**ON A ONE-PARAMETER SUBGROUP OF $U_{st}(\mathfrak{H})$
WHOSE INFINITESIMAL GENERATOR
HAS ONLY A SINGULARLY
CONTINUOUS PART**

By

KAZUO AKUTAGAWA

(Received July 1, 1985)

1. Introduction

For a separable complex Hilbert space $\mathfrak{H}$, let $U_{st}(\mathfrak{H})$ denote the group of all unitary operators of $\mathfrak{H}$, which is endowed with the strong topology and is a metrizable topological group (See Section 2.). Let $\{U_t\}_{t \in \mathbb{R}}$ be a one-parameter subgroup of $U_{st}(\mathfrak{H})$ with infinitesimal generator H .

In this paper, we shall study to what extent the topology of $Cl(\{U_t\}_{t \in \mathbb{R}})$, the closure of $\{U_t\}_{t \in \mathbb{R}}$, is determined by H . In fact, we shall determine the topology of $Cl(\{U_t\}_{t \in \mathbb{R}})$ in terms of H , provided H has an absolutely continuous spectrum or H has no continuous spectrum. As a result, we shall exhibit a concrete example of $\{U_t\}_{t \in \mathbb{R}}$ with the following property (P):

(P) $\{U_t\}_{t \in \mathbb{R}}$ is not closed and $Cl(\{U_t\}_{t \in \mathbb{R}})$ is not locally compact. H has only a singularly continuous part.

Let G be a connected Lie group and L be a topological group. Let $f: G \rightarrow L$ be a continuous homomorphism. The topology of the closure of a one-parameter subgroup of L is the one of those important things that determine the topology of $Cl(f(G))$ in L (cf. [1], [3], [4]).

We are interested especially in the case of $L = U_{st}(\mathfrak{H})$, that is to say, f is a unitary representation. From the previous reason, we consider the topology of $Cl(\{U_t\}_{t \in \mathbb{R}})$.

2. Preliminary

We first note that $U_{st}(\mathfrak{H})$ is a metrizable topological group with the metric

described as follows. Fix a countable dense subset $\{x_j\}_{j \in \mathbf{N}}$ of $\mathfrak{H}$. Then the metric $\rho(\cdot, \cdot)$ on $U_{st}(\mathfrak{H})$ is defined by

$$\rho(U, V) = \sum_{j=1}^{\infty} 2^{-j} \|Ux_j - Vx_j\| \cdot (1 + \|Ux_j - Vx_j\|)^{-1}$$

for $U, V \in U_{st}(\mathfrak{H})$, where $\|\cdot\|$ denotes the norm on $\mathfrak{H}$. It should be remarked that $U_{st}(\mathfrak{H})$ is not complete, but any closed abelian subgroup of $U_{st}(\mathfrak{H})$ is complete. In particular, $Cl(\{U_t\}_{t \in \mathbf{R}})$ is complete.

PROPOSITION. *Let $\{U_t\}_{t \in \mathbf{R}}$ be a one-parameter subgroup of $U_{st}(\mathfrak{H})$ with infinitesimal generator $H = \int_{-\infty}^{+\infty} \lambda dE(\lambda)$ (spectral representation).*

- (i) *If H has no continuous spectrum, then $Cl(\{U_t\}_{t \in \mathbf{R}})$ is locally compact.*
- (ii) *If the absolutely continuous part of H is not empty, then $\{U_t\}_{t \in \mathbf{R}}$ is closed.*
- (iii) *If H has continuous spectrum, then the continuous homomorphism $f: \mathbf{R} \rightarrow U_{st}(\mathfrak{H})$ defined by $f(t) = U_t$ is an immersion (i.e. one to one). In particular, if $\{U_t\}_{t \in \mathbf{R}}$ is closed, then f is an imbedding, namely, $\mathbf{R}$ is homeomorphic to $f(\mathbf{R}) = \{U_t\}_{t \in \mathbf{R}}$.*

PROOF. (i) This result is well-known, so we omit it. (ii) By the assumption, there exists $x_0 (\neq 0) \in \mathfrak{H}$ such that $\langle E(\cdot)x_0, x_0 \rangle$ becomes an absolutely continuous positive measure on $\mathbf{R}$ with respect to the Lebesgue measure of $\mathbf{R}$.

Let $\{U_{t_n}\}_{n \in \mathbf{N}} \subset \{U_t\}_{t \in \mathbf{R}}$ be a Cauchy sequence. Assume that there exists a subsequence $\{t_{v_n}\}_{n \in \mathbf{N}} \subset \{t_n\}_{n \in \mathbf{N}}$ satisfying $t_{v_n} \rightarrow \infty$ (as $n \rightarrow \infty$). For any $\varepsilon (< \|x_0\|^2) > 0$, there exists $N \in \mathbf{N}$ such that

$$\|U_{t_{v_m}}x_0 - U_{t_{v_n}}x_0\|^2 < \varepsilon$$

for $n, m \geq N$. On the other hand, by Riemann-Lebesgue's theorem, we have

$$\lim_{n \rightarrow \infty} \|U_{t_{v_m}}x_0 - U_{t_{v_n}}x_0\|^2 = 2\|x_0\|^2.$$

This gives a contradiction. Hence $\{t_n\}_{n \in \mathbf{N}}$ must be bounded. Therefore $\{U_t\}_{t \in \mathbf{R}}$ is closed.

(iii) The results follow from Category theorem (cf. [1, (2.4)]).

3. Construction of a one-parameter subgroup of $U_{st}(\mathfrak{H})$ with the property (P)

In this section, we construct explicitly a one-parameter subgroup of $U_{st}(\mathfrak{H})$ with the property (P).

(I) Construction of the A -set

Expand every number $x \in [0, 1]$ in the 4-adic system, i.e. write $x = 0.\alpha_1\alpha_2\cdots\alpha_n\cdots$, $\alpha_n = 0, 1, 2$ or 3 for $n \in \mathbb{N}$. If x has two representations, for example $x = 0.1000\cdots = 0.0333\cdots$, for the uniqueness of the expansion we shall always adopt the latter expansion. Let D be the set of those numbers of the type

$$x = 0.\alpha_1|\alpha_2\alpha_2|\cdots|\underbrace{\alpha_n\cdots\alpha_n}_{n \text{ times}}|\cdots,$$

where $\alpha_n = 0$ or 3 for any $n \in \mathbb{N}$. We shall call D the A -set.

Next let us reconstruct the A -set inductively.

1-st step: Divide the interval $[0, 1]$ into four parts with the same length and take off the union $(1/4, 3/4)$ of the two middles. Then the two closed intervals $J_1^{(1)} = [0/4, 1/4]$ and $J_2^{(1)} = [3/4, 4/4]$ remain.

2-nd step: Divide each of the two intervals $J_1^{(1)}$ and $J_2^{(1)}$ into 4^2 parts with the same length and take off the unions $(0/4 + 0/4^2 + 1/4^3, 0/4 + 3/4^2 + 3/4^3)$ and $(3/4 + 0/4^2 + 1/4^3, 3/4 + 3/4^2 + 3/4^3)$ which are the $(4^2 - 2)$ middles of each of them. Then 2^2 closed intervals $J_1^{(2)} = [0/4 + 0/4^2 + 0/4^3, 0/4 + 0/4^2 + 1/4^3]$, $J_2^{(2)} = [0/4 + 3/4^2 + 3/4^3, 1/4 + 0/4^2 + 0/4^3]$, $J_3^{(2)} = [3/4 + 0/4^2 + 0/4^3, 3/4 + 0/4^2 + 1/4^3]$ and $J_4^{(2)} = [3/4 + 3/4^2 + 3/4^3, 4/4 + 0/4^2 + 0/4^3]$ remain.

n -th step: Divide each $J_j^{(n-1)} = [p_j^{(n-1)}, q_j^{(n-1)}]$ of the 2^{n-1} intervals obtained by the $(n-1)$ -th step into 4^n parts with the same length and take off the union $(p_j^{(n-1)} + 1/4^n, q_j^{(n-1)} - 1/4^n)$ of the $(4^n - 2)$ middles out of $J_j^{(n-1)}$. Then the 2^n closed intervals $J_1^{(n)} = [p_1^{(n)}, q_1^{(n)}]$, $J_2^{(n)} = [p_2^{(n)}, q_2^{(n)}]$, ..., $J_{2^n}^{(n)} = [p_{2^n}^{(n)}, q_{2^n}^{(n)}]$ remain and so on ad infinitum.

Denote by X_n the union of all open intervals which are taken off in the n -th step.

$$(3a) \quad D = [0, 1] \sim \bigcup_{n=1}^{\infty} X_n.$$

$$(3b) \quad \mu(D) = 0 \text{ where } \mu \text{ is the Lebesgue measure of } \mathbb{R}.$$

$$(3c) \quad D \text{ has the cardinal number of the continuum.}$$

(II) Construction of the A -function

Keep the notation in (I). For every $x \in [0, 1]$ expanded in the 4-adic system, write $x = 0.\alpha_1\alpha_2\cdots\alpha_n\cdots$ as

$$x = 0.\alpha_1|\alpha_2\alpha_3\cdots|\alpha_{n(n-1)/2+1}\cdots\alpha_{n(n+1)/2}|\cdots.$$

Denote by $n(x)$ the first index n which the n -th term $\alpha_{n(n-1)/2+1}\cdots\alpha_{n(n+1)/2}$ is neither $00\cdots 0$ nor $33\cdots 3$. If there is no such n , i.e. if $x \in D$, write $n(x) = \infty$.

Define the function ϕ by

$$\phi(x) = \sum_{j=1}^{n(x)-1} (3 \cdot 2^j)^{-1} \alpha_{j(j+1)/2} + 2^{-n(x)},$$

and call it the A -function. Then we have:

(3d) If $0 \leq x \leq y \leq 1$, then $0 = \phi(0) \leq \phi(x) \leq \phi(y) \leq \phi(1) = 1$.

(3e) If $x = 0.\alpha_1\alpha_2\cdots\alpha_n\cdots$, $y = 0.\beta_1\beta_2\cdots\beta_n\cdots \in [0, 1]$ and $\alpha_i = \beta_i$ for $1 \leq i \leq r$, then $|\phi(x) - \phi(y)| \leq 2^{-m(r)}$, where $m(r)$ is the greatest number in $\{k \in \mathbb{N} | r \geq k(k+1)/2\}$. Hence the function ϕ is continuous.

(III) A property of the A -set

Choose $n \in \mathbb{N}$ arbitrarily and fix it once and for all. Consider the function $f_n(x) = \cos 2\pi t_n x$ ($t_n = 4^{n(n+1)/2}$) restricted to $J_j^{(n)}$ ($1 \leq j \leq 2^n$). We note that $f_n(p_j^{(n)}) = f_n(q_j^{(n)}) = 1$. Define the function $g_n: [0, 1] \rightarrow [0, 1]$ as follows

$$g_n(x) = \begin{cases} -4 \cdot 4^{n(n+1)/2}(x - p_j^{(n)}) + 1 & (x \in [p_j^{(n)}, p_j^{(n)} + 4^{-1} \cdot 4^{-n(n+1)/2}]) \\ 4 \cdot 4^{n(n+1)/2}(x - q_j^{(n)}) + 1 & (x \in [q_j^{(n)} - 4^{-1} \cdot 4^{-n(n+1)/2}, q_j^{(n)}]) \\ 0 & (x \notin \bigcup_{j=1}^{2^n} \{[p_j^{(n)}, p_j^{(n)} + 4^{-1} \cdot 4^{-n(n+1)/2}] \\ & \quad \cup [q_j^{(n)} - 4^{-1} \cdot 4^{-n(n+1)/2}, q_j^{(n)}]\}) \end{cases}$$

Comparing f_n with g_n , then we have

$$(*) \quad f_n(x) \geq 1 - 4^{-n} \quad \text{for } x \in D.$$

(IV) Construction of a one-parameter subgroup with the property (P)

For any $f \in \mathfrak{H} = L^2([0, 1])$ and for any $\Delta \in \mathfrak{B}([0, 1])$ (all Borel sets of $[0, 1]$), we define a spectral measure $\{E(\cdot)\}$ and a one-parameter subgroup $\{U_t\}_{t \in \mathbb{R}}$ of $U_{st}(\mathfrak{H})$ as follows

$$(E(\Delta)f)(x) = \chi_{\phi(\Delta)}(x) \cdot f(x)$$

and

$$U_t = \int_0^1 e^{\sqrt{-1} \lambda t} dE(\lambda)$$

for $t \in \mathbb{R}$, where χ_B denotes the characteristic function of B . Obviously the infinitesimal generator $H = \int_{-\infty}^{+\infty} \lambda dE(\lambda)$ has only a singularly continuous part.

(V) Topology of $Cl(\{U_t\}_{t \in \mathbb{R}})$

First, we prove that $\{U_t\}_{t \in \mathbb{R}}$ is not closed. For any $x \in \mathfrak{H}$, using $(*)$, we have

$$\|(I - U_{2\pi t_n})x\| \leq (2 \cdot 4^{-n})^{1/2} \|x\|,$$

where $t_n = 4^{n(n+1)/2}$. Thus $\{U_{2\pi t_n}\}_{n \in \mathbb{N}}$ becomes a Cauchy sequence. Moreover $2\pi t_n \rightarrow \infty$ (as $n \rightarrow \infty$). By (iii) of Proposition, $\{U_t\}_{t \in \mathbb{R}}$ is not closed.

Finally we prove that $Cl(\{U_t\}_{t \in \mathbb{R}})$ is not locally compact. For this purpose, we prepare the following lemma.

LEMMA. *The closure of a one-parameter subgroup of $U_{st}(\mathfrak{H})$ is locally compact if and only if it is either compact or isomorphic to $\mathbb{R}$.*

Suppose that $Cl(\{U_t\}_{t \in \mathbb{R}})$ is locally compact. It then follows from Lemma that $Cl(\{U_t\}_{t \in \mathbb{R}})$ is compact. Since any unitary representation of a compact group is a direct sum of finite-dimensional irreducible unitary representations, then H must have only point spectra. This gives a contradiction. Therefore $Cl(\{U_t\}_{t \in \mathbb{R}})$ is not locally compact.

REMARK. We can also construct a one-parameter subgroup $\{V_t\}_{t \in \mathbb{R}}$ of $U_{st}(\mathfrak{H})$ with the following property:

$\{V_t\}_{t \in \mathbb{R}}$ is closed and locally compact. Its infinitesimal generator T has only a singularly continuous part.

Denote the Cantor function by ψ . Define a real spectral measure $\{F(\cdot)\}$ on $\mathfrak{H} = L^2([0, 1])$ and a one-parameter subgroup $\{V_t\}_{t \in \mathbb{R}}$ of $U_{st}(\mathfrak{H})$ by

$$(F(\Delta)f)(x) = \chi_{\psi(\Delta)} \cdot f(x) \text{ for } f \in \mathfrak{H}, \Delta \in \mathfrak{B}([0, 1])$$

and

$$V_t = \int_0^1 e^{\sqrt{-1} \lambda t} dF(\lambda) \quad \text{for } t \in \mathbb{R}.$$

Obviously $T = \int_{-\infty}^{+\infty} \lambda dF(\lambda)$ has only a singularly continuous part. Contrary to $\{U_t\}_{t \in \mathbb{R}}$, $\{V_t\}_{t \in \mathbb{R}}$ is closed. Also, from (iii) of Proposition, $Cl(\{V_t\}_{t \in \mathbb{R}})$ is locally compact.

References

- [1] M. GOTO, *Immersion of Lie groups*, J. Math. Soc. Japan **32** (1980), 727–749.
- [2] T. KATO, *Perturbation theory for linear operators*, Springer, 1966.
- [3] T. C. STEVENS, *Decisive subgroups of analytic groups*, Trans. Amer. Math. Soc., **274** (1982), 101–108.

- [4] T. C. STEVENS, *Weakening the topology of a Lie group*, Trans. Amer. Math. Soc., 276 (1983), 541–549.
- [5] K. YOSIDA, *Functional analysis*, Springer, 1964.

Department of Mathematics
Kyushu University
Hakozaki, Fukuoka 812
Japan