ON THE MCGIBBON AND NEISENDORFER THEOREM RESOLVING THE SERRE CONJECTURE

To the memory of Professor Shichirô Oka

By

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Introduction

In [10] McGibbon and Neisendorfer have proved the conjecture of Serre [13] without the restriction that H_*X is of finite type. Let p be a prime number.

THEOREM 1 [10, Theorem 1]. *Let X be a simply connected space such that $H\mathbb{Z}/p_*X$ is non-zero in at least one and at most finitely many positive degrees. Then $\text{Tor}(\pi_n X, \mathbb{Z}/p)$ is non-zero for infinitely many n .*

Their proof is fairly based on the result of Miller [11]:

THEOREM 2 [11, Theorem C]. *Let X be a connected nilpotent space such that $H\mathbb{Z}/p_*X$ is non-zero in only finitely many degrees. Then $\text{map}_*(B\mathbb{Z}/p, X)$ is weakly contractible.*

In this note we shall prove the following result, which is an improvement of Theorem 1 and Corollary 4.2 of [16].

THEOREM 3. *Let X be a connected nilpotent CW-complex with $H_1 X \otimes \mathbb{Z}_{p^\infty} = 0$ and j be a fixed integer with $0 \leq j < 2(p-1)$. Assume that $H\mathbb{Z}/p_*X$ is non-zero in at least one and at most finitely many positive degrees. Then $\pi_{2(p-1)n+j} X$ contains a cyclic p -group as a direct summand for infinitely many n .*

In the above theorem and throughout this note, a cyclic p -group means a non-zero finite group $\mathbb{Z}/p^r$ for some positive integer r . The proof will be done by directing attention only towards the following fact (Corollary 2): Let X be a simply connected CW-complex and $X_p^\wedge$ be the p -completion of X . Then

$\text{map}_*(CP^\infty, X_p^\wedge)$ is weakly contractible if so is $\text{map}_*(BZ/p, X)$.

In our proof we also adopt the homotopy spectral sequence $E_2^{t,-m} = H^t(Y; \pi_m X) \Rightarrow [S^{m-t} Y, X]$, associated with the skeletons of Y . However, our method is essentially the same as that of McGibbon and Neisendorfer, because our spectral sequence coincides with the homotopy spectral sequence associated with the Postnikov systems of X (cf. [5] and [9]). If one wants to avoid the spectral sequence argument, one can prove our theorem along the line of them entirely.

As a stable version of our theorem we shall show the following result.

THEOREM 4. *Let X be a connective CW -spectrum such that $HZ/p_* X$ is non-zero in at least one and at most finitely many degrees. Fix an integer j with $0 \leq j < 2(p-1)$. Then $\pi_{2(p-1)n+j} X$ contains a cyclic p -group as a direct summand for infinitely many n .*

We first study the p -completion $X_p^\wedge$ and its homotopy group in §1, and next discuss the convergence of the homotopy spectral sequence in §2. In §3 we give several results concerned with $\pi_n(X; Z/p)$ and $\pi_n X \otimes Z/p$ to prove Theorem 3. Combining Propositions 6, 7 and Corollary 9 with Theorem 2 we can immediately show Theorem 3. Finally we consider stable versions in §4 to obtain Theorem 4, which is proved in a parallel argument with the unstable case.

We shall work in the category of based CW -complexes or CW -spectra. So HG_* or $H_*(\ ; G)$ stands for the reduced ordinary homology with coefficients in G and $[\ , \]$ for the set of the homotopy classes of based maps, and so on.

1. p -completion

Let X and Y be connected nilpotent CW -complexes with finite fundamental groups. For a fixed prime p we denote by Y_{τ_p} the homotopy fiber of the $HZ[1/p]_*$ -localization $l: Y \rightarrow Y_{Z[1/p]}$. Then the map $\tau: Y_{\tau_p} \rightarrow Y$ is an HZ/p_* -equivalence, and hence it induces a bijection $[S^k Y, X_p^\wedge] \cong [S^k Y_{\tau_p}, X_p^\wedge]$ for each $k \geq 0$ since the p -completion $X_p^\wedge$ is HZ/p_* -local. Recall from [6] that the arithmetic square

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p^\wedge \\ \downarrow & & \downarrow \\ X_Q & \longrightarrow & (\prod_p X_p^\wedge)_Q \end{array}$$

is up to homotopy a fiber square. Since Y_{τ_p} is $HZ[1/p]_*$ -acyclic, it is HQ_* -acyclic and HZ/q_* -acyclic for any prime $q \neq p$. Then the fiber square implies that the p -completion $c: X \rightarrow X_p^\wedge$ induces a bijection $[S^k Y_{\tau_p}, X] \cong [S^k Y_{\tau_p}, X_p^\wedge]$ for each $k \geq 0$ (see [11, Theorem 1.5]).

Given a connected CW -complex Y , a connected CW -complex X is said to be $[Y,]_*$ -trivial if $[S^k Y, X] = 0$ for all $k \geq 0$. From the above argument we obtain

LEMMA 1. *Let X and Y be connected nilpotent CW -complexes with finite fundamental groups. Then X is $[Y_{\tau_p},]_*$ -trivial if and only if $X_p^\wedge$ is $[Y,]_*$ -trivial.*

The following corollary is a useful tool in our discussion later.

COROLLARY 2. *Let X be a connected nilpotent CW -complex with a finite fundamental group. If X is $[BZ/p,]_*$ -trivial, then $X_p^\wedge$ is $[CP^\infty,]_*$ -trivial.*

Proof. From Miller's Theorem D [11] it follows that X is $[BZ_{p^\infty},]_*$ -trivial when it is $[BZ/p,]_*$ -trivial. The result is now immediate by Lemma 1.

For a connected nilpotent CW -complex X there is a splittable short exact sequence

$$* \longrightarrow \text{Ext}(Z_{p^\infty}, \pi_k X) \longrightarrow \pi_k X_p^\wedge \longrightarrow \text{Hom}(Z_{p^\infty}, \pi_{k-1} X) \longrightarrow *$$

(see [3, Proposition 4.3]). For $n \geq 2$ the mod p homotopy group is defined to be $\pi_n(X; Z/p) = [S^{n-2} M_p, X]$ where M_p is the Moore space of type $(Z/p, 1)$.

An abelian group G is said to be reduced if its only divisible subgroup is zero. Recall that every abelian group is the direct sum of a divisible group and a reduced group [7, Theorem 21.3]. This decomposition, together with some results on pure subgroups of infinite abelian groups, will be used in the proof of the following lemma.

LEMMA 3. *Let X be a simply connected CW -complex and $k \geq 2$.*

- (i) *If $\pi_k X \otimes Z/p \neq 0$, then $\pi_k X_p^\wedge \otimes Z/p \neq 0$.*
- (ii) *If $\pi_k X$ does not contain any cyclic p -group as a direct summand, then $\pi_k X_p^\wedge$ is torsion free and $\pi_{k+1} X_p^\wedge \otimes Z/p \cong \pi_{k+1}(X; Z/p)$.*

Proof. (i) Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_k X \otimes Z/p & \longrightarrow & \pi_k(X; Z/p) & \longrightarrow & \text{Tor}(\pi_{k-1} X, Z/p) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_k X_p^\wedge \otimes Z/p & \longrightarrow & \pi_k(X_p^\wedge; Z/p) & \longrightarrow & \text{Tor}(\pi_{k-1} X_p^\wedge, Z/p) \longrightarrow 0 \end{array}$$

with exact rows. By use of [11, Theorem 1.5] we see that the middle vertical map is an isomorphism, since $S^{k-2}M_p$ is $HZ[1/p]_*$ -acyclic. Hence (i) is immediate.

(ii) Let $\pi_k X = D \oplus G$ where D is a divisible group and G is a reduced group [7, Theorem 21.3]. By our assumption, G is p -torsion free; since if G contains a p -torsion element, then the reduced group G must contain a cyclic p -group as a direct summand by [7, Corollary 27.3], which contradicts our assumption. Then $\text{Ext}(Z_{p^\infty}, \pi_k X) = \text{Ext}(Z_{p^\infty}, G)$ is torsion free. Since $\text{Hom}(Z_{p^\infty}, \pi_{k-1} X)$ is always torsion free, we see that $\pi_k X_p^\wedge$ is torsion free and hence we obtain the result.

2. Convergence of homotopy spectral sequence

Given connected CW -complexes X and Y we consider the exact couple

$$D_!^{t,-m} = [S^{m-t} Y^t, X], \quad E_!^{t,-m} = [S^{m-t}(Y^t/Y^{t-1}), X]$$

for $m \geq t$, where Y^t denotes the t -skeleton of Y . This yields the $(r-1)$ -th derived exact couple $(D_r^{t,-m}, E_r^{t,-m})$ inductively. Thus $D_r^{t,-m} = \text{Im} \{[S^{m-t} Y^{t+r-1}, X] \rightarrow [S^{m-t} Y^t, X]\}$ and there exists a long exact sequence

$$\cdots \rightarrow E_r^{t,-m} \rightarrow D_r^{t,-m} \rightarrow D_r^{t-1,-m+1} \rightarrow E_r^{t+r-1,-m-r+2} \rightarrow \cdots$$

Then we can form a homotopy spectral sequence $\{E_r^{t,-m}(Y, X)\}$, $r \geq 1$ and $m \geq t \geq 1$, with differentials $d_r: E_r^{t,-m} \rightarrow E_r^{t+r,-m-r+1}$ such that

- (1) $E_2^{t,-m} \cong H^t(Y; \pi_m X)$ for $m \geq 2$, and
- (2) $E_\infty^{t,-m} \cong \text{Ker} \{Q_t[S^{m-t} Y, X] \rightarrow Q_{t-1}[S^{m-t} Y, X]\}$

where $Q_t[Y, X] = \text{Im} \{[Y, X] \rightarrow [Y^t, X]\}$ (cf., [5, Ch. IX]). Our $E_\infty^{t,-m}$ is never $\varprojlim_r E_r^{t,-m}$ as in [5], but the small E_∞ -term $e_\infty^{t,m}$ in it. Note that we have an inclusion $E_\infty^{t,-m} \rightarrow \varprojlim_{r \geq t} E_r^{t,-m}$, which is not necessarily a bijection.

As concerns the convergence of the spectral sequence we have a useful result due to Bousfield and Kan [5] (or see Yosimura [14]).

PROPOSITION 4. Fix an integer $n \geq 1$.

- (i) Let $m \geq 1$. If $\varprojlim_{t \geq n} E_r^{t,-m-t} = *$ for all $t \geq n$, then $E_\infty^{t,-m-t} = \varprojlim_{r \geq t} E_r^{t,-m-t}$ for each $t \geq n-1$.
- (ii) Let $m \geq 0$ and $r_0 \geq 2$. If $E_{r_0}^{t,-m-t+1} = *$ for all $t \geq n+r_0$, then $E_\infty^{t,-m-t} = E_{r_1}^{t,-m-t}$ for each $t \geq n$, where $r_1 = \text{Max} \{r_0, t\}$.

Now, we shall remark the following facts for the proof of the next proposition. Let X be a simply connected CW -complex. Since the loop space ΩX is an H -space, the k -invariants $k^{n+2} \in H^{n+2}((\Omega X)(0, n]; \pi_{n+2}X)$ are taken to zero under rationalization. Hence we observe that ΩX_Q splits as the product of the Eilenberg-MacLane spaces $K(\pi_m X \otimes Q, m-1)$. Therefore the spectral sequence $E_2^{t, -m} = H^t(Y; \pi_m X \otimes Q) \Rightarrow [S^{m-t} Y, X_Q]$ collapses when $m \geq t+1$. Thus the differentials $d_r: E_r^{t, -m} \rightarrow E_r^{t+r, -m-r+1}$ vanish for all $r \geq 2$ and $m \geq t+1$, and so $E_2^{t, -m} = E_\infty^{t, -m}$ when $m \geq t+1$.

PROPOSITION 5. *Fix integers k and q with $0 \leq k < q$, and $m \geq 2$. Let Y be a connected CW -complex such that $H_i Y$ is free if $i \equiv k \pmod q$ and zero if otherwise. Suppose that a connected CW -complex X satisfies one of the following conditions:*

- (i) $\pi_{m+j-1} X$ is torsion free for each $j \geq 2$ with $j \equiv 0 \pmod q$,
- (ii) $\pi_m X$ is divisible.

Then in the homotopy spectral sequence $\{E_r^{t, -m}(Y, X)\}$, the differential $d_r: E_r^{t, -m} \rightarrow E_r^{t+r, -m-r+1}$ vanishes for $t \leq m-1$.

Proof. We may assume that X is simply connected, taking the universal cover $\tilde{X}$ of X if necessary.

(i) The proof is by induction on $r \geq 2$. Suppose that $d_s: E_s^{t+r-s, -m-r+s} \rightarrow E_s^{t+r, -m-r+1}$ are trivial for all s with $2 \leq s < r$, and $t \leq m-1$. This means that $E_r^{t+r, -m-r+1} \subset E_2^{t+r, -m-r+1}$. Then we have a commutative diagram

$$\begin{array}{ccccc} E_r^{t, -m} & \xrightarrow{d_r} & E_r^{t+r, -m-r+1} & \subset & E_2^{t+r, -m-r+1} \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{E}_r^{t, -m} & \xrightarrow{\tilde{d}_r} & \tilde{E}_r^{t+r, -m-r+1} & \subset & \tilde{E}_2^{t+r, -m-r+1} \end{array}$$

between the two spectral sequences $\{E_r^{t, -m}(Y, X)\}$ and $\{\tilde{E}_r^{t, -m}(Y, X_Q)\}$. It is sufficient to prove the triviality of d_r only in the case that $t \equiv k$ and $r \equiv 0 \pmod q$. In this case the right vertical map is a monomorphism under our hypothesis on X . Then the triviality of $\tilde{d}_r$ implies the desired result immediately.

(ii) Take a map $f: W \rightarrow X$ from a wedge of m -spheres W such that $f_*: \pi_m W \rightarrow \pi_m X$ is an epimorphism. Note that the rationalization $l: S^m \rightarrow S_Q^m$ induces an epimorphism $[S_Q^m, X] \rightarrow [S^m, X]$ because $\text{Ext}(Q/Z, \pi_m X) = 0$. This implies that $[W_Q, X] \rightarrow [W, X]$ is an epimorphism, too. So we can choose a map $g: W_Q \rightarrow X$ which induces an epimorphism $g_*: \pi_m W \otimes Q \rightarrow \pi_m X$. Now, suppose that $d_s: E_s^{t, -m} \rightarrow E_s^{t+s, -m-s+1}$ are trivial for all s with $2 \leq s < r$, and $t \leq m-1$. This means that there exists an epimorphism $E_2^{t, -m} \rightarrow E_r^{t, -m}$. Considering the two spectral

sequences $\{E_r^{t,-m}(Y, X)\}$ and $\{\tilde{E}_r^{t,-m}(Y, W_Q)\}$ we have a commutative diagram

$$\begin{array}{ccccc} \tilde{E}_2^{t,-m} & \xlongequal{\quad} & \tilde{E}_r^{t,-m} & \xrightarrow{d_r} & \tilde{E}_{r+r}^{t,-m-r+1} \\ \downarrow & & \downarrow & & \downarrow \\ E_2^{t,-m} & \longrightarrow & E_r^{t,-m} & \xrightarrow{d_r} & E_{r+r}^{t,-m-r+1}. \end{array}$$

Since the left vertical map induced by g is an epimorphism, the triviality of $\tilde{d}_r$ implies that of d_r immediately.

REMARK. Proposition 5 is applicable to the case when X and Y are p -local.

3. Homotopy groups of $[BZ/p,]_*$ -trivial spaces

At first we show that for infinitely many n , the mod p homotopy group $\pi_n(X; Z/p)$ is non-zero, although it has been done in [10] when X is simply connected and $[BZ/p,]_*$ -trivial.

PROPOSITION 6. *Let X be a connected nilpotent CW-complex with $HZ/p_*X \neq 0$ and $H_1X \otimes Z_{p^\infty} = 0$. If X is $[BZ/p,]_*$ -trivial, then $\pi_n(X; Z/p)$ is non-zero for infinitely many n .*

Proof. In order to make a contradiction we suppose that $\pi_n(X; Z/p)$ is non-zero for at most a finite number of n .

First consider the case when there exists an integer $m \geq 2$ such that

- 1) $\text{Tor}(\pi_m X, Z/p) \neq 0$ and $\pi_{m+1} X \otimes Z/p = 0 = \pi_k(X; Z/p)$ for all $k \geq m+2$, or
- 2) $\pi_m X \otimes Z/p \neq 0$ and $\pi_k(X; Z/p) = 0$ for all $k \geq m+1$.

Note that $\pi_k X$ is uniquely p -divisible for each $k \geq m+1$ in both cases. We may assume that X is simply connected, for the universal cover $\tilde{X}$ of X is also $[BZ/p,]_*$ -trivial. Take the spectral sequence $E_2^{t,-m} = H^t(BZ/p; \pi_m X) \Rightarrow [S^{m-t} BZ/p, X]$. Obviously $E_2^{s+r, -m-r-s+1} = 0$ for all $r \geq 2$ and $s \geq 0$. Then it follows from Proposition 4 (ii) that $E_2^{t,-m} = E_\infty^{t,-m}$ when $n=1$ or 2 . $E_2^{1,-m} \cong \text{Tor}(\pi_m X, Z/p)$ is non-zero in the first case and $E_2^{2,-m} \cong \pi_m X \otimes Z/p$ is non-zero in the second case. However E_∞ 's must be zero since X is $[BZ/p,]_*$ -trivial. This is a contradiction.

Next we consider the case when

- 3) $\pi_2 X \otimes Z/p = 0 = \pi_k(X; Z/p)$ for all $k \geq 3$.

Thus $\pi_k X$ is uniquely p -divisible for any $k \geq 2$. Take the fibering $X_{\tau_p} \rightarrow X \rightarrow X_{Z[1/p]}$. By applying J. H. C. Whitehead's theorem we observe that X_{τ_p} is the

Eilenberg-MacLane space $K(\pi_1 X_{\tau_p}, 1)$ under the hypothesis that $H_1 X \otimes Z_{p^\infty} = 0$. Note that X_{τ_p} is $[BZ/p,]_*$ -trivial and hence $\text{Tor}(\pi_1 X_{\tau_p}, Z/p)$ must be zero. Since $\pi_1 X_{\tau_p} = \text{Ker} \{l: \pi_1 X \rightarrow \pi_1 X_{Z[1/p]}\}$ is a p -torsion group, we see that it is in fact zero. Hence $l: X \rightarrow X_{Z[1/p]}$ is a homotopy equivalence. But this contradicts the hypothesis that $HZ/p_* X \neq 0$. Q. E. D.

In [10] McGibbon and Neisendorfer have established that for infinitely many n , $\text{Tor}(\pi_n X, Z/p)$ is non-zero whenever X is simply connected and $[BZ/p,]_*$ -trivial. We here improve their result showing that $\pi_n X$ contains cyclic p -group as a direct summand for infinitely many n .

PROPOSITION 7. *Let X be a connected CW-complex which is $[BZ/p,]_*$ -trivial. If $\pi_n(X; Z/p)$ is non-zero for infinitely many n , then $\pi_n X$ contains a cyclic p -group as a direct summand for infinitely many n .*

Proof. We may regard X simply connected, taking the universal cover $\tilde{X}$ if necessary. Suppose that $\pi_n X$ contains a cyclic p -group as a direct summand for at most a finite number of n . Then we can choose a large integer $m \geq 3$ such that $\pi_m(X; Z/p) \neq 0$ and $\pi_k X$ does not contain any cyclic p -group as a direct summand for all $k \geq m-1$. From Lemma 3, it follows that $\pi_m X_p^\wedge \otimes Z/p \cong \pi_m(X; Z/p) \neq 0$ and $\pi_k X_p^\wedge$ is torsion free for any $k \geq m$. Consider the spectral sequence $E_2^{t, -m} = H^t(CP^\infty; \pi_m X_p^\wedge) \Rightarrow [S^{m-t} CP^\infty, X_p^\wedge]$. By making use of Proposition 5 we see that the differentials $d_r: E_r^{t, -m-s} \rightarrow E_r^{t+r, -m-r-s+1}$ are all trivial if $s \geq 0$ and $t \leq m+s-1$. Hence $E_2^{2, -m} = E_r^{2, -m}$ for all $r \geq 2$ and $\varprojlim_{r \geq t} E_r^{t, -m-t+2} = 0$ for any $t \geq 2$. Then Proposition 4(i) implies that $E_2^{2, -m} = E_\infty^{2, -m}$. $E_2^{2, -m} \cong \pi_m X_p^\wedge$ and it is non-zero. But $E_\infty^{2, -m}$ must be just zero since $X_p^\wedge$ is $[CP^\infty,]_*$ -trivial because of Corollary 2. This is a contradiction. Q. E. D.

The p -local space $S^1 CP_{Z(p)}^\infty$ splits as a wedge sum $\bigvee_{1 \leq i < p-1} CP_{1,i}^\infty$ where $H_j CP_{1,i}^\infty = Z_{(p)}$ if $j \equiv 2i+1 \pmod{2(p-1)}$ and zero if otherwise. Using $CP_{1,1}^\infty$ in place of CP^∞ we restate Proposition 7 more precisely than Zabrodsky tried in [16, Corollary 4.2].

PROPOSITION 8. *Let X be a connected CW-complex which is $[BZ/p,]_*$ -trivial. If $\pi_m X \otimes Z/p$ is non-zero for some $m \geq 4$, then there exists an integer $k \geq 1$ such that $\pi_{2(p-1)k+m-1} X$ contains a cyclic p -group as a direct summand.*

Proof. We may assume that X is simply connected. Suppose that $\pi_{2(p-1)k+m-1} X$ does not contain any cyclic p -group as a direct summand for

each $k \geq 1$. Then it follows from Lemma 3 that $\pi_{2(p-1)k+m-1}X_p^\wedge$ is torsion free for any $k \geq 1$. We again consider the spectral sequence $E_2^{t,-m} = H^t(CP_{1,1}^\infty; \pi_m X_p^\wedge) \Rightarrow [S^{m-t}CP_{1,1}^\infty, X_p^\wedge]$. Note that $H_t CP_{1,1}^\infty = 0$ unless $t \equiv 3 \pmod{2(p-1)}$. Then by Proposition 5 we see that the differentials $d_r: E_r^{t,-m-s} \rightarrow E_r^{t+r,-m-r-s+1}$ are all trivial for $t \leq m+s-1$ whenever $s \geq 0$ and $s \equiv 0 \pmod{2(p-1)}$. Therefore $E_2^{3,-m} = E_r^{3,-m}$ for all $r \geq 2$ and $\varinjlim_{r \geq t} E_r^{t,-m-t+3} = 0$ for any $t \geq 3$. Applying Proposition 4(i) we obtain that $E_2^{3,-m} = E_\infty^{3,-m}$, which yields a contradiction. The reason is that $E_2^{3,-m} \cong \pi_m X_p^\wedge \neq 0$ by Lemma 3 and $E_\infty^{3,-m} = 0$ since $X_p^\wedge$ is $[CP_{1,1}^\infty,]_*$ -trivial.

COROLLARY 9. *Let X be a connected CW-complex which is $[BZ/p,]_*$ -trivial and j be a fixed integer with $0 \leq j < 2(p-1)$. If $\pi_m X \otimes Z/p$ is non-zero for some $m \geq 4$, then $\pi_{2(p-1)n+j}X$ contains a cyclic p -group as a direct summand for infinitely many n . (Cf. [16, Corollary 4.2]).*

Proof. Applying Proposition 8 repeatedly $2(p-1)-i$ times, we can find an integer $l \geq 2(p-1)-i-1$ such that $\pi_{2(p-1)l+m+i}X$ contains a cyclic p -group as a direct summand whenever $\pi_m X \otimes Z/p \neq 0$. So it is sufficient to prove our result only in the case $j \equiv m \pmod{2(p-1)}$. Particularly the above observation shows that there exists an integer $l \geq 1$ such that $\pi_{2(p-1)l+m}X$ contains a cyclic p -group as a direct summand, if $\pi_m X \otimes Z/p \neq 0$. By repeating this argument we see easily that for infinitely many n , $\pi_{2(p-1)n+m}X$ contains a cyclic p -group as a direct summand.

4. A stable version

Given a CW-spectrum Y , a CW-spectrum X is said to be $[Y,]_*$ -trivial if $[Y, \sum^k X] = 0$ for all k , $-\infty < k < \infty$.

LEMMA 10. *Let X be a connective CW-spectrum such that $\bigoplus_p HZ/p_* X$ is non-zero in only finitely many degrees. If the sphere spectrum S is $[Y,]_*$ -trivial, then so is X .*

Proof. We first suppose that X is finite dimensional, thus it admits a finite filtration $* = X^m \subset X^{m+1} \subset \dots \subset X^n = X$ by the skeletons. Then it is sufficient to show our result only in the case when X is a wedge of the sphere spectrum. When $X = \bigvee_\alpha S$, it is the sphere spectrum with coefficients in the free abelian group $\bigoplus_\alpha Z$, thus the Moore spectrum of type $\bigoplus_\alpha Z$. We can construct a CW-spectrum $\bigvee S$ so that $\bigvee S$ and SG are related by the universal coefficient sequence

$$0 \longrightarrow \text{Ext}(\bigvee S_{*-1} Y, G) \longrightarrow SG^* Y \longrightarrow \text{Hom}(\bigvee S_* Y, G) \longrightarrow 0$$

for any abelian group G , where SG denotes the Moore spectrum of type G (see [15, (2.4)]). When $G=Z$, our hypothesis means that $\text{Hom}(\bigvee S_*Y, Z)=0=\text{Ext}(\bigvee S_*Y, Z)$. Then it follows from [2, Proposition 2.3] that $\bigvee S_*Y=0$, and hence $SG^*Y=0$ for any G . Consequently we see that X is $[Y,]_*$ -trivial if it is finite dimensional.

We now choose an integer n such that $\bigoplus HZ/p_k X=0$ for all $k \geq n+1$. Putting $W=X \wedge SQ/Z$, which is shortly written XQ/Z , we see obviously that $\Sigma^{-n-1}W/W^n$ is homotopy equivalent to the Moore spectrum of type G where $G=\text{Ker}\{H_n W^n \rightarrow H_n W\}$. Thus there is a cofiber sequence $\Sigma^n SG \rightarrow W^n \rightarrow W$. Then the above observation implies that $W=XQ/Z$ is $[Y,]_*$ -trivial. On the other hand, $XQ=X \wedge SQ$ splits as the product of the Eilenberg-MacLane spectra $\Sigma^n H(\pi_n X \otimes Q)$ since the canonical map $XQ^*V \rightarrow \prod_n \text{Hom}(\pi_{*+n}V, \pi_n X \otimes Q)$ is an isomorphism because of Dold's theorem. Note that $H(\pi_n X \otimes Q)$ is homotopy equivalent to the Moore spectrum of type $\pi_n X \otimes Q$. Then we see easily that XQ is $[Y,]_*$ -trivial, too. From these results it follows that X is $[Y,]_*$ -trivial as desired. Q. E. D.

According to Ravenel [12, Lemma 3.2] the sphere spectrum S is $[BPZ/p,]_*$ -trivial. Using this fact we obtain

COROLLARY 11. *Let X be a connective CW-spectrum such that HZ/p_*X is non-zero in only finitely many degrees. Then X is $[BPZ/p,]_*$ -trivial.*

Let Y be a ring spectrum and W be a Y -module spectrum with unit. Obviously a CW-spectrum X is $[W,]_*$ -trivial if it is $[Y,]_*$ -trivial. In particular, X is $[HZ/p,]_*$ -trivial whenever it is $[BPZ/p,]_*$ -trivial.

For a CW-spectrum X , its p -completion $X_p^\wedge$ is constructed as the function spectrum $F(\Sigma^{-1}SZ_{p^\infty}, X)$ and there is a splittable short exact sequence

$$0 \longrightarrow \text{Ext}(Z_{p^\infty}, \pi_k X) \longrightarrow \pi_k X_p^\wedge \longrightarrow \text{Hom}(Z_{p^\infty}, \pi_{k-1} X) \longrightarrow 0$$

(see [4]). It is easy to show that the p -completion $X_p^\wedge$ is $[Y,]_*$ -trivial if X is $[YZ/p,]_*$ -trivial. Moreover we notice about the homotopy group of $X_p^\wedge$ that (i) and (ii) in Lemma 3 remain true for any k .

As for convergence of the Atiyah-Hirzebruch spectral sequence $E_2^{t,-m} = H^t(Y; \pi_m X) \Rightarrow [Y, \Sigma^{t-m} X]$, a similar result to Proposition 4 is shown when a CW-spectrum Y is connective (see Adams [1] or Yosimura [14]). Of course we may eliminate the restriction on m and n in this stable case. Since XQ splits as the product of the Eilenberg-MacLane spectra $\Sigma^n H(\pi_n X \otimes Q)$, the Atiyah-

Hirzebruch spectral sequence $E_2^{t,-m} = H^t(Y; \pi_m X \otimes Q) \Rightarrow [Y, \Sigma^{t-m} XQ]$ always collapses. Therefore Proposition 5 is also valid for any t and m in the stable case.

The above observations allow us to prove the following results corresponding to Propositions 6, 7 and Corollary 9 by parallel arguments to those in §3.

PROPOSITION 12. *Let X be a CW-spectrum with $\pi_* XZ/p \neq 0$. If X is $[HZ/p,]_*$ -trivial, then $\pi_n XZ/p$ is non-zero for infinitely many n . (Cf. Lin [8, Corollary 1.3] or Ravenel [12, Theorem 4.12]).*

PROPOSITION 13. *Let X be a CW-spectrum with $\pi_* XZ/p \neq 0$, and j be a fixed integer with $0 \leq j < 2(p-1)$. Assume that X is $[BPZ/p,]_*$ -trivial. Then*

(i) *$\pi_n X$ contains a cyclic p -group as a direct summand for infinitely many n , and more precisely*

(ii) *$\pi_{2(p-1)n+j} X$ contains a cyclic p -group as a direct summand for infinitely many n .*

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