ELLIPTIC UNITS AND TWO VARIABLE p -ADIC L -FUNCTIONS

By

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§1. Introduction

Let p be an odd prime number and for each $n \geq 0$, let $\zeta_{p^{n+1}}$ be a primitive p^{n+1} -th root of unity. Let $\mathbf{Q}_p$ be the rational p -adic number field and denote by U_n the group of principal units in $\mathbf{Q}_p(\zeta_{p^{n+1}})$. Let $\bar{C}_n \subseteq U_n$ be the closure of the group of cyclotomic units of $\mathbf{Q}(\zeta_{p^{n+1}})$ which are congruent to 1 modulo $(\zeta_{p^{n+1}} - 1)$, and put $Y_\infty = \varprojlim U_n / \bar{C}_n$ where the projective limit is taken relative to the norm maps. Then the Iwasawa algebra $\mathbf{Z}_p[[T]]$ operates naturally on Y_∞ and it is known that for each non-trivial even character κ of $\text{Gal}(\mathbf{Q}_p(\zeta_p)/\mathbf{Q}_p)$, the κ -eigenspace of Y_∞ is isomorphic to $\mathbf{Z}_p[[T]]/(g_\kappa)$, where g_κ is a power series which is essentially the p -adic L -function.

The elliptic analogue of this fact in one variable case is treated in Coates-Wiles [1] and Goldstein [3]. An analogous result to Theorem 1 of Coates-Wiles [1] in two variable case is established in Yager [6].

In the present paper, making a minor modification of the definition of the elliptic units in [6], we shall relate two variable power series which interpolate special values of primitive L -functions to the structure of a certain Iwasawa module. Consequently we shall obtain analogous results to Theorem 3.10.1) and Theorem 3.11.1) of Goldstein [3] in two variable case.

§2. Notations

Let K be an imaginary quadratic field with class number 1 and discriminant $-d_k$ lying inside the complex plane $\mathbf{C}$, and denote by $\mathcal{O}$ the ring of integers of K . Let E be an elliptic curve defined over K with complex multiplication by $\mathcal{O}$. We denote by ψ_E the Grossencharacter of E over K , and by $\mathfrak{f}$ the conductor of ψ_E . Fix a Weierstrass model for E .

$$(2.1) \quad y^2 = 4x^3 - g_2x - g_3$$

such that $g_2, g_3 \in O$ and the discriminant of (2.1) is divisible only by primes dividing 6f. Let $\mathfrak{P}(z)$ be the Weierstrass function associated with (2.1), and L the period lattice of $\mathfrak{P}(z)$. Fix an element $\Omega_\infty \in L$ such that $L = \Omega_\infty O$. Put $\xi(z) = (\mathfrak{P}(z), \mathfrak{P}'(z))$. As usual, we identify O with the endomorphism ring of E in such a way that the endomorphism corresponding to α in O is given by $\xi(z) \mapsto \xi(\alpha z)$.

We fix a prime $\mathfrak{p}$ of K lying above p such that $(p, 6d_k f) = (1)$ and $\mathfrak{p}$ is of degree 1. Hence $(p) = \mathfrak{p}\mathfrak{p}^*$. Put $\pi = \psi_E(\mathfrak{p})$ and $\pi^* = \psi_E(\mathfrak{p}^*)$. For each $\alpha \in O$, let E_α be the kernel of the endomorphism α of E . For any $0 \leq m, n \in \mathbb{Z}$, put

$$F_m = K(E_{\pi^{*m+1}}), \quad K_n = K(E_{\pi^{n+1}}) \quad \text{and} \quad K_{n,m} = F_m(E_{\pi^{n+1}}).$$

The extension F_m/K is unramified at $\mathfrak{p}$ and the extension K_n/K is totally ramified at $\mathfrak{p}$. Further, the number of primes of F_m lying above $\mathfrak{p}$ is bounded for m .

We fix a prime $\mathfrak{p}_m$ of F_m in such a way that $\mathfrak{p}_m$ lies above $\mathfrak{p}$ and that $\mathfrak{p}_{m+1}$ lies above $\mathfrak{p}_m$. For any prime v of F_m lying above $\mathfrak{p}$ let $\mathfrak{F}_m^{(v)}$ be the completion of F_m at v , and $\mathfrak{K}_{n,m}^{(v)}$ the completion of $K_{n,m}$ at the unique prime above v , and denote by $O_m^{(v)}$ the ring of integers of $\mathfrak{F}_m^{(v)}$. For simplicity, we shall omit the superscript for the prime when referring to completions at or above $\mathfrak{p}_m$. We denote by $K_{\mathfrak{p}}$ the completion of K at $\mathfrak{p}$ and by $O_{\mathfrak{p}}$ the ring of integers of $K_{\mathfrak{p}}$. We shall identify $O_{\mathfrak{p}}$ with $\mathbb{Z}_p$, the ring of integers of $\mathbb{Q}_p$.

Put

$$K_\infty = \bigcup_{n,m \geq 0} K_{n,m}, \quad F_\infty = \bigcup_{m \geq 0} F_m, \quad \text{and} \quad \mathfrak{F}_\infty = \bigcup_{m \geq 0} \mathfrak{F}_m.$$

Let $\tau_{\mathfrak{p}}$ denote the Artin symbol $(\mathfrak{p}, F_\infty/K)$. We always view our global fields as lying inside the complex numbers and equipped with embeddings into their completions. Then $\tau_{\mathfrak{p}}$ induces the Frobenius automorphism for the extension $\mathfrak{F}_\infty/K_{\mathfrak{p}}$.

We put $G_\infty = \text{Gal}(K_\infty/K)$ and $\Gamma = \text{Gal}(K_\infty/K_{0,0})$. Let $E_{\pi^\infty} = \bigcup_{n \geq 0} E_{\pi^{n+1}}$ and $E_{\pi^{*\infty}} = \bigcup_{m \geq 0} E_{\pi^{*m+1}}$. Let $\kappa_1: G_\infty \rightarrow \mathbb{Z}_p^\times$ and $\kappa_2: G_\infty \rightarrow \mathbb{Z}_p^\times$ be respectively the characters giving the action of G_∞ on E_{π^∞} and $E_{\pi^{*\infty}}$. Then, $(\kappa_1, \kappa_2): G_\infty \cong \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ and $\Gamma \cong (1+p\mathbb{Z}_p) \times (1+p\mathbb{Z}_p)$ and we have $G_\infty = \Gamma \times \Delta$, where Δ is the product of the two cyclic group of order $p-1$ which can be identified with $\text{Gal}(K_{0,0}/K)$. Let χ_1 and χ_2 denote the restriction of κ_1 and κ_2 to Δ respectively. Then χ_1 and χ_2 generate $\text{Hom}(\Delta, \mathbb{Z}_p^\times)$.

For any $\mathbb{Z}_p[\Delta]$ -module A and any $(i_1, i_2) \in (\mathbb{Z}/(p-1)\mathbb{Z})^2$, we denote by

$A^{(i_1, i_2)}$ the $\chi_1^{i_1} \chi_2^{i_2}$ -eigenspace of A , that is, the submodule of A on which Δ acts via $\chi_1^{i_1} \chi_2^{i_2}$. We have the canonical decomposition,

$$(2.2) \quad A = \bigoplus_{(i_1, i_2) \in (\mathbf{Z}/(p-1)\mathbf{Z})^2} A^{(i_1, i_2)}.$$

Let $A = \mathbf{Z}_p[[T_1, T_2]]$ be the ring of formal power series in indeterminates T_1 and T_2 with coefficients in $\mathbf{Z}_p$. Choose a topological generator u of $1 + p\mathbf{Z}_p$ and let γ_1 and γ_2 be the unique elements of Γ such that $\kappa_1(\gamma_1) = \kappa_2(\gamma_2) = u$ and $\kappa_1(\gamma_2) = \kappa_2(\gamma_1) = 1$. Then γ_1 and γ_2 are topological generators of Γ and any compact $\mathbf{Z}_p$ -module B on which Γ acts continuously admits a unique structure of A -module such that

$$(1 + T_1)x = \gamma_1 x \quad \text{and} \quad (1 + T_2)x = \gamma_2 x \quad \text{for all } x \in B.$$

We put $\omega_{n,1} = (1 + T_1)^{p^n} - 1$ and $\omega_{n,2} = (1 + T_2)^{p^n} - 1$.

The Galois group G_∞ acts naturally on the rings $\prod_{v|p} \mathfrak{R}_{n,m}^{(v)}$ and $\prod_{v|p} \mathfrak{F}_m^{(v)}$. We embed $K_{n,m}$ and F_m in these rings via the diagonal map. We put $\mathbf{O}_m = \prod_{v|p} \mathbf{O}_m^{(v)}$.

Let $U_{n,m}'^{(v)}$ denote the units of $\mathfrak{R}_{n,m}^{(v)}$ and $U_{n,m}^{(v)}$ the subgroup consisting of those units which are congruent to 1 modulo the maximal ideal. Put $\mathbf{U}_{n,m}' = \prod_{v|p} U_{n,m}'^{(v)}$, $\mathbf{U}_{n,m} = \prod_{v|p} U_{n,m}^{(v)}$, $\mathbf{U}_\infty' = \varprojlim \mathbf{U}_{n,m}'$ and $\mathbf{U}_\infty = \varprojlim \mathbf{U}_{n,m}$, where the projective limits are taken relative to the usual norm maps on the $\prod_{v|p} \mathfrak{R}_{n,m}^{(v)}$. The ring $\mathbf{Z}_p[G_\infty]$ operates naturally on $\mathbf{U}_\infty$. Hence $\mathbf{U}_\infty$ is a compact Γ -module and also a A -module.

Let $\hat{E}$ be the formal group giving the kernel of reduction modulo p on E . The parameter of $\hat{E}$ is

$$(2.3) \quad t = -2x/y = -2\mathfrak{P}(z)/\mathfrak{P}'(z) = e(z).$$

Viewing z as a parameter of the formal additive group G_a , $e(z)$ is the exponential map of $\hat{E}$. We write $\lambda: \hat{E} \rightarrow G_a$ for the logarithm of $\hat{E}$ which is the inverse of (2.3).

Let $\mathbf{C}_p$ be the completion of the algebraic closure of $\mathfrak{F}_\infty$. We denote by ord_p the additive valuation of $\mathbf{C}_p$ normalized such that $\text{ord}_p(p) = 1$.

For any integral ideal $\mathfrak{g}$ of K , we write $\text{ord}_p \mathfrak{g}$ for the ordinal of the ideal $\mathfrak{g}$ with respect to p . We denote by $k_{\mathfrak{g}}$ the positive rational integer which generates $\mathfrak{g} \cap \mathbf{Z}$. Let $Cl(\mathfrak{g})$ denote the group of ray classes modulo $\mathfrak{g}$ of K , and $R_{\mathfrak{g}}$ the ray class field modulo $\mathfrak{g}$.

For any character χ of $\text{Gal}(K_{n,m}/K)$, we denote by $\mathfrak{f}_\chi$ the conductor of the fixed field of $\text{Ker}\chi$ over K . For any integral ideal $\mathfrak{g}$ of K divisible by $\mathfrak{f}_\chi$, χ induces a character of $Cl(\mathfrak{g})$ via the reciprocity map. We denote this induced character by χ' . In the special case $\mathfrak{g}=\mathfrak{f}_\chi$, χ' is a primitive character of $Cl(\mathfrak{f}_\chi)$, which we denote by $\tilde{\chi}'$.

§3. Elliptic units

For any lattice $\mathfrak{Q}$ in the complex plane, we put $\sigma(z, \mathfrak{Q}) = z \prod_{\substack{\omega \in \mathfrak{Q} \\ \omega \neq 0}} (1 - (z/\omega)) \exp((z/\omega) + (1/2)(z/\omega)^2)$ and $\theta(z, \mathfrak{Q}) = \Delta(\mathfrak{Q}) \exp(-6g_2(\mathfrak{Q})z^2) \sigma(z, \mathfrak{Q})^{12}$, where $\Delta(\mathfrak{Q})$ is the discriminant of $\mathfrak{Q}$ and $g_2(\mathfrak{Q}) = \lim_{s \rightarrow 0^+} \sum_{\substack{\omega \in \mathfrak{Q} \\ \omega \neq 0}} \omega^{-2} |\omega|^{-2s}$.

For any integral ideal $\mathfrak{a}$ of K , we define

$$\Theta(z, \mathfrak{a}) = \theta(z, L)^{N\mathfrak{a}} / \theta(z, \mathfrak{a}^{-1}L)$$

where $N\mathfrak{a}$ is the absolute norm of $\mathfrak{a}$. It is known that $\Theta(z, \mathfrak{a})$ is an elliptic function for the lattice L .

For each $(i_1, i_2) \in (\mathbb{Z}/(p-1)\mathbb{Z})^2$, we denote the greatest common divisor of $\mathfrak{f}$ and $\mathfrak{f}_{\chi^{i_1, i_2}}$ by $\mathfrak{f}^{(i_1, i_2)}$ and fix an ideal $\mathfrak{c}^{(i_1, i_2)}$ of K such that $\mathfrak{f} \subseteq \mathfrak{c}^{(i_1, i_2)} \subseteq \mathfrak{f}^{(i_1, i_2)}$. The case $\mathfrak{c}^{(i_1, i_2)} = \mathfrak{f}$ was treated in Yager [6]. In the present paper, we shall treat only two cases of $\mathfrak{c}^{(i_1, i_2)} = \mathfrak{f}^{(i_1, i_2)}$ and $\mathfrak{c}^{(i_1, i_2)} = \mathfrak{f}$.

Fix a generator $c^{(i_1, i_2)}$ of $\mathfrak{c}^{(i_1, i_2)}$. For any $n, m \geq 0$, we put $F_m^{(i_1, i_2)} = F_m(E_{c^{(i_1, i_2)}})$ and $K_{n,m}^{(i_1, i_2)} = K_{n,m}(E_{c^{(i_1, i_2)}})$. By complex multiplication theory, we have $R_{c^{(i_1, i_2)}p^{*m+1}} \subseteq F_m^{(i_1, i_2)} \subseteq R_{\mathfrak{f}p^{*m+1}}$ and $R_{c^{(i_1, i_2)}p^{n+1}p^{*m+1}} \subseteq K_{n,m}^{(i_1, i_2)} \subseteq R_{\mathfrak{f}p^{n+1}p^{*m+1}}$. Put $\rho_m^{(i_1, i_2)} = \Omega_\infty / c^{(i_1, i_2)} \pi^{*m+1}$ and let $B_m^{(i_1, i_2)}$ be a set of integral ideals of K prime to $\mathfrak{f}p^*$ such that $\{(\mathfrak{b}, F_m^{(i_1, i_2)}/K) : \mathfrak{b} \in B_m^{(i_1, i_2)}\}$ is precisely $\text{Gal}(F_m^{(i_1, i_2)}/F_m)$. Let $\mathfrak{I}$ denote the set of integral ideals of K which are prime to $6p\mathfrak{f}$. For each $\mathfrak{a} \in \mathfrak{I}$ we define

$$A_m^{(i_1, i_2)}(z, \mathfrak{a}) = \prod_{\mathfrak{b} \in B_m^{(i_1, i_2)}} \Theta(z + \psi_E(\mathfrak{b})\rho_m^{(i_1, i_2)}, \mathfrak{a}).$$

Then $A_m^{(i_1, i_2)}(z, \mathfrak{a})$ is a rational function of $\mathfrak{P}(z)$ and $\mathfrak{P}'(z)$ with coefficients in F_m , and is independent of the choice of the set $B_m^{(i_1, i_2)}$. We put

$$\delta = \{\mu : \mathfrak{I} \rightarrow \mathbb{Z} \mid \mu(\mathfrak{a}) = 0 \text{ for almost all } \mathfrak{a} \in \mathfrak{I} \text{ and } \sum_{\mathfrak{a} \in \mathfrak{I}} (N\mathfrak{a} - 1)\mu(\mathfrak{a}) = 0\}$$

For each $\mu \in \delta$, we define

$$\Theta(z: \mu) = \prod_{a \in \mathfrak{Z}} \Theta(z, a)^{\mu(a)} \quad \text{and} \quad \Lambda_m^{(i_1, i_2)}(z: \mu) = \prod_{a \in \mathfrak{Z}} \Lambda_m^{(i_1, i_2)}(z, a)^{\mu(a)}.$$

We put $\tau_n = \Omega_\infty / \pi^{n+1}$ and $u_n = \varepsilon(\tau_n)$. Choose $\varepsilon_n \in O$ such that $\varepsilon_n \pi^* \equiv 1 \pmod{\mathfrak{p}^{n+1}}$. Put $C_{n,m}'^{(i_1, i_2)} = \{ \Lambda_m^{(i_1, i_2)}(\varepsilon_n^{m+1} \tau_n: \mu) \mid \mu \in \delta \}$. Then, $C_{n,m}'^{(i_1, i_2)}$ is a subgroup of the group of units in $K_{n,m}$ and is stable under the action of G_∞ .

For any $\mu \in \delta$, $\Lambda_m^{(i_1, i_2)}(z: \mu)$ belongs to $F_m(\mathfrak{P}(z), \mathfrak{P}'(z))$, and so, for any $\sigma \in \text{Gal}(F_\infty/K)$, we can define a function $\Lambda_m^{(i_1, i_2)\sigma}(z: \mu)$ by applying σ to the coefficients of $\Lambda_m^{(i_1, i_2)}(z: \mu)$. Put

$$e_{n,m}^{(i_1, i_2)}(\mu) = \Lambda_m^{(i_1, i_2)\tau_n^{-n}}(\varepsilon_n^{m+1} \tau_n: \mu).$$

Then, embedding $C_{n,m}'^{(i_1, i_2)}$ into $U_{n,m}'$ via diagonal map, $(e_{n,m}^{(i_1, i_2)}(\mu))_{n,m \geq 0} \in U_\infty'$. We denote $(e_{n,m}^{(i_1, i_2)}(\mu))_{n,m \geq 0}$ by $e^{(i_1, i_2)}(\mu)$.

If $\beta = (\beta_{n,m}^{(v)}) \in U_\infty'$, then for each integer $m \geq 0$ and each prime v of F_m lying above p , there is a unique unit power series $C_{m,\beta}^{(v)}(T)$ in $O_m^{(v)}[[T]]$ such that $\beta_{n,m}^{(v)} = (C_{m,\beta}^{(v)})^{\tau_p^{-n}}(u_n)$ for all $n \geq 0$ (Yager [6] and Coleman [2]). We write $\prod_{v|p} O[[T]]$ as $O_m[[T]]$, which has the natural structure of G_∞ -module. We put $C_{m,\beta}(T) = (C_{m,\beta}^{(v)}(T)) \in O_m[[T]]$.

THEOREM 3.1. *For each $\mu \in \delta$, we have*

$$C_{m,e^{(i_1, i_2)}(\mu)}(T) = \Lambda_m^{(i_1, i_2)}(\pi^{*-(m+1)}\lambda(T): \mu).$$

§ 4. The two variable Γ -transform

In this section, we summarize some of the basic properties of the two variable Γ -transform which will be used later.

Let $O_{\mathbb{C}_p}$ denote the ring of integers of $\mathbb{C}_p$, and let μ be a measure on $\mathbb{Z}_p^2$ taking values in $O_{\mathbb{C}_p}$. Then μ corresponds to a power series

$$(4.1) \quad f_\mu(T_1, T_2) = \sum_{n,m \geq 0} \left(\int_{\mathbb{Z}_p^2} \binom{x_1}{n} \binom{x_2}{m} d\mu \right) T_1^n T_2^m \in O_{\mathbb{C}_p}[[T_1, T_2]].$$

This correspondence is bijective. For $f(T_1, T_2) \in O_{\mathbb{C}_p}[[T_1, T_2]]$ we denote by μ_f the $O_{\mathbb{C}_p}$ -valued measure which corresponds to f .

For any $x \in \mathbb{Z}_p^\times$, we write $x = \omega(x)\langle x \rangle$, where $\omega(x)$ is a $(p-1)$ -st root of unity and $\langle x \rangle \equiv 1 \pmod{p}$, and let $\ell(x) \in \mathbb{Z}_p$ be such that $\langle x \rangle = u^{\ell(x)}$.

Let $(i_1, i_2) \in (\mathbb{Z}/(p-1)\mathbb{Z})^2$. For any $f \in O_{\mathbb{C}_p}[[T_1, T_2]]$, we define a Γ -transform $\Gamma_f^{(i_1, i_2)}: \mathbb{Z}_p^2 \rightarrow O_{\mathbb{C}_p}$ by

$$\Gamma_f^{(i_1, i_2)}(s_1, s_2) = \int_{\mathbf{z}_p^\times \times \mathbf{z}_p^\times} \langle x_1 \rangle^{s_1} \langle x_2 \rangle^{s_2} \omega^{i_1}(x_1) \omega^{i_2}(x_2) d\mu_f.$$

Then $\Gamma_f^{(i_1, i_2)}(s_1, s_2) = f^{(i_1, i_2)}(u^{s_1} - 1, u^{s_2} - 1)$, where $f^{(i_1, i_2)}(T_1, T_2) = \sum_{n, m \geq 0} \left(\int_{\mathbf{z}_p^\times \times \mathbf{z}_p^\times} \binom{\ell(x_1)}{n} \binom{\ell(x_2)}{m} \omega^{i_1}(x_1) \omega^{i_2}(x_2) d\mu_f \right) T_1^n T_2^m \in O_{\mathbf{C}_p}[[T_1, T_2]]$.

Let ν be a Dirichlet character with conductor p^m . For any $f \in O_{\mathbf{C}_p}[[T_1, T_2]]$, we define power series $f_{(\nu, 1)}$ and $f_{(1, \nu)}$ by

$$(4.2) \quad f_{(\nu, 1)} = (1/\tau(\nu^{-1}, \zeta_{p^m})) \sum_{a=1}^{p^m} \nu^{-1}(a) f(\zeta_{p^m}^a(1 + T_1) - 1, T_2)$$

and

$$(4.3) \quad f_{(1, \nu)} = (1/\tau(\nu^{-1}, \zeta_{p^m})) \sum_{a=1}^{p^m} \nu^{-1}(a) f(T_1, \zeta_{p^m}^a(1 + T_2) - 1)$$

where ζ_{p^m} is an arbitrary primitive p^m -th root of unity and $\tau(\nu^{-1}, \zeta_{p^m}) = \sum_{a=1}^{p^m} \nu^{-1}(a) \zeta_{p^m}^a$, the Gauss sum attached to ν^{-1} and ζ_{p^m} . It is easy to see that the right hand sides of (4.2) and (4.3) are independent of the choice of ζ_{p^m} and that they belong to $O_{\mathbf{C}_p}[[T_1, T_2]]$. If ν_1 and ν_2 are Dirichlet characters with conductor a power of p , then $(f_{(\nu_1, 1)})_{(1, \nu_2)} = (f_{(1, \nu_2)})_{(\nu_1, 1)}$. We denote this power series by $f_{(\nu_1, \nu_2)}$. If (j, k) is in $(\mathbf{Z}/(p-1)\mathbf{Z})^2$, then $\Gamma_{f_{(\omega^j, \omega^k)}}^{(i_1, i_2)}(s_1, s_2) = \Gamma_f^{(i_1+j, i_2+k)}(s_1, s_2)$. If φ_1 and φ_2 are Dirichlet characters of the second kind, then $\Gamma_{f_{(\varphi_1, \varphi_2)}}^{(i_1, i_2)}(s_1, s_2) = f^{(i_1, i_2)}(\varphi_1(u)u^{s_1} - 1, \varphi_2(u)u^{s_2} - 1)$. Further, we have $(f_{(\varphi_1, \varphi_2)})_{(\omega^j, \omega^k)} = f_{(\varphi_1 \omega^j, \varphi_2 \omega^k)}$.

For any $f \in O_{\mathbf{C}_p}[[T_1, T_2]]$, put

$$U_1 f = f(T_1, T_2) - 1/p \sum_{a=1}^p f(\zeta_p^a(1 + T_1) - 1, T_2)$$

and

$$U_2 f = f(T_1, T_2) - 1/p \sum_{a=1}^p f(T_1, \zeta_p^a(1 + T_2) - 1).$$

Then, $U_1 f$ and $U_2 f$ are in $O_{\mathbf{C}_p}[[T_1, T_2]]$. Let D_j be the operator $(1 + T_j)\partial/\partial T_j$ on $\mathbf{C}_p[[T_1, T_2]]$ for $j=1, 2$. We have

$$\begin{aligned} \int_{\mathbf{z}_p^\times \times \mathbf{z}_p} \phi d\mu_f &= \int_{\mathbf{z}_p^2} \phi d\mu_{(U_1 f)}, \quad \int_{\mathbf{z}_p \times \mathbf{z}_p^\times} \phi d\mu_f = \int_{\mathbf{z}_p^2} \phi d\mu_{(U_2 f)} \\ \int_{\mathbf{z}_p^\times \times \mathbf{z}_p^\times} \phi d\mu_f &= \int_{\mathbf{z}_p^2} \phi d\mu_{(U_1 U_2) f} \end{aligned}$$

and $\int_{\mathbb{Z}_p^2} \phi d\mu_{(D_j f)} = \int_{\mathbb{Z}_p^2} \phi(x_1, x_2) x_j d\mu_f$ ($j=1, 2$) for any measurable function $\phi: \mathbb{Z}_p^2 \rightarrow O_{C_p}$.

By direct calculation, it is easy to verify that operators $f \mapsto f_{(v_1, v_2)}$, $f \mapsto U_i f$, $f \mapsto D_j f$ for $i, j=1, 2$ are commutative with one another.

LEMMA 4.1. Let $f \in O_{C_p}[[T_1, T_2]]$ and let $g \in C_p[[T_1, T_2]]$ be such that $D_1 g = f$. Then we have

$$\int_{\mathbb{Z}_p^2 \times \mathbb{Z}_p} x_1^{-1} d\mu_f = g(0, 0) - 1/p \sum_{a=1}^p g(\zeta_p^a - 1, 0).$$

PROOF. Let $g = \sum_{l, m \geq 0} a_{l, m} T_1^l T_2^m$. For each $n \geq 0$, put $g_n = \sum_{0 \leq l, m \leq n} a_{l, m} T_1^l T_2^m$, and choose an integer $r_n \geq 0$ such that $p^{r_n} g_n \in O_{C_p}[[T_1, T_2]]$, then,

$$\begin{aligned} \int_{\mathbb{Z}_p^2 \times \mathbb{Z}_p} x_1^{-1} d\mu_f &= \lim_{n \rightarrow \infty} \int_{\mathbb{Z}_p^2 \times \mathbb{Z}_p} x_1^{-1} d\mu_{D_1 g_n} \\ &= \lim_{n \rightarrow \infty} p^{-r_n} \int_{\mathbb{Z}_p^2} d\mu_{U_1(p^{r_n} g_n)} \end{aligned}$$

Hence by (4.1),

$$\int_{\mathbb{Z}_p^2 \times \mathbb{Z}_p} x_1^{-1} d\mu_f = g(0, 0) - 1/p \sum_{a=1}^p g(\zeta_p^a - 1, 0).$$

§5. Calculation of $G_{\beta}^{(i_1, i_2)}(\zeta - 1, \zeta' - 1)$

Let $\hat{O}_{\infty}$ denote the completion of the ring of integers of $\mathfrak{F}_{\infty}$. Let $\eta(T) = \Omega_p T + \dots$ be the isomorphism from $\hat{E}$ to the formal multiplicative group G_m over $\hat{O}_{\infty}$ given in Yager [6], and denote by $i(T)$ the inverse of $\eta(T)$.

For each $\beta \in U_{\infty}$, let $g_{\beta}(T_1, T_2)$ be the two variable power series given by Theorem 5 of Yager [6] and put $h_{\beta}(T_1, T_2) = g_{\beta}(i(T_1), T_2)$. Let $(i_1, i_2) \in (\mathbb{Z}/(p-1)\mathbb{Z})^2$ and put

$$G_{\beta}^{(i_1, i_2)}(T_1, T_2) = (U_1 h_{\beta})^{(i_1-1, -i_2)}(u^{-1}(1+T_1)-1, (1+T_2)^{-1}-1).$$

Then the map $\beta \mapsto G_{\beta}^{(i_1, i_2)}(T_1, T_2)$ is a A -homomorphism from U_{∞} to $\hat{O}_{\infty}[[T_1, T_2]]$ (Theorem 22 of [6]). Further, $G_{\beta}^{(i_1, i_2)}(T_1, T_2) = G_{\beta^{(i_1, i_2)}}^{(i_1, i_2)}(T_1, T_2)$, where $\beta^{(i_1, i_2)}$ denotes the $U_{\infty}^{(i_1, i_2)}$ -component of β under the canonical decomposition (2.2) for $A = U_{\infty}$.

For each $n \geq 0$, let V_n denote the group of all p^n -th roots of unity, and put $V_\infty = \bigcup_{n \geq 0} V_n$.

Let $(\zeta, \zeta') \in V_\infty^2$. In the rest of this section, using results described in the previous section, we calculate $G_\beta^{(i_1, i_2)}(\zeta - 1, \zeta' - 1)$, especially in the case $\beta = \langle e^{(i_1, i_2)}(\mu) \rangle^{(i_1, i_2)}$ for $\mu \in \delta$.

Let φ_ζ and $\varphi_{\zeta'}$ denote the unique Dirichlet characters of the second kind such that $\varphi_\zeta(u) = \zeta$ and $\varphi_{\zeta'}(u) = \zeta'$ and let $p^{n_{\zeta}(i_1)}$ and $p^{n_{\zeta'}(i_2)}$ be the conductors of $\varphi_\zeta \omega^{i_1}$ and $\varphi_{\zeta'} \omega^{i_2}$, respectively. We assume that either $\varphi_\zeta \omega^{i_1}$ or $\varphi_{\zeta'} \omega^{i_2}$ is non-trivial.

For each $m \geq 0$ and $\sigma \in \text{Gal}(F_m/K)$, put $h_{m, \beta, \sigma}(T) = (g_{m, \beta}^\sigma(W))_{\mathfrak{p}_m|W=i(T)}$, where $g_{m, \beta}(T) = \lambda'(T)^{-1} d/dT \log C_{m, \beta}(T) \in \mathbf{O}_m[[T]]$ and $(g_{m, \beta}^\sigma(W))_{\mathfrak{p}_m}$ denotes the $\mathfrak{p}_m$ -component of the power series in $\mathbf{O}_m[[W]]$ given by applying σ to $g_{m, \beta}(W)$. Note that $\kappa_2(\sigma)$ is well defined modulo p^{m+1} , and so, $(1 + T_2)^{\kappa_2(\sigma)}$ is well defined modulo $\omega_{m+1, 2} \mathbf{Z}_p[[T_2]]$. If $m+1 \geq n_{\zeta'}(i_2)$, then we have

$$\begin{aligned} & (U_1 h_\beta)_{(\varphi_\zeta \omega^{i_1}, \varphi_{\zeta'} \omega^{-i_2})} \\ & \equiv \sum_{\sigma \in \text{Gal}(F_m/K)} (U_1 h_{m, \beta, \sigma}(T_1))_{(\varphi_\zeta \omega^{i_1, 1}, (1 + T_2)^{\kappa_2(\sigma)})_{(1, \varphi_{\zeta'} \omega^{-i_2})}} \\ & \pmod{\omega_{m+1, 2} \hat{\mathcal{O}}_\infty[[T_1, T_2]]}. \end{aligned}$$

From the definition of $h_{m, \beta, \sigma}(T)$, it is easy to see

$$D_1 \cdot \log(C_{m, \beta}^\sigma(W))_{\mathfrak{p}_m|W=i(T_1)} = \Omega_p^{-1} h_{m, \beta, \sigma}(T_1).$$

In what follows, we assume that $\zeta_{p^{n+1}}$ is chosen to satisfy

$$i(\zeta_{p^{n+1}} - 1) = u_n = \varepsilon(\Omega_\infty / \pi^{n+1}).$$

Then, taking account also of Lemma 2 of Yager [6], we see for any m with $m+1 \geq n_{\zeta'}(i_2)$ that

$$\begin{aligned} & G_\beta^{(i_1, i_2)}(\zeta - 1, \zeta' - 1) \\ & = \int_{\mathbf{Z}_p^\times \times \mathbf{Z}_p} x_1^{-1} d\mu_{(U_1 h_\beta)_{(\varphi_\zeta \omega^{i_1}, \varphi_{\zeta'} \omega^{-i_2})}} \\ & = \Omega_p \sum_{\sigma \in \text{Gal}(F_m/K)} \{ U_1 (\log(C_{m, \beta}^\sigma(W))_{\mathfrak{p}_m|W=i(T_1)})_{(\varphi_\zeta \omega^{i_1, 1})} \\ & \quad \times ((1 + T_2)^{\kappa_2(\sigma)})_{(1, \varphi_{\zeta'} \omega^{-i_2})} \} |_{(0,0)} \end{aligned}$$

$$= \begin{cases} \frac{\Omega_p}{\tau(\varphi_\zeta^{-1}\omega^{-i_1}, \zeta_{p^{n_\zeta(i_1)}})} \sum_{\sigma \in \text{Gal}(F_m/K)} \sum_{a=1}^{p^{n_\zeta(i_1)}} \varphi_\zeta^{-1}\omega^{-i_1}(a) \varphi_{\zeta'}^{-1}\omega^{-i_2}(\kappa_2(\sigma)) \\ \quad \times \log_p (C_{m,\beta}^\sigma(T))_{\mathfrak{p}_m} |_{T=[a]_E(u_{n_\zeta(i_1)}-1)} \\ \text{if } \varphi_\zeta \omega^{i_1} \text{ is non-trivial} \\ \\ (\Omega_p/p) \sum_{\sigma \in \text{Gal}(F_m/K)} \varphi_\zeta^{-1}\omega^{-i_2}(\kappa_2(\sigma)) \sum_{a=1}^{p-1} \log_p \{(C_{m,\beta}(0))^\sigma / \\ (C_{m,\beta}^\sigma(T))_{\mathfrak{p}_m} |_{T=[a]_E(u_0)}\} \\ \text{if } \varphi_\zeta \omega^{i_1} \text{ is trivial} \end{cases}$$

where $\log_p$ means the p -adic logarithm defined on $\mathbf{C}_p^\times$.

Let us consider the case $\beta = \langle e^{(i_1, i_2)}(\mu) \rangle^{(i_1, i_2)}$. We first have $G_{<e^{(i_1, i_2)}(\mu) > (i_1, i_2)}(\zeta - 1, \zeta' - 1) = G_{<e^{(i_1, i_2)}(\mu) > (\zeta - 1, \zeta' - 1)}$.

It is easy to see

$$(C_{m, e^{(i_1, i_2)}(\mu)}^\sigma(T))_{\mathfrak{p}_m} |_{T=[a]_E(u_n)} = A_m^{(i_1, i_2)\sigma}(\varepsilon_n^{m+1} a \tau_n; \mu)$$

and

$$(C_{m, e^{(i_1, i_2)}(\mu)}(0))^\sigma = A_m^{(i_1, i_2)}(0; \mu)^\sigma$$

for any $n, m \geq 0$ and $\sigma \in \text{Gal}(F_m/K)$ and $a \geq 1$.

If $n+1 \geq n_\zeta(i_1)$ and $m+1 \geq n_{\zeta'}(i_2)$, then $\varphi_\zeta \omega^{i_1}$ and $\varphi_{\zeta'} \omega^{i_2}$ naturally defines a character $\varphi_\zeta^{(1)} \chi_1^{i_1} \varphi_{\zeta'}^{(2)} \chi_2^{i_2}$ of $\text{Gal}(K_{n,m}/K)$ by

$$\varphi_\zeta^{(1)} \chi_1^{i_1} \varphi_{\zeta'}^{(2)} \chi_2^{i_2}(\sigma) = \varphi_\zeta \omega^{i_1}(\kappa_1(\sigma)) \varphi_{\zeta'} \omega^{i_2}(\kappa_2(\sigma))$$

for any $\sigma \in \text{Gal}(K_{n,m}/K)$.

For any integral ideal $\mathfrak{g} \neq (1)$ of K , and any ray class $C \in Cl(\mathfrak{g})$, let $\varphi_g(C) \in R_g$ be the ray class invariant defined in Robert [5]. Then

$$\begin{aligned} & A_m^{(i_1, i_2)}(\varepsilon_n^{m+1} \tau_n, \alpha)^{k_{n,m}^{(i_1, i_2)}} \\ &= N_{K_{n,m}^{(i_1, i_2)}/K_{n,m}}(\varphi_{c(i_1, i_2)} \mathfrak{p}^{n+1} \mathfrak{p}^{*m+1}(C_0)^{N\alpha} / \\ & \quad \varphi_{c(i_1, i_2)} \mathfrak{p}^{n+1} \mathfrak{p}^{*m+1}(C_0)^{\sigma(\alpha)})^{\sigma_{n,m}^{(i_1, i_2)}} \end{aligned}$$

and

$$\begin{aligned} & A_m^{(i_1, i_2)}(0, \alpha)^{k_{c(i_1, i_2)} \mathfrak{p}^{*m+1}} \\ &= N_{F_m^{(i_1, i_2)}/F_m}(\varphi_{c(i_1, i_2)} \mathfrak{p}^{*m+1}(C'_0)^{N\alpha} / \varphi_{c(i_1, i_2)} \mathfrak{p}^{*m+1}(C'_0)^{\sigma(\alpha)}) \end{aligned}$$

for any $\alpha \in \mathfrak{I}$, where $k_{n,m}^{(i_1, i_2)} = k_{c(i_1, i_2)p^{n+1}p^{*m+1}}$, $\sigma_{n,m}^{(i_1, i_2)} = ((\varepsilon_n^{m+1}c^{(i_1, i_2)}\pi^{*m+1} + \pi^{n+1}), R_{c(i_1, i_2)p^{n+1}p^{*m+1}/K})$, $\sigma(\alpha) = (\alpha, R_{c(i_1, i_2)p^{n+1}p^{*m+1}/K})$ and C_0 (resp. C'_0) is the unit ray class of $Cl(c^{(i_1, i_2)}p^{n+1}p^{*m+1})$ (resp. $Cl(c^{(i_1, i_2)}p^{*m+1})$). For any character χ of $Cl(\mathfrak{g})$, we define

$$S_{\mathfrak{g}}^{(p)}(\chi) = \sum_{C \in Cl(\mathfrak{g})} \chi^{-1}(C) \log_p \varphi_{\mathfrak{g}}(C).$$

By class field theory, it is easy to see that $[K_{n,m}^{(i_1, i_2)} : R_{c(i_1, i_2)p^{n+1}p^{*m+1}}]$ is independent of n and m . We denote this degree by $d^{(i_1, i_2)}$.

We put $\eta_{\zeta}^{(i_1, i_2)} = (\chi_1^{i_1} \chi_2^{i_2})'((\varepsilon_{n_{\zeta}}^{m+1}c^{(i_1, i_2)}\pi^{*m+1} + \pi^{n_{\zeta}}(i_1)))$, which is independent of m .

For each $\mu \in \delta$, we define

$$h_{\mu}^{(i_1, i_2)}(T_1, T_2) = \sum_{\alpha \in \mathfrak{I}} \mu(\alpha) (N\alpha - \omega^{i_1}(\psi_E(\alpha))\omega^{i_2}(\bar{\psi}_E(\alpha))(1+T_1)^{\ell(\psi_E(\alpha))}(1+T_2)^{\ell(\bar{\psi}_E(\alpha))}).$$

Then, from direct calculation and Theorem 2 of section 2.3 of Robert [5], we obtain the following

THEOREM 5.1. *Let $(i_1, i_2) \in (\mathbf{Z}/(p-1)\mathbf{Z})^2$, $\mu \in \delta$ and $(\zeta, \zeta') \in V_{\infty}^2$. We assume that either $\varphi_{\zeta}\omega^{i_1}$ or $\varphi_{\zeta'}\omega^{i_2}$ is non-trivial. Let m be an integer such that $m+1 \geq n_{\zeta'}(i_2)$. Then we have*

(1) *If $\varphi_{\zeta}\omega^{i_1}$ is non-trivial*

$$\begin{aligned} G_{<e^{(i_1, i_2)}(\mu)>^{(i_1, i_2)}}(\zeta-1, \zeta'-1) \\ = \frac{\Omega_p d^{(i_1, i_2)} \eta_{\zeta}^{(i_1, i_2)} \zeta^{\ell(c^{(i_1, i_2)})} \zeta'^{\ell(\bar{\pi}^{n_{\zeta}}(i_1))}}{k_{c(i_1, i_2)p^{n_{\zeta}}(i_1)p^{*m+1}} \tau(\varphi_{\zeta}^{-1} \omega^{-i_1}, \zeta_{p^{n_{\zeta}}(i_1)})} \\ \times h_{\mu}^{(i_1, i_2)}(\zeta-1, \zeta'-1) S_{c(i_1, i_2)p^{n_{\zeta}}(i_1)p^{*m+1}}^{(p)}((\varphi_{\zeta}^{(1)} \chi_1^{i_1} \varphi_{\zeta'}^{(2)} \chi_2^{i_2})'). \end{aligned}$$

(2) $G_{<e^{(0, i_2)}(\mu)>^{(0, i_2)}}(0, \zeta'-1)$

$$\begin{aligned} = \frac{\Omega_p d^{(0, i_2)} \omega^{i_2}(\bar{\pi}) \zeta'^{\ell(\bar{\pi})}}{p k_{c(0, i_2)p^{*m+1}}} h_{\mu}^{(0, i_2)}(0, \zeta'-1) \\ \times (p - \zeta'^{\ell(\bar{\pi})} \omega^{i_2}(\bar{\pi})) S_{c(0, i_2)p^{*m+1}}^{(p)}((\varphi_{\zeta'}^{(2)} \chi_2^{i_2})'). \end{aligned}$$

Now let us consider two cases of $c^{(i_1, i_2)} = \bar{f}^{(i_1, i_2)}$ and $c^{(i_1, i_2)} = \bar{f}$. In the first case, we write $f^{(i_1, i_2)}$ for $c^{(i_1, i_2)}$, $\tilde{e}^{(i_1, i_2)}(\mu)$ for $e^{(i_1, i_2)}(\mu)$, $\tilde{d}^{(i_1, i_2)}$ for $d^{(i_1, i_2)}$ and $\tilde{\eta}_\zeta^{(i_1, i_2)}$ for $\eta_\zeta^{(i_1, i_2)}$. In the second case, we write f for $c^{(i_1, i_2)}$, $\tilde{e}^{(i_1, i_2)}(\mu)$ for $e^{(i_1, i_2)}(\mu)$ and $\tilde{\eta}_\zeta^{(i_1, i_2)}$ for $\eta_\zeta^{(i_1, i_2)}$. Note that $d^{(i_1, i_2)} = 1$ in the second case.

From Theorem 2 of section 2.3 of Robert [5], we see

$$\begin{aligned} & \frac{S_{\bar{f}^{(i_1, i_2)} p^{n_\zeta(i_1)} p^{n_{\zeta'}(i_2)}}^{(p)}((\varphi_\zeta^{(1)} \chi_1^{i_1} \varphi_{\zeta'}^{(2)} \chi_2^{i_2})')}{K_{\bar{f}^{(i_1, i_2)} p^{n_\zeta(i_1)} p^{n_{\zeta'}(i_2)}}} \\ &= \frac{S_{\bar{f}^{(i_1, i_2)} p^{n_\zeta(i_1)} p^{n_{\zeta'}(i_2)}}^{(p)}((\varphi_\zeta^{(1)} \chi_1^{i_1} \varphi_{\zeta'}^{(2)} \chi_2^{i_2})')}{K_{\bar{f}^{(i_1, i_2)} p^{n_\zeta(i_1)} p^{n_{\zeta'}(i_2)}}} \\ &\times \prod_{\substack{I \mid \bar{f} \\ I \nmid \bar{f}^{(i_1, i_2)}}} (1 - (\varphi_\zeta^{(1)} \chi_1^{i_1} \varphi_{\zeta'}^{(2)} \chi_2^{i_2})'^{-1}(I)) \end{aligned}$$

where the product on the right is taken over all prime ideals of I dividing $\bar{f}$ but not dividing $\bar{f}^{(i_1, i_2)}$. For any prime ideal I prime to $\bar{f}^{(i_1, i_2)} p$, let α_I be an arbitrary generator of I . Put $v_I^{(i_1, i_2)} = (\chi_1^{i_1} \chi_2^{i_2})'(I)$. Further we put $\varepsilon^{(i_1, i_2)} = \tilde{\eta}_\zeta^{(i_1, i_2)} / \eta_\zeta^{(i_1, i_2)}$, which is independent of ζ . Comparing values of $G_{<\tilde{e}^{(i_1, i_2)}(\mu)>^{(i_1, i_2)}}^{(i_1, i_2)}(T_1, T_2)$ and $G_{<\tilde{e}^{(i_1, i_2)}(\mu)>^{(i_1, i_2)}}^{(i_1, i_2)}(T_1, T_2)$ at $(T_1, T_2) = (\zeta - 1, \zeta' - 1)$, we obtain the following

COROLLARY 5.2. *We have*

$$\begin{aligned} & G_{<\tilde{e}^{(i_1, i_2)}(\mu)>^{(i_1, i_2)}}^{(i_1, i_2)}(T_1, T_2) \\ &= (\tilde{d}^{(i_1, i_2)})^{-1} \varepsilon^{(i_1, i_2)} (T_1 + 1)^{\ell(f) - \ell(f^{(i_1, i_2)})} \\ &\times \left\{ \prod_{\substack{I \mid \bar{f} \\ I \nmid \bar{f}^{(i_1, i_2)}}} (1 - (v_I^{(i_1, i_2)})^{-1} (1 + T_1)^{-\ell(\alpha_I)} (1 + T_2)^{-\ell(\bar{\alpha}_I)}) \right\} \\ &\times G_{<\tilde{e}^{(i_1, i_2)}(\mu)>^{(i_1, i_2)}}^{(i_1, i_2)}(T_1, T_2). \end{aligned}$$

§ 6. The structure of $\mathfrak{Y}_\infty^{(i_1, i_2)}$

In this section we assume that $c^{(i_1, i_2)} = \bar{f}^{(i_1, i_2)}$ for any $(i_1, i_2) \in (\mathbf{Z}/(p-1)\mathbf{Z})^2$.

Let $n, m \geq 0$ and viewing $C_{n, m}'^{(i_1, i_2)}$ as a subgroup of $U_{n, m}'$, we put $C_{n, m}^{(i_1, i_2)} = C_{n, m}'^{(i_1, i_2)} \cap U_{n, m}$. Let $\overline{C_{n, m}^{(i_1, i_2)}}$ denote the closure of $C_{n, m}^{(i_1, i_2)}$, which is precisely the Λ -submodule of $U_{n, m}$ generated by $C_{n, m}^{(i_1, i_2)}$. Let $\overline{C_{n, m}^{(i_1, i_2)}}$ denote the $\chi_1^{i_1} \chi_2^{i_2}$ -eigenspace of $\overline{C_{n, m}^{(i_1, i_2)}}$ and put

$$\overline{C_{n, m}} = \bigoplus_{(i_1, i_2) \in (\mathbf{Z}/(p-1)\mathbf{Z})^2} \overline{C_{n, m}^{(i_1, i_2)}} \subseteq U_{n, m}$$

It is clear that $\chi_1^{i_1}\chi_2^{i_2}$ -eigenspace of $\overline{C_{n,m}}$ is precisely $\overline{C_{n,m}^{(i_1, i_2)}}$. We define a Λ -module $\mathfrak{Y}_{\infty}^{(i_1, i_2)}$ by

$$\mathfrak{Y}_{\infty}^{(i_1, i_2)} = \varprojlim (\mathbf{U}_{n,m}/\overline{C_{n,m}})^{(i_1, i_2)}$$

where the projective limit is taken relative to the norm maps.

Let $D^{(i_1, i_2)}$ be the Λ -submodule of $\mathbf{U}_{\infty}$ generated by $\{\langle \tilde{e}^{(i_1, i_2)}(\mu) \rangle^{(i_1, i_2)} | \mu \in \delta\}$. Then, denoting $p_{n,m}^{(i_1, i_2)}: \mathbf{U}_{\infty}^{(i_1, i_2)} \rightarrow \mathbf{U}_{n,m}^{(i_1, i_2)}$ the projection map, we have $p_{n,m}^{(i_1, i_2)}(D^{(i_1, i_2)}) = \overline{C_{n,m}^{(i_1, i_2)}}$ and $\bigcap_{n,m \geq 0} \text{Ker } p_{n,m}^{(i_1, i_2)} = \{1\}$. Hence $\mathfrak{Y}_{\infty}^{(i_1, i_2)} \cong \mathbf{U}_{\infty}^{(i_1, i_2)} / D^{(i_1, i_2)}$.

It was shown in Yager [6] that there is an injective homomorphism $W^{(i_1, i_2)}: \mathbf{U}_{\infty}^{(i_1, i_2)} \rightarrow \Lambda$ and that there is a unit power series $\phi^{(i_1, i_2)}(T_1, T_2)$ in $\hat{\mathcal{O}}_{\infty}[[T_1, T_2]]$ such that $G_{\beta}^{(i_1, i_2)}(T_1, T_2) = \phi^{(i_1, i_2)}(T_1, T_2)W^{(i_1, i_2)}(\beta)$ for all $\beta \in \mathbf{U}_{\infty}^{(i_1, i_2)}$. We put $\mathfrak{H}^{(i_1, i_2)} = W^{(i_1, i_2)}(\mathbf{U}_{\infty}^{(i_1, i_2)})$.

Let $H^{(i_1, i_2)}$ denote the ideal of Λ generated by $\{h_{\mu}^{(i_1, i_2)}(T_1, T_2) : \mu \in \delta\}$. Then $H^{(i_1, i_2)} = \Lambda$ unless $(i_1, i_2) = (0, 0)$ or $(1, 1)$, $H^{(0,0)} = T_1\Lambda + T_2\Lambda$ and $H^{(1,1)} = (T_1 + 1 - u)\Lambda + (T_2 + 1 - u)\Lambda$ (Lemma 28 of Yager [6]).

THEOREM 6.1. *For each $(i_1, i_2) \in (\mathbf{Z}/(p-1)\mathbf{Z})^2$, there is an element $g^{(i_1, i_2)}(T_1, T_2)$ of Λ such that*

$$\mathfrak{Y}_{\infty}^{(i_1, i_2)} \cong \mathfrak{H}^{(i_1, i_2)} / g^{(i_1, i_2)}(T_1, T_2)H^{(i_1, i_2)}.$$

PROOF. It is sufficient to construct an element $g^{(i_1, i_2)}(T_1, T_2) \in \Lambda$ such that $W^{(i_1, i_2)}(D^{(i_1, i_2)}) = g^{(i_1, i_2)}(T_1, T_2)H^{(i_1, i_2)}$.

If $(i_1, i_2) = (0, 0)$, then, by Theorem 5.1, there exists $e^{(0,0)} \in D^{(0,0)}$ such that $G_{e^{(0,0)}}^{(0,0)}(T_1, T_2)/T_1 \in \hat{\mathcal{O}}_{\infty}[[T_1, T_2]]$ and

$$\begin{aligned} & \frac{G_{e^{(0,0)}}^{(0,0)}(\zeta - 1, \zeta' - 1)}{\zeta - 1} \\ &= \frac{\Omega_p \tilde{d}^{(0,0)} \zeta'^{\ell(\bar{\pi}^n \zeta^{(0)})}}{k_{p^n \zeta^{(0)} p^n \zeta'^{(0)}} \tau(\varphi_{\zeta}^{-1}, \zeta_{p^n \zeta^{(0)}})} S_{p^n \zeta^{(0)} p^n \zeta'^{(0)}}^{(p)}((\varphi_{\zeta}^{(1)} \varphi_{\zeta'}^{(2)})') \end{aligned}$$

for any $(\zeta, \zeta') \in (V_{\infty} - \{1\})^2$. We have further

$$G_{\tilde{e}^{(0,0)}(\mu)}^{(0,0)}(T_1, T_2) = h_{\mu}^{(0,0)}(T_1, T_2)(1/T_1)G_{e^{(0,0)}}^{(0,0)}(T_1, T_2)$$

for any $\mu \in \delta$. Hence, putting $g^{(0,0)}(T_1, T_2) = W^{(0,0)}(e^{(0,0)})/T_1$, we have

$$W^{(0,0)}(D^{(0,0)}) = g^{(0,0)}(T_1, T_2)H^{(0,0)}.$$

If $(i_1, i_2) = (1, 1)$, then there are $\mu_1, \dots, \mu_t \in \delta$ and $f_1, \dots, f_t \in \Lambda$ such that

$T_1 + 1 - u = \sum_{j=1}^t f_j(T_1, T_2) h_{\mu_j}^{(1,1)}(T_1, T_2)$. Put $\tilde{e}^{(1,1)} = \prod_{j=1}^t f_j \langle \tilde{e}^{(1,1)}(\mu_j) \rangle^{(1,1)}$ and $\hat{e}^{(1,1)} = \prod_{j=1}^t f_j \langle \hat{e}^{(1,1)}(\mu_j) \rangle^{(1,1)}$. Then as is shown in the proof of Theorem 29 of Yager [6], $G_{\hat{e}^{(1,1)}}^{(1,1)}(T_1, T_2) \in (T_1 + 1 - u) \hat{O}_{\infty}[[T_1, T_2]]$. Hence from Corollary 5.2, we have $G_{\hat{e}^{(1,1)}}^{(1,1)}(u - 1, u^{k_2} - 1) = 0$ for any $k_2 \in \mathbb{Z}$. Therefore $G_{\hat{e}^{(1,1)}}^{(1,1)}(T_1, T_2) / (T_1 + 1 - u) \in \hat{O}_{\infty}[[T_1, T_2]]$. Put $g^{(1,1)}(T_1, T_2) = W^{(1,1)}(\tilde{e}^{(1,1)}) / T_1 + 1 - u$. Then, $W^{(1,1)}(D^{(1,1)}) = g^{(1,1)}(T_1, T_2) H^{(1,1)}$.

Unless $(i_1, i_2) = (0, 0)$ or $(1, 1)$, then it is easy to see that there exists $e^{(i_1, i_2)} \in D^{(i_1, i_2)}$ such that

$$G_{\langle \tilde{e}^{(i_1, i_2)}(\mu) \rangle^{(i_1, i_2)}}^{(i_1, i_2)}(T_1, T_2) = h_{\mu}^{(i_1, i_2)}(T_1, T_2) G_{e^{(i_1, i_2)}}^{(i_1, i_2)}(T_1, T_2).$$

Put $g^{(i_1, i_2)}(T_1, T_2) = W^{(i_1, i_2)}(e^{(i_1, i_2)})$. Then, $W^{(i_1, i_2)}(D^{(i_1, i_2)}) = g^{(i_1, i_2)}(T_1, T_2) \cdot H^{(i_1, i_2)}$.

§7. Relation to p -adic L -functions

For each integer $k \geq 1$, let $L(\overline{\psi_E^k}, s)$ be the primitive complex Hecke L -function attached to $\overline{\psi_E^k}$, and let $L_{\mathfrak{f}}(\overline{\psi_E^k}, s)$ be the complex Hecke L -function of $\overline{\psi_E^k}$, where $\overline{\psi_E^k}$ is viewed as a Grossencharacter modulo $\mathfrak{f}$. For any $0 \leq j < k$, put

$$(7.1) \quad L_{\infty}(\overline{\psi_E^{k+j}}, k) = (1 - \psi_E(\mathfrak{p})^{k+j} / (N\mathfrak{p})^{j+1}) (1 - \overline{\psi_E}(\mathfrak{p}^*)^{k+j} / (N\mathfrak{p}^*)^k) \times (2\pi / \sqrt{d_k})^j \Omega_{\infty}^{-(k+j)} L(\overline{\psi_E^{k+j}}, k)$$

and

$$(7.2) \quad L_{\mathfrak{f}, \infty}(\overline{\psi_E^{k+j}}, k) = (1 - \psi_E(\mathfrak{p})^{k+j} / (N\mathfrak{p})^{j+1}) (1 - \overline{\psi_E}(\mathfrak{p}^*)^{k+j} / (N\mathfrak{p}^*)^k) \times (2\pi / \sqrt{d_k})^j \Omega_{\infty}^{-(k+j)} L_{\mathfrak{f}}(\overline{\psi_E^{k+j}}, k).$$

Damerell's Theorem states that the right hand sides of (7.1) and (7.2) are algebraic. Let $(i_1, i_2) \in (\mathbb{Z}/(p-1)\mathbb{Z})^2$. Then, there is a power series $G_{\mathfrak{f}}^{(i_1, i_2)}(T_1, T_2) \in \hat{O}_{\infty}[[T_1, T_2]]$ such that for all integers $k_1 > -k_2 \geq 0$ with $(k_1, k_2) \equiv (i_1, i_2) \pmod{p-1}$,

$$(7.3) \quad G_{\mathfrak{f}}^{(i_1, i_2)}(u^{k_1} - 1, u^{k_2} - 1) = (k_1 - 1)! \Omega_p^{k_2 - k_1} L_{\mathfrak{f}, \infty}(\overline{\psi_E^{k_1 - k_2}}, k_1)$$

(Theorem 29 of Yager [6]). We put

$$(7.4) \quad G^{(i_1, i_2)}(T_1, T_2) = \prod_{\substack{f \mid f \\ \mathfrak{f}(i_1, i_2)}} (1 - (v_f^{(i_1, i_2)})^{-1} (1 + T_1)^{-\ell(\alpha_f)} (1 + T_2)^{-\ell(\bar{\alpha}_f)})^{-1} G_f^{(i_1, i_2)}(T_1, T_2).$$

THEOREM 7.1. For each $(i_1, i_2) \in (\mathbf{Z}/(p-1)\mathbf{Z})^2$, $G^{(i_1, i_2)}(T_1, T_2)$ is an associate element of $g^{(i_1, i_2)}(T_1, T_2)$ in $\hat{\mathcal{O}}_\infty[[T_1, T_2]]$. Further, for any integers k_1 and k_2 with $k_1 > -k_2 \geq 0$ and $(k_1, k_2) \equiv (i_1, i_2) \pmod{p-1}$, we have

$$(7.5) \quad G^{(i_1, i_2)}(u^{k_1} - 1, u^{k_2} - 1) = (k_1 - 1)! \Omega_p^{k_2 - k_1} L_\infty(\overline{\psi_E^{k_1 - k_2}}, k_1).$$

PROOF. Observing the construction of $g^{(i_1, i_2)}(T_1, T_2)$ in the proof of Theorem 6.1 and that of $G_f^{(i_1, i_2)}(T_1, T_2)$ in Theorem 29 of Yager [6], we see from Corollary 5.2 that $G^{(i_1, i_2)}(T_1, T_2)/g^{(i_1, i_2)}(T_1, T_2)$ is a unit in $\hat{\mathcal{O}}_\infty[[T_1, T_2]]$. Hence we obtain the first assertion.

It is easy to see that $\mathfrak{f}^{(i_1, i_2)}$ is precisely the conductor of $\psi_E^{k_1 - k_2}$ and that $(\chi_1^{i_1} \chi_2^{i_2})((\alpha), K_{0,0}/K) = \omega^{k_1}(\alpha) \omega^{k_2}(\bar{\alpha}) \psi_E^{k_1 - k_2}((\alpha)) / \alpha^{k_1 - k_2}$ holds for any $\alpha \in \mathcal{O}$ prime to $\mathfrak{f}^{(i_1, i_2)} p$. Hence from (7.3) and (7.4) and the Euler product decomposition of L -functions, the second assertion is obvious.

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