A REMARK ON THE GROUP OF SELF- HOMOTOPY EQUIVALENCES

By

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§ 0. Introduction

Let X be a based topological space. We denote the group of (based) self homotopy equivalences of X as $\mathcal{E}(X)$. When X is simply connected complex, $\mathcal{E}(X)$ has been studied by various authors (see [5] and its references). On the other hand, when X is not simply connected, it seems that we do not have enough information. But for special spaces there are some papers [7], [8] and so on. In these papers they study the kernel of the natural map from $\mathcal{E}(X)$ to $\text{Aut } \pi_1 \times \text{Aut } \pi_n$ for a space which has trivial i -th homotopy groups for $1 < i < n$ ($n = \dim X$).

In this paper we shall describe the image of this homomorphism by the direct application of the classical theory [2], [4] and [9] so that we shall determine $\mathcal{E}(X)$ for some complexes.

§ 1. Image of the natural map ϕ .

Let X be a n -dimensional CW-complex such that $\pi_i(X) = 0$ for $1 < i < n$, and let

$$(1.1) \quad C(\tilde{X}): 0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} Z \longrightarrow 0$$

be the augmented chain complex of $\tilde{X}$, the universal cover of X . Since $\tilde{X}$ is $n-1$ connected, the above complex is exact at C_i for $i < n$. Then we can define a cohomology class $I^{n+1} \in H^{n+1}(\pi_1, \pi_n(X))$ as in ([2], Theorem 5.1).

Let $\phi: [X, X] \rightarrow \text{Hom}(\pi_1, \pi_1) \times \text{Hom}(\pi_n, \pi_n)$ be the natural map, $\phi(f) = (f_* | \pi_1, f_* | \pi_n)$.

LEMMA 1 (Theorem 3 [4], c.f. Theorem 4.8 [10]). $\text{Im } \phi = \{(\alpha, \beta) | \beta \text{ is an } \alpha\text{-homomorphism and } \alpha^* I^{n+1} = \beta_* I^{n+1}\}$

PROOF. By the naturality of I^{n+1} , an each element of $\text{Im } \phi$ satisfies above conditions. Conversely, if we are given α, β which satisfy the conditions, we obtain a chain map $\lambda: C(\tilde{X}) \rightarrow C(\tilde{X})$ which induces (α, β) by Theorem 4 of [4]. By Theorem 16 [9], λ has a geometrical realization, for X is a J_n -complex. This is an outline of the proof.

§ 2. The exact sequence for $\mathcal{E}(X)$

Let X be a space as in section 1. We recall the the kernel of the homomorphism given by the restriction of ϕ to the group of self homotopy equivalences, denoted by $\mathcal{E}(X)$, can be represented by the cohomology group according to [7] or [8] as follows.

In the chain complex (1.1) of §1, we should note that $\text{Hom}_R(B_{n-1}, \pi_n)/\text{Hom}_R(C_{n-1}, \pi_n) = H^n(\pi_1, \pi_n)$, where $R = \mathbb{Z}[\pi_1]$, the group ring of π_1 and $B_{n-1} = \partial_n(C_n)$.

Thus if we are given $\gamma \in H^n(\pi_1, \pi_n)$ we can represent γ as an element of $\text{Hom}_R(B_{n-1}, \pi_n)$. If $i \leq n-1$, let $\tilde{f}_i = 1: C_i \rightarrow C_i$, and let $\tilde{f}_n = 1 + \gamma \partial_n$. Then $\{\tilde{f}_n\}$ is a chain map. Again by Theorem 16 in [9], there is a geometric realization $f: X \rightarrow X$ which induces the identity on π_1 . We define $\kappa(\gamma) = f$ (see [7]). It is easy to see that homomorphism and is monic. Now we obtain the following.

THEOREM 2 (cf. T. Yamanoshita [10], Theorem 4.8). *Let M be a subgroup of $\text{Aut } \pi_1 \times \text{Aut } \pi_n$ of the form $\{(\alpha, \beta) \in \text{Aut } \pi_1 \times \text{Aut } \pi_n \mid \alpha \text{ and } \beta \text{ satisfy conditions of Lemma 1}\}$. Then the following is exact.*

$$0 \longrightarrow H^n(\pi_1, \pi_n) \xrightarrow{\kappa} \mathcal{E}(X) \xrightarrow{\phi} M \longrightarrow 1.$$

PROOF. Exactness at $\mathcal{E}(X)$ is already known by above mentioned papers [7], [8]. If we restrict Lemma 1 to $\mathcal{E}(X)$, we obtain the result, since there is the fact that $f: X \rightarrow X$ is a homotopy equivalence if and only if $f_1: \pi_1 \rightarrow \pi_1$ and $f_*: H_*(\tilde{X}) \rightarrow H_*(\tilde{X})$ are isomorphisms.

§ 3. Examples

Let X be a n -dimensional CW-complex such that $\pi_i(X) = 0$ for $1 < i < n$ with $\pi_1(X)$ a free group. Then,

$$(3.1) \quad \mathcal{E}(X) = \text{Im } \phi = \{(\alpha, \beta) \in \text{Aut } \pi_1 \times \text{Aut } \pi_n \mid \beta \text{ is an } \alpha\text{-homomorphism}\}.$$

PROOF. Since the cohomological dimension of a free group is at most one,

$H^n(\pi_1, \pi_n) = 0 = H^{n+1}(\pi_1, \pi_n)$. Then i). The kernel of ϕ is trivial by the exact sequence (2.1), and ii). The k -invariant I^{n+1} vanishes. These lead us to (3.1).

For example, let $X = S^1 \vee S^n$. If $n=1$, $X = K(Z * Z, 1)$. So we see that $\mathcal{E}(X) = \text{Aut}(Z * Z)$. When $n \geq 2$, $\pi_1(X) = Z$, $\pi_n = Z[\pi_1] = Z[Z]$. As $Z[Z]$ has only trivial units and $\text{Aut } Z = \pm 1$, it is not hard to see that $\mathcal{E}(X) = D(Z) \times Z_2$ (c.f. [3]). Thus,

$$(3.2) \quad \mathcal{E}(S^1 \vee S^n) = \begin{cases} \text{Aut}(Z * Z) & \text{when } n=1, \\ D(Z) \times Z_2 & \text{when } n \geq 2. \end{cases}$$

We recover the well known result.

$$(3.3) \quad \mathcal{E}(RP^n) = Z_2 \text{ ([1], [6])}.$$

PROOF. Let δ be a generator of $\pi_1 = Z_2$. Since σ acts on $\pi_n = Z$ as an inversion if n is even and acts trivially if n is odd, $H^n(\pi_1, \pi_n) = 0$ for any n by the classical results of group cohomology. So, $\mathcal{E}(RP^n)$ is isomorphic to $\text{Im } \phi$. As we also see that $H^{n+1}(\pi_1, \pi_n) = Z_2$, $\text{Im } \phi = Z_2$.

For the space $RP^n \vee S^n$, we also obtain that $\mathcal{E}(RP^n \vee S^n) = \text{Im } \phi = \text{Aut}_{Z[\pi_1]} \pi_n$ since $H^*(\pi_1, \pi_n)$ are isomorphic to the cohomologies of (3.3) by easy calculations. Since $\pi_n(RP^n \vee S^n) = Z \oplus Z \oplus Z$ for $n \geq 2$, we obtain the following.

(3.4) For $n \geq 2$, $\mathcal{E}(RP^n \vee S^n)$ is isomorphic to the subgroups of $GL_3(Z)$ of the following forms.

$$\left\{ \begin{pmatrix} a & b & -b \\ c & d & e \\ -c & e & d \end{pmatrix} \in GL_3(Z) \right\} \text{ if } n \text{ is even, } \left\{ \begin{pmatrix} a & b & b \\ c & d & e \\ c & e & d \end{pmatrix} \in GL_3(Z) \right\} \text{ otherwise.}$$

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