_k} & \Sigma^8 KP^{k-1} & \xrightarrow{i_{k-1,k}} & \Sigma^8 KP^k & \xrightarrow{p_{1,k}} & S^{4k+8} \\ \downarrow d(k) & & \downarrow f & & \downarrow f & & \downarrow d(k) \\ S^{4k+7} & \xrightarrow{\phi_{k+2}} & KP^{k+1} & \xrightarrow{i_{k+1,k+2}} & KP^{k+2} & \xrightarrow{p_{1,k+2}} & S^{4k+8}. \end{array}$$

where $d(k)$ is a map of degree $(2k+4)!/(2k)!$ (resp. $(2k+3)!/(2k-1)!$) for the case $KP^i = HP^i$ (resp. ΣQP^i).

Let $s(k)$ be the stable order of the attaching map $\phi_k: S^{4k-1} \rightarrow KP^{k-1}$ of the top cell in KP^k . Then $s(k) = (2k)!/a(k)$ if $KP^{k-1} = HP^{k-1}$ and $a(k-1)((2k-1)!)$ if $KP^{k-1} = \Sigma QP^{k-1}$ (cf. [7]), where $a(j) = 1$ if j is even and $= 2$ if j is odd. Let $M_k = S^0 \cup_{s(k)} e^1$ be the mod $s(k)$ Moore spectrum, and $i_0: S^0 \rightarrow M_k$ and $p_1: M_k \rightarrow S^1$ be the inclusion map and the collapsing map respectively. Then there are an extension $\bar{\phi}_k: \Sigma^{4k-1} M_k \rightarrow KP^{k-1}$ of ϕ_k and a map $x(k): S^{4k} \rightarrow KP^k$ which enable the following diagram to commute up to sign:

$$(1.5) \quad \begin{array}{ccccccc} S^{4k-1} & \xrightarrow{i_0} & \Sigma^{4k-1} M_k & \xrightarrow{p_1} & S^{4k} & \xrightarrow{s(k)} & S^{4k} \\ \parallel & & \downarrow \bar{\phi}_k & & \downarrow x(k) & & \parallel \\ S^{4k-1} & \xrightarrow{\phi_k} & KP^{k-1} & \xrightarrow{i_{k-1,k}} & KP^k & \xrightarrow{p_{1,k}} & S^{4k} \end{array}$$

Then we have $x(k)^*(1_{4k}) = s(k)\beta_k$ for a generator $1_{4k} \in H_{4k}(S^{4k}; \mathbb{Z})$. Thus $x(k)$ is a generator of a free part of $\pi_{4k}^S(KP^k)$. Let $g_i \in \widetilde{KO}^{4i}(S^0) = \mathbb{Z}$ be a generator. Then, concerning the homomorphism $x(k)^*: \widetilde{KO}^{4i}(KP^k) \rightarrow \widetilde{KO}^{4i}(S^{4k})$, we have the following:

LEMMA 1.6. (i) For $KP^k = HP^k$ and $1 \leq s \leq k$,

$$x(k)^*(X^s) = (a(k-1)/a(k-s)) \left(\sum_{i=0}^{s-1} (-1)^i \binom{2s}{i} (s-i)^{2k} \right) g_{k-s}.$$

(ii) For $KP^k = \Sigma QP^k$ and $0 \leq s \leq k-1$,

$$x(k)^*(UX^s) = \frac{(ak-1)}{(s+1)a(k-s+1)} \left(\sum_{i=0}^s (-1)^i \binom{2s+2}{i} (s+1-i)^{2k} \right) g_{k-s-1}.$$

PROOF. We set $x(k)^*(Y) = dg_{k-s}$ for some integer d , where $Y = X^s$ or UX^{s-1} . Applying the Pontrjagin character, we have $x(k)^*ph(Y) = da(k-s)$. Thus, by Lemma 1.1, we have the desired equalities. Q. E. D.

REMARK. In the preceding lemma, the necessary property for $x(k)$ is only that it is a generator of a free part of $\pi_{4k}^S(KP^k)$. Hence the formulas for $x(k)$ in the lemma are also valid for any generator of a free part of $\pi_{4k}^S(KP^k)$.

§2. The main theorem

In this section, the Moore spectrum M is always the mod 8 Moore spectrum,

that is $M = S^0 \cup_8 e^1$. Since $\langle 8, 2j_7, 8 \rangle = 0$ in the 2-component of the stable stem, there is a map $A: \Sigma^8 M \rightarrow M$ which satisfies $p_1 A i_0 = 2j_7$, where j_{4i-1} denotes a generator of the image of the stable J -homomorphism $\pi_{4i-1}(SO) \rightarrow \pi_{4i-1}^S(S^0)$. Then we define an element $\omega(m) \in \pi_{8m+3}^S(KP^2)$ as follows:

$$\omega(m) = \bar{\phi}_3 \circ A^{m-1} \circ i_0: S^{8m+3} \longrightarrow \Sigma^{8m+3} M \longrightarrow \Sigma^{11} M \longrightarrow KP^2.$$

Let $h: \pi_{8n+4}^S(KP_3^{2n+1}) \rightarrow H_{8n+4}(KP_3^{2n+1})$ be the stable Hurewicz homomorphism. Then we have

THEOREM 2.1. *For $n \geq 1$, there is an element $X_n \in \pi_{8n+4}^S(KP_3^{2n+1})$ which satisfies the following:*

$$(i) \quad h(X_n) = \begin{cases} ((4n+2)!/6!)\beta_n & \text{if } KP_3^{2n+1} = HP_3^{2n+1}, \\ ((4n+1)!/5!)\beta_n & \text{if } KP_3^{2n+1} = \Sigma QP_3^{2n+1}. \end{cases}$$

(ii) $\partial_2 X_n = u \cdot \omega(n)$ in $\pi_{8n+3}^S(KP^2)$ for $\partial_2: KP_3^{2n+1} \rightarrow \Sigma KP^2$, where u is an odd integer.

REMARK. Let M be the mod 360 (resp. 120) Moore spectrum and take $A: \Sigma^8 M \rightarrow M$ to satisfy $p_1 A i_0 = 2j_7$ (resp. $10j_7$) for the case $KP_3^{2n+1} = HP_3^{2n+1}$ (resp. ΣQP_3^{2n+1}) instead of the above definitions of M and A , and define $\omega(m)$ similarly. Then, by taking $X'_n = 3^{n-1}X_n$ (resp. $5^{n-1}X_n$) in the above theorem, we can show that $\partial_2 X'_n = \omega(n)$. But we are only discussing the 2-primary result, so it is not significant to discuss such detailed values concerning odd primary data. Thus, in the proof of the theorem, we shall often omit the odd multiple u in the equality $C = u \cdot D$ and simply denote it by $C = D$ to make expressions plain. Then we can easily check that it will be sufficient to prove the theorem and cause no confusion.

To prove the theorem, we consider the following diagram:

(2.2)

$$\begin{array}{ccccc}
& S^{8n+12} & \xrightarrow{f} & S^{8n+12} & \\
& \uparrow p_{3,2n+1} & & \uparrow p_{5,2n+3} & \\
& & KP_3^{2n+3} & & \\
& & \downarrow p_{3,5} & & \\
S^{8n+12} & \xrightarrow{X_n} & \Sigma^8 KP_3^{2n+1} & \xrightarrow{f} & KP_5^{2n+3} \\
& \searrow \omega(n) & \downarrow \partial_2 & \downarrow \partial_2 & \downarrow \partial_4 \\
& & \Sigma^9 KP^2 & \xrightarrow{f} & \Sigma KP^4 \\
& \nearrow \bar{\phi}_3 & & & \nearrow p_{1,3} \\
& & \Sigma^9 KP^2 & & \Sigma KP^4 \\
& & \downarrow & & \downarrow \\
& & \Sigma^{12} M & \xrightarrow{\bar{\phi}_3} & \Sigma KP^2 \\
& \nearrow A^{n-1}i_0 & & & \nearrow i_{2,4} \\
& & \Sigma^{20} M & \xrightarrow{A} & \Sigma^{12} M
\end{array}$$

By the cellular approximation, the central squares are homotopy commutative, that is, $p_{5,2n+3}f = \bar{f}p_{3,2n+1}$ and $\partial_4 f = f\partial_2$. Also, $p_{1,3}\partial_4 = \partial'$ and $i_{2,4}\partial_2 = \partial_4 p_{3,5}$. We notice that the composition $\bar{\phi}_3 \circ A^{n-1}i_0$ is $\omega(n+1)$ by definition. We shall show the following

PROPOSITION 2.3. $f \circ \omega(n) = i_{2,4} \circ \omega(n+1)$.

Then we can establish the proof of Theorem 2.1 by induction on n as follows: When $n=1$, we may take X_1 to be the identity. Assume that we have an element $X_n \in \pi_{8n+4}^S(KP_3^{2n+1})$ satisfying the conditions (i) and (ii) in the theorem for n . Then, by chasing the diagram (2.2) and using (1.3) and Proposition 2.3, we can construct an element $X_{n+1} \in \pi_{8n+12}^S(KP_3^{2n+3})$ which satisfies the conditions (i) and (ii) for $n+1$. Its method is quite similar to the proofs of [3; Theorem 5] and [4; Theorem 3.2], so we omit the details. Thus the remainder of this section is devoted to the proof of Proposition 2.3.

We prepare the following diagram:

$$\begin{array}{ccccc}
& & S^{12} = KP_3^3 & \xrightarrow{i_{3,4}} & KP_3^4 \\
& & \uparrow & & \uparrow p_{1,3} \\
& & S^{20} & & \\
& \uparrow p_1 & & \nearrow \bar{\phi}_3 & \Sigma^8 KP^2 \xrightarrow{f} KP^4 \\
S^{8n+11} \xrightarrow{A^{n-1}i_0} \Sigma^{19}M & & & \nwarrow A & \uparrow i_{2,4} \\
& \uparrow i_0 & & \Sigma^{11}M \xrightarrow{\bar{\phi}_3} KP^2 \xrightarrow{p_{1,2}} S^8
\end{array}$$

LEMMA 2.4. *The composition $p_{1,3} \circ f \circ \bar{\phi}_3$ is null homotopic.*

PROOF. By the commutativity of the left square in (1.4), we have $f \circ \phi_3 = \phi_5 \circ d(3)$. Since KP_3^4 is a mapping cone of $3j_3$ (resp. $4j_3$) for $KP_3^4 = HP_3^4$ (resp. $KP_3^4 = \Sigma QP_3^4$), we have

$$p_{1,3} \circ \phi_5 \circ d(3) \in \begin{cases} i_{3,4} * \langle 3j_3, 4j_3, d(3) \rangle & \text{if } KP_3^4 = HP_3^4, \\ i_{3,4} * \langle 4j_3, 5j_3, d(3) \rangle & \text{if } KP_3^4 = \Sigma QP_3^4. \end{cases}$$

But by [2], both brackets contain only zeros. Therefore $p_{1,3} \circ f \circ \phi_3 = 0$. Thus there is a map $\alpha: S^{20} \rightarrow KP_3^4$ with $\alpha \circ p_1 = p_{1,3} \circ f \circ \bar{\phi}_3$. Since $\pi_4^S(S^0) = 0$, $\alpha = i_{3,4} \circ \alpha'$ for some map $\alpha': S^{20} \rightarrow S^{12} = KP_3^3$. Let $\eta \in \pi_1^S(S^0)$ be the generator, and $\bar{\eta}: KP_3^4 \rightarrow S^{11}$ be an extension of $\eta: S^{12} \rightarrow S^{11}$. Then $\eta \circ \alpha' \circ p_1 = \bar{\eta} \circ \alpha \circ p_1 = \bar{\eta} \circ p_{1,3} \circ f \circ \bar{\phi}_3$. But we see that $\bar{\eta} \circ p_{1,3} \circ f = 0$ by using (1.3) and $\pi_5^S(S^0) = 0$. Thus $\eta \circ \alpha' \circ p_1 = 0$. Because all the elements in $\pi_9^S(S^0)$ are of order at most 2 and M is the mod 8 Moore spectrum, we have $\eta \circ \alpha' = 0$. Moreover, because $\eta^*: \pi_8^S(S^0) \rightarrow \pi_9^S(S^0)$ is monomorphic, it follows that $\alpha' = 0$. Thus $\alpha = 0$ and $p_{1,3} \circ f \circ \bar{\phi}_3 = 0$. Q. E. D.

By Lemma 2.4, there is a map $\psi: \Sigma^{19}M \rightarrow KP^2$ which satisfies

$$i_{2,4} \circ \psi = f \circ \bar{\phi}_3.$$

PROPOSITION 2.5. *Let $Y = X, X^2, U$ or $UX \in \widetilde{KO}^*(KP^2)$. Then*

$$e(\psi \circ i_0)(Y) = e(\bar{\phi}_3 \circ A \circ i_0)(Y),$$

where $e(a)$ denotes the e -invariant of a defined in §1.

PROOF. Using (1.4), we have

$$e(\psi \circ i_0)(Y) = e(f \circ \phi_3)(Y) = d(3)e(\phi_5)(Y).$$

On the other hand, using (1.5), we have

$$e(\bar{\phi}_3 \circ A \circ i_0)(Y) = e(x(3) \circ 2j_7)(Y) = e(2j_7)(x(3)^*(Y)).$$

We remark that $e(j_7)(g_i) = e'_R(j_7) = 1/240$. Then, by Proposition 1.2, Lemma 1.6 and routine calculations, we have

$$d(3)e(\phi_5)(Y) = e(2j_7)(x(3)^*(Y)) \quad \text{up to odd multiples.}$$

Thus we have the desired result.

Q. E. D.

COROLLARY 2.6. $p_{1,2} \circ \psi = p_{1,2} \circ \bar{\phi}_3 \circ A$.

PROOF. $p_{1,2} \circ \psi \circ i_0$ and $p_{1,2} \circ \bar{\phi}_3 \circ A \circ i_0$ are the elements in $\pi_{11}^S(S^0)$. But all the elements in $\pi_{11}^S(S^0)$ are detected by their e'_R -invariants, since $\pi_{11}^S(S^0) = \text{Im } J_{11}$. By Proposition 2.5,

$$\begin{aligned} e'_R(p_{1,2} \circ \psi \circ i_0) &= e(\psi \circ i_0)(Y) = e(\bar{\phi}_3 \circ A \circ i_0)(Y) \\ &= e'_R(p_{1,2} \circ \bar{\phi}_3 \circ A \circ i_0) \end{aligned}$$

for $Y = X^2$ or UX . Thus $p_{1,2} \circ \psi \circ i_0 = p_{1,2} \circ \bar{\phi}_3 \circ A \circ i_0$. Since $\pi_{12}^S(S^0) = 0$, we have the desired result.

Q. E. D.

By this corollary, we have a map $g: \Sigma^{19}M \rightarrow S^4$ which satisfies

$$(2.7) \quad i_{1,2}g = \psi - \bar{\phi}_3 \circ A \quad \text{for } i_{1,2}: S^4 \longrightarrow KP^2.$$

PROPOSITION 2.8. $e'_R(g \circ i_0) = 0$.

PROOF. Let $Y = X$ or U . Then, by Proposition 2.5, $e(\psi \circ i_0)(Y) = e(\bar{\phi}_3 \circ A \circ i_0)(Y)$, and we have the desired result as follows:

$$\begin{aligned} e'_R(g \circ i_0) &= e(g \circ i_0)(\iota) = e(g \circ i_0)(i_{1,2}^*(Y)) = e(i_{1,2} \circ g \circ i_0)(Y) \\ &= e(\psi \circ i_0)(Y) - e(\bar{\phi}_3 \circ A \circ i_0)(Y) = 0. \end{aligned}$$

where $\iota \in \widetilde{KO}^4(S^4)$ is the generator.

Q. E. D.

By [10], $\pi_{15}^S(S^0) \cong Z_{32}\{j_{15}\} \oplus Z_2\{\eta\kappa\}$ for some element $\kappa \in \pi_{14}^S(S^0)$. Therefore, by Proposition 2.8, we have

$$(2.9) \quad g \circ i_0 = \eta\kappa \quad \text{or} \quad 0.$$

LEMMA 2.10. $(i_{1,3})^*(\eta\kappa)=0$ for $i_{1,3}: S^4 \rightarrow KP^3$.

PROOF. The cell structure of KP^3 is as follows:

$$HP^3 = S^4 \cup_{j_3} e^8 \cup_{2j_3} e^{12} \quad \text{and} \quad \Sigma QP^3 = S^4 \cup_{2j_3} e^8 \cup_{3j_3} e^{12}.$$

By [10], it holds $\langle j_3^2, 2, \varepsilon \rangle \ni \eta\kappa$, where ε is some class of $\pi_8^S(S^0)$. Since $\langle j_3, 2j_3, \varepsilon \rangle = \langle 2j_3, 3j_3, \varepsilon \rangle = \langle j_3^2, 2, \varepsilon \rangle$, we see that there is a coextension $\varepsilon': S^{20} \rightarrow KP_2^3$ of ε such that $\partial_1 \varepsilon' = \eta\kappa$ for $\partial_1: KP_2^3 \rightarrow KP^1 = S^4$. This yields the desired result. Q. E. D.

By (2.9) and Lemma 2.10, $i_{1,3} \circ g \circ i_0 = 0$. Therefore, we have a map $h: S^{20} \rightarrow KP^3$ which satisfies $h \circ p_1 = i_{1,3} \circ g$.

LEMMA 2.11. $2h=0$.

PROOF. We have $p_{1,2} \circ h \circ p_1 = p_{1,2} \circ i_{1,3} \circ g = 0$ for $p_{1,2}: KP^3 \rightarrow KP_2^3$. But $p_1^*: \pi_{20}^S(KP_2^3) \rightarrow \{\Sigma^{19}M, KP_2^3\}$ is monomorphic, because $\pi_{20}^S(KP_2^3) \cong \pi_{20}^S(S^{12}) \cong Z_2 \oplus Z_2$ (cf. [10]). Thus $p_{1,2}h=0$. Therefore, there is a map $h': S^{20} \rightarrow S^4 = KP^1$ with $i_{1,3}h' = h$. Since $\pi_{16}^S(S^0) \cong Z_2 \oplus Z_2$ (cf. [10]), we have $2h'=0$. Thus $2h=0$. Q. E. D.

LEMMA 2.12. $p_1 \circ A^l \circ i_0$ ($l \geq 0$) is divisible by 2.

PROOF. We prove by induction on l . For $l=0$, it is clear. When $l=1$, $p_1 \circ A \circ i_0 = 2j_7$ and the lemma holds. We assume that $p_1 \circ A^l \circ i_0 = 2z$ for some $z \in \pi_{8l-1}^S$ ($l \geq 1$). Then we have

$$p_1 \circ A^{l+1} \circ i_0 \in \langle 2j_7, 8, 2z \rangle \subseteq \langle 2, 8j_7, 2z \rangle$$

and

$$\langle 2, 8j_7, 2z \rangle \subseteq \langle 2, 8j_7, 2 \rangle z \ni 0$$

by [10]. The last relation implies that the bracket $\langle 2, 8j_7, 2z \rangle$ consists of elements all of which are divisible by 2. Thus by the first relation we have that $p_1 \circ A^{l+1} \circ i_0$ is divisible by 2, and we have the desired result. Q. E. D.

By Lemmas 2.11 and 2.12, we have $h \circ p_1 \circ A^l \circ i_0 = 0$ for $l \geq 0$.

Thus we can complete the proof of Proposition 2.3 as follows:

$$f \circ \omega(n) = f \circ \bar{\phi}_3 \circ A^{n-1} \circ i_0 = i_{2,4} \circ \psi \circ A^{n-1} \circ i_0$$

$$\begin{aligned}
&= i_{2,4} \circ \bar{\phi}_3 \circ A^n \circ i_0 + i_{1,4} \circ g \circ A^{n-1} \circ i_0 \\
&= i_{2,4} \circ \bar{\phi}_3 \circ A^n \circ i_0 + i_{3,4} \circ h \circ p_1 \circ A^{n-1} \circ i_0 \\
&= i_{2,4} \circ \omega(n+1).
\end{aligned}$$

§3. Applications

For the stable Hurewicz homomorphism $h: \pi_{4m}^S(KP_l^m) \rightarrow H_{4m}(KP_l^m; \mathbb{Z})$, we denote the order of the cokernel of h by $Sp^K(m, l)$, and for $KP_l^m = HP_l^m$ (resp. ΣQP_l^m) we denote it by $Sp^H(m, l)$ (resp. $Sp^Q(m, l)$). Then $Sp^K(m, l)$ is equal to the stable order of the attaching map of the top cell in KP_l^m . $Sp^H(m, 1)$ is given by D. M. Segal [9], and $Sp^Q(m, 1)$ is given by G. Walker [11], Mukai [8] and [7]. For an odd prime p , $v_p(Sp^K(m, 2))$ is determined by using the result of K. Knapp [5], and we determined $v_2(Sp^K(m, 2))$ in [3] and [4], where $v_p(i)$ is the index of p in the prime power decomposition of an integer i .

As an application of Theorem 2.1, we have

THEOREM 3.1. *Let $n \geq 1$. Then*

$$v_2(Sp^H(2n+1, 3)) = v_2((4n+2)!/16) \quad \text{and}$$

$$v_2(Sp^Q(2n+1, 3)) = v_2((4n+1)!/8).$$

PROOF. By Theorem 2.1, each value of the left hand sides of the desired equalities is equal to or smaller than each value of the right hand side. Since $Sp^K(m, l)\phi_m$ is stably trivial for the attaching map $\phi_m: S^{4m-1} \rightarrow KP_l^{m-1}$ of the top cell of KP_l^m , we have $Sp^K(m, l)e(\phi_m)(y) \in a(m-s)\mathbb{Z}$ for $l \leq s \leq m-1$ and for any $y \in \widetilde{KO}^{4s}(KP_l^{m-1})$, where $e(\phi_m)$ denotes the e -invariant of ϕ_m . Especially, we have $Sp^K(2n+1, 3)e(\phi_{2n+1})(Y) \in \mathbb{Z}$ for $Y = X^3$ and UX^2 . Thus, using Proposition 1.2 and an elementary arithmetic, we have the opposite inequalities.

Q. E. D.

Let $U(m, l)$ be the complex stable James number as in [5]. That is, $U(m, l)$ is the stable order of the attaching map of the top cell of CP_{m-l}^{m-1} . Then, by Theorem 3.1, we have the following corollary, which is obtained using the similar methods in [3; section 6] and [4; section 4] and we omit its proof:

COROLLARY 3.2. (i) $v_2(U(4m+2, 4m-2)) = v_2((4m+1)!/8)$ for $m \geq 1$.

(ii) $v_2(U(4m+3, 4m-2)) = v_2((4m+2)!/8)$ for $m \geq 1$.

References

- [1] J. F. ADAMS, *On the groups $J(X)$ —II*, *Topology*, **3** (1965), 131–171.
- [2] J. F. ADAMS, *On the groups $J(X)$ —IV*, *Topology*, **5** (1966), 21–71.
- [3] M. IMAOKA and K. MORISUGI, *On the stable Hurewicz image of some stunted projective spaces, I*, *Publ. RIMS, Kyoto Univ.*, **20** (1984), 839–852.
- [4] M. IMAOKA and K. MORISUGI, *On the stable Hurewicz image of some stunted projective spaces, II*, *Publ. RIMS, Kyoto Univ.*, **20** (1984), 853–866.
- [5] K. KNAPP, *Some applications of K-theory to framed bordism: E-invariant and transfer*, *Habilitationsschrift*, Bonn, 1979.
- [6] C. A. MCGIBBON, *Self maps of projective spaces*, *Trans. Amer. Math. Soc.*, **271** (1982), 326–346.
- [7] K. MORISUGI, *Stable self maps of the quaternionic (quasi-) projective space*, *Publ. RIMS, Kyoto Univ.*, **20** (1984), 971–976.
- [8] J. MUKAI, *The order of attaching class in the suspended quaternionic quasi-projective space*, *Publ. RIMS, Kyoto Univ.*, **20** (1984), 717–725.
- [9] D. M. SEGAL, *On the stable homotopy of quaternionic and complex projective spaces*, *Proc. Amer. Math. Soc.*, **25** (1970), 838–841.
- [10] H. TODA, *Composition methods in homotopy groups of spheres*, *Annals of Math. Studies* **49** (Princeton Univ. Press, 1962).
- [11] G. WALKER, *Estimates for the complex and quaternionic James numbers*, *Quart. J. Math. Oxford* (2), **32** (1981), 467–489.

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