ON THE FIXED POINT SET OF A SEMI-FREE S^1 -ACTION ON A WEAKLY ALMOST COMPLEX MANIFOLD

Dedicated to Professor Nobuo Shimada on his 60th birthday

By

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§ 1. Introduction

In this note we shall study the maximum dimension of the fixed point set of a semi-free S^1 -action on a weakly almost complex manifold. In [3] P. E. Conner and E. E. Floyd demonstrated the effectiveness of bordism techniques to obtain the results concerning the fixed point set of the involution. This case has also been studied by J. M. Boardman [2]. Semi-free S^1 -actions on oriented manifolds were studied in [5] and [7]. The results about Z_{p^r} -actions have been obtained in [6] and [11].

Let $\hat{x} \in H^2(BU(1); Z)$ be the first Chern class of the canonical line bundle over $BU(1)$. Define $b_i \in H_{2i}(BU(1); Z)$ by $\langle \hat{x}^i, b_i \rangle = 1$. Let $i: BU(1) \rightarrow BU$ be the canonical inclusion and abusing notations write b_i for $i_*(b_i)$. Then there is given an isomorphism $H_*(BU; Z) \cong Z[b_1, b_2, \dots]$. To each sequence $\omega = (i_1, i_2, \dots, i_k)$ of non-negative integers with $i_1 \leq i_2 \leq \dots \leq i_k$, let $c_\omega \in H^*(BU; Z)$ be the dual for $b_{i_1} b_{i_2} \dots b_{i_k}$. We may assign the characteristic class $c_\omega(v_M)$ to the normal bundle v_M of a weakly almost complex manifold M . The main result of this paper will be:

THEOREM 1.1. *Let M^{2n} be a weakly almost complex manifold with the semi-free S^1 -action preserving the complex structure. If for some partition ω of n containing no positive integer of the form $2^s - 1$*

$$\langle c_\omega(v_M), [M^{2n}] \rangle \not\equiv 0 \pmod{2}$$

then the dimension of some component of the fixed point set is greater than or

equal to $4n/5$.

In Section 2 we give a ring homomorphism $J: \mathfrak{M}_*^U(S^1) \rightarrow U^0(BU(1))$ between the bordism group of complex vector bundles and the complex cobordism ring of $BU(1)$. In Section 3 we analyze the composition of J and the Boardman map $\beta: U^0(BU(1)) \rightarrow H^*(BU(1))[[t_1, t_2, \dots]]$ and we obtain a nice system of ring generators of $\mathfrak{M}_*^U(S^1) \otimes \mathbb{Z}_2$ in Theorem 3.7. In Section 4 we use Theorem 3.7 to prove Theorem 1.1.

§2. The ring homomorphism $J: \mathfrak{M}_*^U(S^1) \rightarrow U^0(BU(1))$

Denote by $SF_m^U(S^1)$ the bordism group of semi-free S^1 -actions on m -dimensional weakly almost complex manifolds and denote by $F_m^U(S^1)$ the bordism group of free S^1 -actions on m -dimensional weakly almost complex manifolds which is isomorphic to the bordism group $U_{m-1}(BU(1))$ of singular weakly almost complex manifolds in $BU(1)$. Let

$$\mathfrak{M}_m^U(S^1) = \sum_{k+2l=m} U_k(BU(l)).$$

Consider a semi-free action $\phi: S^1 \times M^n \rightarrow M^n$ on a weakly almost complex manifold M^n . Denote by F^k the union of the k -dimensional component of the fixed point set and denote by v_k the normal bundle to F^k which is a complex vector bundle. Define $i_*: SF_m^U(S^1) \rightarrow \mathfrak{M}_m^U(S^1)$ by

$$i_*([M^m, \phi]) = \sum_k [v_k].$$

An l -dimensional complex vector bundle ξ over a weakly almost complex closed manifold M represents an element $[\xi]$ of $U_k(BU(l))$. The sphere bundle $S(\xi)$ has a free S^1 -action $\phi: S^1 \times S(\xi) \rightarrow S(\xi)$, $\phi(\lambda, v) = \lambda v$ the scalar multiplication of $\lambda \in S^1 \subset \mathbb{C}^1$. Then we have the homomorphism $\partial: U_k(BU(l)) \rightarrow F_{k+2l-1}^U(S^1)$ given by $\partial[\xi] = [S(\xi), \phi]$ which induces the homomorphism $\partial: \mathfrak{M}_m^U(S^1) \rightarrow F_{m-1}^U(S^1)$. After the same fashion of [7] and [11], we obtain the split exact sequence

$$(2.1) \quad 0 \longrightarrow SF_{2m}^U(S^1) \xrightleftharpoons[CP]{i_*} \mathfrak{M}_{2m}^U(S^1) \xrightarrow{\partial} F_{2m-1}^U(S^1) \longrightarrow 0.$$

Here the homomorphism CP with $CP \cdot i_* = id$ is given as follows. For a complex vector bundle ξ over M , take the Whitney sum $\theta^1 \oplus \xi$, θ^1 the 1-dimensional complex trivial bundle, and let S^1 act on the projective space bundle $P(\theta^1 \oplus \xi)$ by $\phi: S^1 \times P(\theta^1 \oplus \xi) \rightarrow P(\theta^1 \oplus \xi)$, $\phi(\lambda, [z, v]) = [z, \lambda v]$, where v and z belong to

the fibers ξ_x and θ_x^1 , respectively, and let $CP[\xi] = [P(\theta^1 \oplus \xi), \phi]$. Let S^1 act on $S^{2n+1} \times C^1 \rightarrow CP^n$ by $\lambda(v, z) = (\lambda v, \lambda z)$. Then we have a line bundle $\xi_n: S^{2n+1} \times_{S^1} C^1 \rightarrow CP^n$ over the complex projective space $CP^n = S^{2n+1}/S^1$. The tangent bundle of CP^n is stably equivalent to $(n+1)\xi_n$. The conjugate bundle $\bar{\xi}_n$ is referred to the canonical line bundle. Then we have

PROPOSITION 2.2 (cf. [7], [9]). *The bordism groups $\mathfrak{M}_*^U(S^1)$ is isomorphic to the polynomial algebra $U_*[[\bar{\xi}_0], [\bar{\xi}_1], \dots, [\bar{\xi}_n], \dots]$.*

Given a complex vector bundle ξ , there is the line bundle $\eta(\xi): S(\xi) \times_{S^1} C^1 \rightarrow P(\xi)$, where $S(\xi) \times_{S^1} C^1$ is the quotient space of $S(\xi) \times C^1$ under the identification $(v, z) \equiv (\lambda v, \lambda^{-1} z)$, $\lambda \in S^1$. For sufficiently large N , we have the classifying map

$$f_\xi: P(\xi) \longrightarrow CP^N$$

for the line bundle $\eta(\xi) \rightarrow P(\xi)$. Then there is the homomorphism

$$\tilde{d}_*: U_{2n}(BU(l)) \longrightarrow U_{2n+2l-2}(CP^N)$$

defined by $\tilde{d}_*([\xi]) = [P(\xi) \xrightarrow{f_\xi} CP^N]$. The bordism group $U_*(CP^N)$ is a free U_* -module with a base $\{[CP^i \subset CP^N]; i=0, 1, \dots, N\}$, and the cobordism ring $U^*(CP^N)$ is a truncated polynomial algebra $U^*[x_N]/(x_N^{N+1})$ where x_N is the cobordism first Chern class of ξ_N . If X is a weakly almost complex closed manifold, there is given the Thom Atiyah duality theorem of the type of Poincare duality $D: U^*(X) \cong U_*(X)$. Under $D: U^*(CP^N) \cong U_*(CP^N)$, the element x_N^i goes into $[CP^{N-i} \subset CP^N]$. Consider now the composition

$$\tilde{J}_N: U_{2n}(BU(l)) \xrightarrow{\tilde{d}_*} U_{2(n+l-1)}(CP^N) \xrightarrow{D^{-1}} U^{2(N-n-l+1)}(CP^N).$$

Given $[\xi] \in U_{2n}(BU(l))$, for a non-negative integer k there is an element $[f_\xi: P(\xi \times C^k) \rightarrow CP^N] \in U_*(CP^N)$ and

$$\tilde{J}_{N_k}[\xi \times C^k]/x_{N_k}^{n_k} \in U^0(CP^{N_k}),$$

where $2n_k = 2N_k - \dim P(\xi \times C^k)$ and $\dim P(\xi \times C^k) = 2(n+l+k-1)$. Let $i_k: CP^{n+l+k-1} \subset CP^{N_k}$ be the inclusion. Then there is the sequence

$$\{i_k^* \{ \tilde{J}_{N_k}[\xi \times C^k]/x_{N_k}^{n_k} \}; k=0, 1, \dots\}$$

which determines an element of the inverse limit $\varprojlim_N U^0(CP^N)$. We thus get a homomorphism

$$J: U_{2n}(BU(l)) \longrightarrow U^0(BU(1)) = \varprojlim_N U^0(CP^N)$$

given by $J([\xi]) = \{i_k^* \{ \tilde{J}_{N_k}[\xi \times C^k] / x_{N_k}^n \} \}$.

We recall the Boardman map

$$\beta: U^*(X) \longrightarrow H^*(X)[[t_1, t_2, \dots]]$$

which is the multiplicative natural transformation with the following properties:

(2.3) For a complex line bundle ξ ,

$$\beta(c_b^1(\xi)) = c^1(\xi) + \{c^1(\xi)\}^2 t_1 + \{c^1(\xi)\}^3 t_2 + \dots$$

where c_b^i denotes the cobordism Chern class and c^i denotes the ordinary cohomology Chern class.

(2.4) For $X = \{\text{a point}\}$, $\beta: U^* \rightarrow Z[t_1, t_2, \dots]$ coincides with the Hurewicz homomorphism and

$$\beta([M]) = \sum_{i_1 \leq i_2 \leq \dots \leq i_l} c_{(i_1, i_2, \dots, i_l)}(v_M) [M] t_{i_1} t_{i_2} \dots t_{i_l},$$

where $c_{(i_1, i_2, \dots, i_l)}(v_M)[M]$ is the Chern number for the Chern class of the normal bundle of M which corresponds to the symmetric polynomial $\sum y_1^{i_1} y_2^{i_2} \dots y_l^{i_l}$.

(2.5) If a CW-complex X has no torsion in the homology group, then β is injective.

Given a complex vector bundle ξ over X , there is the characteristic class $c_t(\xi)$ in $H^*(X)[[t_1, t_2, \dots]]$ satisfying

- (1) $c_t(f^* \xi) = f^* c_t(\xi)$
- (2) $c_t(\xi \oplus \eta) = c_t(\xi) c_t(\eta)$
- (3) if ξ is a complex line bundle, then

$$c_t(\xi) = 1 + c^1(\xi) t_1 + \{c^1(\xi)\}^2 t_2 + \dots$$

and

$$\beta(\xi) = c^1(\xi) c_t(\xi).$$

A complex oriented map $f: X \rightarrow Y$ between weakly almost complex manifolds defines the Gysin homomorphism of degree $\dim Y - \dim X$

$$f_!: U^*(X) \longrightarrow U^*(Y)$$

and

$$f_!: H^*(X)[t_1, t_2, \dots] \longrightarrow H^*(Y)[t_1, t_2, \dots].$$

Then we have

THEOREM 2.6 (cf. [1], [4], [8]). $\beta f_!(1) = f_! c_t(\tilde{v}_f)$, where $\tilde{v}_f = f^! \tau_Y - \tau_X$, and τ_M denotes the tangent bundle of M .

Let ξ be a complex vector bundle over a weakly almost complex closed manifold M , and let $f_\xi: P(\xi) \rightarrow CP^N$ be the classifying map for the canonical line bundle $\eta(\xi)$ over $P(\xi)$. Then we have

LEMMA 2.7.

$$\begin{aligned} \beta(\tilde{J}_N([\xi])) &= \beta f_{\xi!}(1) \\ &= \{c_t(\xi_N)\}^{N+1} f_{\xi!} \left\{ \frac{1}{c_t(\pi^!(\tau_M)) c_t(\pi^! \xi \otimes \overline{\eta(\xi)})} \right\} \end{aligned}$$

where $\pi: P(\xi) \rightarrow M$ is the projection and τ_M is the tangent bundle of M .

PROOF. The map f_ξ is a representative of $\tilde{d}_*([\xi])$ in $U_*(CP^N)$. Theorem 2.6 implies that $\beta \tilde{J}_N([\xi]) = f_{\xi!} c_t(\tilde{v})$, $\tilde{v} = f_\xi^!((N+1)\xi_N) - \pi^! \tau_M - \pi^! \xi \otimes \overline{\eta(\xi)}$, and

$$\beta(\tilde{J}_N([\xi])) = f_{\xi!} \left\{ \frac{(f_\xi^*(c_t(\xi_N)))^{N+1}}{c_t(\pi^! \tau_M) c_t(\pi^! \xi \otimes \overline{\eta(\xi)})} \right\}.$$

Q. E. D.

The definition of J is assured by the following proposition which has been proved by Ossa [7], but here we shall use Lemma 2.7 to verify

PROPOSITION 2.8. If for the complex vector bundle $\xi \rightarrow M^{2n}$ $\tilde{d}_*([\xi]) = b_m[\epsilon_0] + b_{m-1}[\epsilon_1] + \dots + b_0[\epsilon_m] \in U_*(CP^N)$, $m = \dim P(\xi)/2$, then

$$\begin{aligned} \tilde{d}_*([\xi \times C^k]) &\equiv b_m[\epsilon_k] + b_{m-1}[\epsilon_{k+1}] + \dots + b_1[\epsilon_{m+k-1}] + b_0[\epsilon_{m+k}] \\ &\text{modulo the } U_*\text{-submodule of } U_*(CP^N) \text{ generated by} \\ &[\epsilon_0], [\epsilon_1], \dots, [\epsilon_{k-1}], \end{aligned}$$

where $\epsilon_l: CP^1 \rightarrow CP^N$ is the natural inclusion.

PROOF. We take the classifying maps $f: P(\xi) \rightarrow CP^N$ and $f': P(\xi \times C^k) \rightarrow CP^N$ for bundles $\eta(\xi)$ and $\eta(\xi \times C^k)$, respectively. Then we have the commutative diagram

$$\begin{array}{ccc}
 P(\xi) & \xrightarrow{f} & CP^N \\
 \downarrow i & \nearrow f' & \\
 P(\xi \times C^k) & &
 \end{array}$$

where i is the natural injection. Then

$$\beta f_!(1) = c_i(\xi_N)^{N+k+1} \left\{ f'_i \left\{ \frac{1}{c_i(\tilde{\pi}^1 \tau_M) c_i(\tilde{\pi}^1 \xi \otimes \eta(\xi \times C^k))} i_!(1) \right\} \right\}$$

where $\tilde{\pi}: P(\xi \times C^k) \rightarrow M$ is the bundle projection. Since $i_*([P(\xi)]) = \{c^1(\eta(\xi \times C^k))\}^k \cap [P(\xi \times C^k)]$, it follows that $i_!(1) = \{c^1(\eta(\xi \times C^k))\}^k$ and

$$\beta f_!(1) = \beta(x_N^k f'_i(1)).$$

Making use of (2.5) and the fact that $D(x_N^{N-i}) = [\iota_i]$, the proposition follows.

Q. E. D.

Now we summarize properties of $\tilde{d}_*$. The properties of oriented bordism version have been verified by Ossa [7].

(2.9) Let θ^1 be a trivial complex line bundle. Then

$$\tilde{d}_*[\theta^1 \oplus \xi] - \tilde{d}_*[\theta^1 \oplus \eta(\xi)] = [P(\theta^1 \oplus \xi)] [\iota_0].$$

(2.10) Given $[M^{2n}, \phi] \in SF_*^U(S^1)$,

$$J(i^*[M^{2n}, \phi]) \equiv [M^{2n}] x^n \text{ modulo } x^{n+1},$$

where $U^*(BU(1)) = U^*[[x]]$.

(2.11) Let $\tilde{d}_*([\xi \times C^t] = \sum_{u=0}^m a_{m-u} [\iota_u]$, $2m = \dim P(\xi \times C^t)$ and $\tilde{d}_*([\eta \times C^s]) = \sum_{v=0}^n b_{n-v} [\iota_v]$, $2n = \dim P(\eta \times C^s)$. Then

$$\begin{aligned}
 \tilde{d}_*([\xi \times \eta \times C^{t+s}]) &\equiv \sum_{\substack{0 \leq u \leq m \\ 0 \leq v \leq n}} a_{m-u} b_{n-v} [\iota_{u+v+1}] \\
 &\text{modulo } U_*\{[\iota_{m_0}], [\iota_{m_0-1}], \dots, [\iota_0]\},
 \end{aligned}$$

where $m_0 = \max\{m, n\}$.

By (2.11) we can see the following

THEOREM 2.12. $J: \mathfrak{M}_*^U(S^1) \rightarrow U^*(BU(1))$ is a ring homomorphism.

§ 3. On the ring structure of $\mathfrak{M}_*^U(S^1) \otimes Z_2$

Let $f_m: P(\xi_m \times C^k) \rightarrow CP^N$ be the classifying map for the canonical line bundle $\eta(\xi_m \times C^k)$. The cohomology ring $H^*(P(\xi_m \times C^k); Z)$ is isomorphic to the $H^*(CP^m)$ -module generated by $1, y, y^2, \dots, y^k, y = c^1(\eta(\xi_m \times C^k))$, with the relation

$$(*) \quad y^{k+1} = -\pi^*(\hat{x}_m)y^k,$$

where $\hat{x}_m = c^1(\xi_m)$ and $\pi: P(\xi_m \times C^k) \rightarrow CP^m$ is the projection. Then we have

LEMMA 3.1.

$$f_{m!}(\pi^*(\hat{x}_m^s)y^t) = \begin{cases} (-1)^t \hat{x}_m^{s+t+N-m-k} & \text{if } 0 \leq s \leq m \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let $f_{m!}(\pi^*(\hat{x}_m^s)y^t) = b\hat{x}_N^l$ where $b \in Z$ and $l = s + t + N - m - k$. Denote by $[M]$ the homology fundamental class of an orientable manifold M . Noting that $f_m^!(\xi_N) = \eta(\xi_m \times C^k)$, we have

$$b = \langle (-1)^{N-l} y^{N-l+t} \pi^*(\hat{x}_m^s), [P(\xi_m \times C^k)] \rangle.$$

Since $[P(\xi_m \times C^k)]$ is the dual class for $\{c^1(\overline{\eta(\xi_m \times C^k)})^k \cup \pi^*(\hat{x}_m^m) = (-1)^k y^k \pi^*(\hat{x}_m^m)\}$, it follows from $(*)$ that

$$b = \begin{cases} (-1)^t & \text{if } 0 \leq s \leq m \\ 0 & \text{otherwise.} \end{cases} \quad \text{Q. E. D.}$$

PROPOSITION 3.2. Let $\hat{x}$ be the first cohomology Chern class of the canonical line bundle over $BU(1)$. Then

$$\beta J[\xi_m] \equiv 1 + \sum_{n \geq m+1} \left\{ (m+1) - \sum_{i=0}^m (-1)^i \binom{n}{i} \right\} t_n \hat{x}^n$$

modulo decomposable elements of t_i .

PROOF. Let $f_m: P(\xi_m \times C^k) \rightarrow CP^N$ be the classifying map for the canonical line bundle $\eta(\xi_m \times C^k)$. Then

$$\beta \left\{ \frac{1}{x^{N-(m+k)}} \tilde{J}_N[[\xi \times C^k]] \right\}$$

$$\begin{aligned}
&= \frac{c_t(\xi_N)^{m+1}}{c^1(\xi_N)^{N-(m+k)}} f_{m!} \left\{ \frac{1}{\pi^* \{c_t(\xi_m)\}^{m+1} c_t(\pi^1 \xi_m \otimes \eta(\xi_m \times C^k))} \right\} \\
&\equiv \frac{1}{\hat{x}_N^{N-(m+k)}} \{1 + \sum_{n \geq 1} (m+1) t_n \hat{x}_N^n\} \cup \\
&\quad f_{m!} \{ \{1 - \sum_{n \geq 1} (m+1) t_n \pi^*(\hat{x}_m^n)\} \cup \{1 - \sum_{n \geq 1} t_n (-\pi^*(\hat{x}_m) - y)^n\} \}
\end{aligned}$$

modulo decomposable elements of t_i .

By virtue of Lemma 3.1 the proposition follows.

Q. E. D.

PROPOSITION 3.3. Given $[M^{2n}] \in \mathfrak{M}_*^U(S^1)$,

$$\beta J[M^{2n}] = \langle c_t(v_M), [M^{2n}] \rangle (\hat{x}^n + \text{higher terms of } \hat{x}).$$

PROOF. $[M^{2n}] \in \mathfrak{M}_*^U(S^1)$ is represented by a trivial $(k+1)$ -dimensional complex vector bundle $\underline{C}^{k+1}$ over M^{2n} and $\tilde{d}_*[\underline{C}^{k+1}]$ has a representative

$$P(\underline{C}^{k+1}) = M^{2n} \times CP^k \xrightarrow{\pi_2} CP^k \subset {}^i CP^{n+k}$$

where π_2 is the projection and i is the inclusion. The Gysin homomorphism $\pi_{2!}: H^*(M^{2n} \times CP^k) \rightarrow H^*(CP^k)$ maps $\alpha \times \hat{x}_k^s$ into $\langle \alpha, [M^{2n}] \rangle \hat{x}_k^s$ and $i_!: H^*(CP^k) \rightarrow H^*(CP^{n+k})$ maps $\hat{x}_k^s$ into $\hat{x}_{n+k}^{s+n}$. Hence

$$\begin{aligned}
&\beta(\tilde{d}_*[\underline{C}^{k+1}]) \\
&= i_! \pi_{2!}(c_t(\tilde{v}_{i\pi_2})) \\
&= i_! \pi_{2!}(c_t(v_M) \times c_t(\xi_k)^n) \\
&= \langle c_t(v_M), [M^{2n}] \rangle c_t(\xi_k)^n \{c^1(\xi_k)\}^n.
\end{aligned}$$

Q. E. D.

J. Milnor has showed that the complex cobordism ring U_* is the integral polynomial ring on classes $[M_i]$ of dimension $2i$. The $2i$ -dimensional weakly almost complex manifold M_i may be taken to be

$$c_{(i)}(v_{M_i})[M_i] = \begin{cases} 1 & \text{if } i \neq p^s - 1 \text{ for any prime } p \\ p & \text{if } i = p^s - 1 \text{ for some prime } p \text{ and integer } s \end{cases}$$

(cf. [10]). Moreover if $i = p^s - 1$ for some prime p and integer s , then the M_i may be taken to be $c_{\omega}(v_M)[M_i] \equiv 0$ modulo p (cf. [4], [10]).

COROLLARY 3.4. The complex cobordism ring U_* is the integral ring on

classes $[M_i]$ of dimension $2i$ which satisfy

- (1) if $i \neq p^s - 1$ for any prime p , then

$$\beta J[M_i] \equiv t_i \hat{x}^i \text{ modulo decomposable elements of } t_i.$$

- (2) if $i = p^s - 1$ for some prime p and s , then

$$\beta J[M_i] \equiv 0 \text{ modulo } p$$

Consider the modulo 2 reduction of the formal power series in Proposition 3.2. Let

$$\beta J[\xi_m] \times 1 = 1 + \sum_{n \geq m+1} a_{m,n} t_n \hat{x}^n \text{ in } U^*(BU(1)) \otimes Z_2$$

where $a_{m,n} \equiv (m+1) - \sum_{i=0}^m (-1)^i \binom{n}{i}$ in Z_2 .

We then have:

LEMMA 3.5. Suppose that $n+1 \leq l_1 < l_2 < \dots < l_n \leq 2n+1$ or $n+2 \leq l_1 < l_2 < \dots < l_n \leq 2n+2$, and suppose that each l_j is not of form $2^s - 1$. Then the matrix

$$\begin{bmatrix} a_{1,l_1} & a_{1,l_2} & \dots & a_{1,l_n} \\ a_{2,l_1} & a_{2,l_2} & \dots & a_{2,l_n} \\ \dots & \dots & \dots & \dots \\ a_{n,l_1} & a_{n,l_2} & \dots & a_{n,l_n} \end{bmatrix}$$

is non-singular.

Let $J_2 = \beta J \otimes 1: \mathfrak{M}_*(S^1) \otimes Z_2 \rightarrow H^*(BU(1))[[t_1, t_2, \dots]] \otimes Z_2$. Making use of Lemma 3.5 and Corollary 3.4 ($p=2$), we obtain the following:

PROPOSITION 3.6. (1) Suppose that $n \neq 2^s - 1$ for any s . There is a linear combination $l_n(\xi, M)$ of $[\xi_j]$, $j < n$ and $[M_k]$, $k < n$, such that

$$J_2\{[M_n] + l_n(\xi, M)\} \equiv t_{2n} \hat{x}^{2n} + \text{higher terms of } \hat{x} \\ \text{modulo decomposable elements of } t_i.$$

(2) There is a linear combination $\tilde{l}_n(\xi, M)$ of $[\xi_j]$, $j < n$ and $[M_k]$, $k < n$ such that if $n \neq 2^t - 1$ for any t ,

$$J_2\{[\xi_n] + \tilde{l}_n(\xi, M)\} \equiv t_{2n+1} \hat{x}^{2n+1} + \text{higher terms of } \hat{x} \\ \text{modulo decomposable elements of } t_i$$

and if $n = 2^t - 1$ for some t ,

$$J_2\{[\bar{\xi}_n] + \bar{l}_n(\xi, M)\} \equiv t_{2^n} \hat{x}^{2^n} + \text{higher terms of } \hat{x}$$

modulo decomposable elements of t_i .

Let $F_n = \bigoplus_{i \leq n} \{ \bigoplus_k U_{2i}(BU(k)) \}$. Then $[M_n] + l_n(\xi, M) \in F_n$ but $[M_n] + l_n(\xi, M) \notin F_{n-1}$, and $[\bar{\xi}_n] + \bar{l}_n(\xi, M) \in F_n$ but $[\bar{\xi}_n] + \bar{l}_n(\xi, M) \notin F_{n-1}$. We thus have the following:

THEOREM 3.7. *There exist $\eta_2, \eta_4, \dots, \eta_k$ ($k \neq 2^t - 1$ for any t), $\dots$, and η_0 such that*

- (1) $\mathfrak{M}_*(S^1) \otimes Z_2 \cong Z_2[\eta_0, \eta_2, \dots, \eta_k, \dots, [M_1], \dots, [M_{2^s-1}], \dots]$,
- (2) $J_2[\eta_0] = 1$,
 $J_2[\eta_k] = (t_k + d_k(t))\hat{x}^k + \text{higher terms of } \hat{x}$, where $d_k(t)$ is a decomposable element of t_i ,
 $J_2[M_{2^s-1}] = 0$,
- (3) $\eta_{2^s} \in F_s$ but $\eta_{2^s} \notin F_{s-1}$ and $\eta_{2^s+1} \in F_s$ but $\eta_{2^s+1} \notin F_{s-1}$.

§4. A proof of Theorem 1.1

Let (M^{2^n}, ϕ) be a weakly almost complex manifold with a semi-free S^1 -action preserving the complex structure. Applying the homomorphism i_* of Section 2 to the $[M^{2^n}, \phi]$, it is seen by Theorem 3.7 that

$$i_*([M^{2^n}, \phi]) \otimes 1 \equiv \sum \alpha_\omega \eta_\omega \quad \text{modulo the submodule generated by} \\ \text{polynomials of } [M_1], [M_3], \dots, [M_{2^s-1}], \dots$$

where $\alpha_\omega \in Z_2$ and η_ω denotes a monomial $\eta_{i_1} \eta_{i_2} \cdots \eta_{i_k}$ for $\omega = (i_1, i_2, \dots, i_k)$. (2.4) and (2.10) imply

$$J_2(i_*([M^{2^n}, \phi]) \otimes 1) \\ = \sum \langle c_\omega(v_M), [M^{2^n}] \rangle t_\omega \hat{x}^{\|\omega\|} \otimes 1 + \text{higher terms of } \hat{x}$$

where t_ω denotes a monomial $t_{i_1} t_{i_2} \cdots t_{i_k}$ for $\omega = (i_1, i_2, \dots, i_k)$ and $\|\omega\| = i_1 + i_2 + \cdots + i_k$. On the other hand,

$$J_2(\sum \alpha_\omega \eta_\omega) = \sum \alpha_\omega (t_\omega \hat{x}^\omega \otimes 1 + \text{higher terms of } \hat{x}).$$

We note that each partition ω contains no number of type $2^s - 1$ and there is a

partition $\tilde{\omega} = (2j_1, 2j_2, \dots, 2j_s, 2j'_1 + 1, 2j'_2 + 1, \dots, 2j'_t + 1)$ of $2n$ with $\alpha_{\tilde{\omega}} \neq 0$. Let $l = j_1 + \dots + j_s + j'_1 + \dots + j'_t$. Then $\eta_{\tilde{\omega}} \in F_l$ but $\eta_{\tilde{\omega}} \notin F_{l-1}$. Since $5(j_1 + j_2 + \dots + j_s + j'_1 + \dots + j'_t) \geq 2n$, we get that some component of the fixed set is of dimension $\geq 4n/5$.

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