PROPER HOLOMORPHIC MAPPINGS BETWEEN ANNULI IN $\mathbb{C}^n$

By

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(Received Feb. 5, 1986)

1. Introduction

The proper holomorphic self-maps of the unit ball $B^n = \{z \in \mathbb{C}^n; |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < 1\}$, $n \geq 2$, were shown to be automorphisms by H. Alexander [1]. The automorphisms of the unit ball were examined by W. Rudin [2]. The purpose of this paper is to show that the proper holomorphic mappings between annuli extend to automorphisms of the unit ball and to determine all of them.

2. Preliminaries

The inner product

$$(1) \quad \langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j \quad (z, w \in \mathbb{C}^n)$$

and the associated norm

$$(2) \quad \|z\| = \langle z, z \rangle^{1/2} \quad (z \in \mathbb{C}^n)$$

make $\mathbb{C}^n$ into an n -dimensional Hilbert space. The ball $B(a, r)$ is defined to be the set of all $z \in \mathbb{C}^n$ which satisfy the inequality $\|z - a\| < r$. Taking $a=0$ and $r=1$ gives the unit ball B , consisting of all $z \in \mathbb{C}^n$ with $\|z\| < 1$. We let $\mathcal{U} = \mathcal{U}(n)$ be the group of all unitary operators on the Hilbert space $\mathbb{C}^n$.

Fix $a \in B$. Let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[a]$ generated by a , and let $Q_a = I - P_a$ be the projection onto the orthogonal complement of $[a]$. To be explicit, $P_0 = 0$ and

$$(3) \quad P_a z = \frac{\langle z, a \rangle}{\langle a, a \rangle} a \quad \text{if } a \neq 0.$$

Put $s_a = (1 - \|a\|^2)^{1/2}$ and define

$$(4) \quad \phi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle}.$$

If $\Omega = \{z \in \mathbb{C}^n; \langle z, a \rangle \neq 1\}$ then $\phi_a: \Omega \rightarrow \mathbb{C}^n$ is holomorphic. Clearly, $\Omega \supset \bar{B}$, since $\|a\| < 1$. Moreover, ϕ_a is a homeomorphism of $\bar{B}$ onto $\bar{B}$ and ϕ_a is an automorphism of B . The following proposition is contained in [2].

PROPOSITION. *If ψ is an automorphism of B and $a = \psi^{-1}(0)$, then there is a unique $U \in \mathfrak{U}$ such that*

$$(5) \quad \psi = U\phi_a.$$

3. Statement of the theorems

We always assume $n \geq 2$. Let $D_1 = B \setminus \overline{B(a, r)}$ and $D_2 = B \setminus \overline{B(b, R)}$, where $\overline{B(a, r)}, \overline{B(b, R)} \subset B$. Our main results are the following theorems.

THEOREM 1. *Let F be a proper holomorphic mapping of D_1 onto D_2 . Then $\|a\| = \|b\|$, $r = R$, and F extends to an automorphism of B .*

THEOREM 2. *Let $\|a\| = \|b\|$, $r = R$, and F is an automorphism of B . Then F is a biholomorphic mapping of D_1 onto D_2 if and only if there exist a $d_0 \in \mathbb{C}$ with*

$$(6) \quad |d_0 - 1/(1 + \|a\|^2 - r^2)| = 1/(1 + \|a\|^2 - r^2)$$

and a $U \in \mathfrak{U}$ with

$$(7) \quad U(a) = (d_0/d_0)b \quad (d_0 \neq 0)$$

$$(8) \quad U(a) = b \quad (d_0 = 0)$$

such that $F = U\phi_a$, where we put $d = d_0 a$.

From the above two theorems, we obtain the following corollary.

COROLLARY. *F is a proper holomorphic self-mapping of $\{z \in \mathbb{C}^n; r < \|z\| < 1\}$ if and only if F is a unitary operator on $\mathbb{C}^n$.*

4. Proof of the theorems

We prove Theorem 1 and the necessity part of the condition of Theorem 2.

The converse of Theorem 2 is immediate.

F extends to a holomorphic mapping from B into B . Since F is a proper holomorphic mapping of D_1 onto D_2 , F is a proper holomorphic mapping of $B(a, r)$ onto $B(b, R)$. Therefore F is a biholomorphic mapping of $B(a, r)$ onto $B(b, R)$ by [1]. By the proposition, there exist a $c \in B$ and a $U_1 \in \mathcal{U}$ such that

$$(9) \quad F(z) = RU_1\phi_c((z-a)/r) + b \quad (z \in B(a, r)).$$

Multiplying the equality by $1 - \langle (z-a)/r, c \rangle$, we obtain

$$(10) \quad (1 - \langle (z-a)/r, c \rangle)F(z) = RU_1(c - P_c((z-a)/r) - s_c Q_c((z-a)/r)) \\ + (1 - \langle (z-a)/r, c \rangle)b_1 \quad (z \in B),$$

since both sides of the equality (10) are holomorphic on B .

(I) the case $c=0$

In this case, the equality (10) becomes

$$(11) \quad F(z) = -RU_1((z-a)/r) + b_1 \quad (z \in B).$$

Since $F(B) \subset B$, we have

$$(12) \quad R = r, (R/r)U_1a + b = 0$$

and so

$$(13) \quad F(z) = -U_1z, -U_1a = b.$$

(II) the case $c \neq 0$

We put

$$(14) \quad H_1 = \{z \in \mathbb{C}^n; \langle (z-a)/r, c \rangle = 1\}.$$

On H_1 , the inner product of the right-hand side of (10) and U_1c is equal to

$$(15) \quad R(\|c\|^2 - 1).$$

Therefore, $H_1 \cap \bar{B} = \emptyset$. So F is holomorphic on $\bar{B}$, and moreover proper holomorphic mapping of B onto B . By [1], F is an automorphism of B . By the proposition, there exist a $d \in B$ and a $U_2 \in \mathcal{U}$ such that

$$(16) \quad F(z) = U_2\phi_d(z) \quad (z \in B).$$

From (10) and (16), we have

$$\begin{aligned}
(17) \quad & R(1 - \langle z, d \rangle)(c - P_c((z-a)/r) - s_c Q_c((z-a)/r)) \\
& + (1 - \langle z, d \rangle)(1 - \langle (z-a)/r, c \rangle)b_1 \\
& = (1 - \langle (z-a)/r, c \rangle)V(d - P_d(z) - s_d Q_d(z)) \quad (z \in B),
\end{aligned}$$

where we put

$$(18) \quad b_1 = U_1^{-1}b,$$

$$(19) \quad V = U_1^{-1}U_2.$$

Since the components of both sides of the equality (17) are polynomials, we have the following three identities putting $z = a + rw$ and comparing the terms of the same order.

$$\begin{aligned}
(20) \quad & R(1 - \langle a, d \rangle)c + (1 - \langle a, d \rangle)b_1 \\
& = V(d - P_d(a) - s_d Q_d(a)),
\end{aligned}$$

$$\begin{aligned}
(21) \quad & -R\langle rw, d \rangle c - R(1 - \langle a, d \rangle)(P_c(w) + s_c Q_c(w)) \\
& - \langle rw, d \rangle b_1 - (1 - \langle a, d \rangle)\langle w, c \rangle b_1 \\
& = -\langle w, c \rangle V(d - P_d(a) - s_d Q_d(a)) \\
& + V(-P_d(rw) - s_d Q_d(rw)),
\end{aligned}$$

$$\begin{aligned}
(22) \quad & R\langle rw, d \rangle(P_c(w) + s_c Q_c(w)) + \langle rw, d \rangle\langle w, c \rangle b_1 \\
& = \langle w, c \rangle V(P_d(rw) + s_d Q_d(rw)).
\end{aligned}$$

On $H_2 = \{w \in \mathbb{C}^n; \langle w, d \rangle = 0\}$, the equality (22) becomes

$$(23) \quad 0 = \langle w, c \rangle V s_d r w \quad (w \in H_2).$$

If $d=0$, then $H_2 = \mathbb{C}^n$ and from (23) $c=0$. This is a contradiction, so $d \neq 0$, and from (23) there exists an $\alpha (\neq 0) \in \mathbb{C}$ such that

$$(24) \quad c = \alpha d.$$

Substituting (24) in (22), we have

$$\begin{aligned}
(25) \quad & R(P_d(w) + s_{\alpha d} Q_d(w)) + \langle w, \alpha d \rangle b_1 \\
& = \bar{\alpha} V(P_d(w) + s_d Q_d(w)).
\end{aligned}$$

On H_2 , (25) becomes

$$(26) \quad R s_{ad} w = \bar{\alpha} V s_d w.$$

Putting

$$(27) \quad V = (R s_{ad}) / (\bar{\alpha} s_d),$$

we have

$$(28) \quad V w = v w \quad (w \in H_2).$$

Since $d \neq 0$ and $V \in \mathfrak{U}$, $|v| = 1$ and there exists a $u \in \mathbb{C}$ with $|u| = 1$ such that

$$(29) \quad V d = u d.$$

Putting $w = d$ in (25) and using (29), we have

$$(30) \quad b_1 = \beta d,$$

where we put

$$(31) \quad \beta = u / \|d\|^2 - R / (\bar{\alpha} \|d\|^2).$$

Substituting (24) and (30) in (21), we have

$$(32) \quad a = \gamma d$$

for some constant $\gamma \in \mathbb{C}$. From (20), (21), (24), (27), (28), (29), (30), (31), and (32), $\gamma \neq 0$. Therefore, there exists a $d_0 (\neq 0) \in \mathbb{C}$ such that

$$(33) \quad d = d_0 a.$$

On H_2 , (21) becomes

$$(34) \quad -R(1 - \langle a, d \rangle) s_c Q_c(w) = V(-s_d Q_d(rw)).$$

From (27), (28), and (33), (34) yields

$$(35) \quad \alpha = r / (1 - d_0 \|a\|^2).$$

Putting $w = d$ in (21), we have

$$(36) \quad \begin{aligned} & -Rr\|d\|^2 c - R(1 - \langle a, d \rangle) d - r\|d\|^2 b_1 - (1 - \langle a, d \rangle) \langle d, c \rangle b_1 \\ & = -\langle d, c \rangle V(d - P_d(a) - s_d Q_d(a)) - V P_d(rd). \end{aligned}$$

From (24), (27), (29), (30), (31), and (33), (36) yields

$$(37) \quad u = rv^2/R.$$

Since $|u| = |v| = 1$, we have

$$(38) \quad r = R$$

and

$$(39) \quad u = v^2.$$

From (27), (31), (33), (35), (38), and (39), we have

$$(40) \quad \beta = (1 - \bar{d}_0 \|a\|^2) / \{(1 - d_0 \|a\|^2) \bar{d}_0\}.$$

Therefore

$$(41) \quad \|a\| = \|b_1\|.$$

From (24), (27), (33), (35), and (38), we obtain (6) since $|v| = 1$. From (18), (19), (24), (27), (33), (35), and (38), we have (7) for $U = U_2$ and

$$(42) \quad \|a\| = \|b\|.$$

This completes the proof.

References

- [1] H. ALEXANDER, *Proper holomorphic mappings in $\mathbb{C}^n$* , Indiana Univ. Math. J. **26** (1977), 137–146.
- [2] W. RUDIN, *Function theory in the unit ball of $\mathbb{C}^n$* , Springer Verlag, 1980.

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