ON THE EQUIVARIANT HOPF STRUCTURES OF A SPHERE WITH AN S^1 -ACTION

By

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1. Introduction

Let V be a complex S^1 -module (an S^1 -complex representation space) with a trivial S^1 -module $\mathbb{C}^n$ ($n > 0$) as the direct summand. The aim of this paper is to study the equivariant Hopf structure of the unit sphere $S(V)$. If $S(V)$ is equipped with the equivariant Hopf structure, the equivariant projective plane XP_2 for $X = S(V)$ is constructed by the ordinary way [6]. As well-known, the structure of the representation ring R_{S^1} is isomorphic to $\mathbb{Z}[t, t^{-1}]$, where t^i corresponds to a one dimensional complex S^1 -module $\mathbb{C}^1$ with an S^1 -action defined by $z \cdot v = z^i v$, $z \in S^1$ and $v \in \mathbb{C}^1$. The equivariant complex K -group $K_{S^1}(X)$ is an R_{S^1} -module. In this paper we obtain

THEOREM 1.1. *$\tilde{K}_{S^1}(XP_2)$, for $X = S(V)$, is isomorphic to a free R_{S^1} -module with a basis $\{u, v\}$ and the elements u and v satisfy*

$$u^2 = v, \quad uv = 0 \quad \text{and} \quad v^2 = 0.$$

Let L be the canonical one dimensional complex S^1 -module with an S^1 -action $z \cdot v = zv$, $z \in S^1$, $v \in \mathbb{C}$. The i -th fold tensor product module L^i is a one dimensional complex S^1 -module with an S^1 -action:

$$z \cdot v = z^i v.$$

According to [1], [2], [4], the unit sphere S^n has the Hopf structure if and only if $n = 1, 3$ or 7 . The result of Adams implies that if $S(V)$ has the Hopf structure then the dimension of $S(V)$ is $1, 3$ or 7 , and the stationary spheres are of dimension $1, 3$ or 7 .

Making use of Theorem 1.1, we have

THEOREM 1.2. *Let L be the canonical one dimensional complex S^1 -module*

which corresponds to t in $R_{S^1} = \mathbb{Z}[t, t^{-1}]$.

(1) If $V = L^{l_1} \oplus \cdots \oplus L^{l_s} \oplus \mathbb{C}^n$ and $S(V)$ has an equivariant Hopf structure then $n=1, 2$ or 4 .

(2) If $V = L^a \oplus L^b \oplus L^c \oplus \mathbb{C}$ and abc is odd, then $S(V)$ doesn't have an equivariant Hopf structure.

In the quaternion space $H \cong \mathbb{C} \oplus \mathbb{C}$, a multiplication is given by

$$(z_1 + z_2 j)(z'_1 + z'_2 j) = (z_1 z'_1 - \bar{z}'_2 z_2) + (z'_2 z_1 + z_2 \bar{z}'_1)j$$

and the S^1 -action

$$z \cdot (z_1 + z_2 j) = z_1 + z^a z_2 j, \quad z \in S^1$$

is compatible with the multiplication:

$$\begin{aligned} \{z \cdot (z_1 + z_2 j)\} \{z \cdot (z'_1 + z'_2 j)\} &= (z_1 + z^a z_2 j)(z'_1 + z^a z'_2 j) \\ &= (z_1 z'_1 - \bar{z}'_2 z_2) + z^a (z'_2 z_1 + z_2 \bar{z}'_1)j \\ &= z \cdot \{(z_1 z'_1 - \bar{z}'_2 z_2) + (z'_2 z_1 + z_2 \bar{z}'_1)j\} \\ &= z \cdot \{(z_1 + z_2 j)(z'_1 + z'_2 j)\}. \end{aligned}$$

Hence $H \cong L^a \oplus \mathbb{C}$ as the complex S^1 -modules and the sphere $S(H)$ has an equivariant Hopf structure.

Next we consider the Cayley algebra $\mathcal{L} = \{q + q'e \mid q, q' \in H\}$ with a multiplication

$$(q + q'e)(s + s'e) = (qs - \bar{s}'q') + (s'q + q'\bar{s})e.$$

The S^1 -action

$$z \cdot (q + q'e) = q + z^b q'e, \quad z \in S^1$$

is compatible with the multiplication by the same way as above. Hence the sphere $S(\mathcal{L})$, $\mathcal{L} \cong L^b \oplus L^b \oplus \mathbb{C} \oplus \mathbb{C}$ has an equivariant Hopf structure. Moreover we can define another S^1 -action on $\mathcal{L}$ as following:

$$z \cdot (q + q'e) = (x_1 + z^a x_2 j) + z^b (x'_1 + z^a x'_2 j)e, \quad z \in S^1$$

where $q = x_1 + x_2 j$, $q' = x'_1 + x'_2 j$. We calculate for $q = x_1 + x_2 j$, $q' = x'_1 + x'_2 j$, $s = y_1 + y_2 j$, $s' = y'_1 + y'_2 j$,

$$\begin{aligned}
& \{z \cdot (q + q'e)\} \{z \cdot (s + s'e)\} \\
&= \{(x_1 + z^a x_2 j) + z^b (x'_1 + z^a x'_2 j)e\} \{(y_1 + z^a y_2 j) + z^b (y'_1 + z^a y'_2 j)e\} \\
&= \{(x_1 + z^a x_2 j)(y_1 + z^a y_2 j) - (\bar{y}'_1 - z^a y'_2 j)(x'_1 + z^a x'_2 j)\} \\
&\quad + z^b \{(y'_1 + z^a y'_2 j)(x_1 + z^a x_2 j) + (x'_1 + z^a x'_2 j)(\bar{y}_1 - z^a y_2 j)\} e \\
&= z \cdot \{(qs - \bar{s}'q') + (s'q + q'\bar{s})e\} = z \cdot \{(q + q'e)(s + s'e)\}.
\end{aligned}$$

Then $\mathcal{L} \cong L^a \oplus L^b \oplus L^{a+b} \oplus C$ and $S(\mathcal{L})$ has an equivariant Hopf structure.

Throughout this paper G denotes the circle group S^1 .

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2. The ring structure of $\tilde{K}_G(S^1(S(V)))$.

Let $V = L_1 \oplus \cdots \oplus L_m \oplus C$ where L_i is a one dimensional complex G -module. We consider the exact sequence for the equivariant K_G -theory:

$$\longrightarrow K_G^{*-1}(S(V)) \longrightarrow K_G^*(*) \xrightarrow{\cdot e(V)} K_G^*(*) \longrightarrow K_G^*(S(V)) \longrightarrow K_G^{*+1}(*) \longrightarrow$$

The euler class $e(V)$ is zero because V has a trivial representation space as a direct summand. This fact implies that

$$(2.1) \quad K_G(S(V)) \cong R_G, \quad K_G^{-1}(S(V)) \cong R_G$$

and $\tilde{K}_G(S(V))$ is isomorphic to zero and $\tilde{K}_G^{-1}(S(V)) \cong R_G$.

Here we consider the K_G -group of the suspension space $S^1(S(V))$.

PROPOSITION 2.2. *Let $V = L_1 \oplus \cdots \oplus L_s \oplus C^n$, where each L_j is a nontrivial one dimensional complex G -module. Then the equivariant K -group $\tilde{K}_G(S^1(S(V)))$ for the suspension space of $S(V)$ is isomorphic to a free R_G -module with a generator u' , and $u'^2 = 0$. Applying the Adams operation ψ^k : $\tilde{K}_G(S^1(S(V))) \rightarrow \tilde{K}_G(S^1(S(V)))$, we have*

$$\psi^k(u') = k^n r(k) u',$$

where $r(k) = (L_1^{k-1} + L_1^{k-2} + \cdots + 1) \cdots (L_s^{k-1} + L_s^{k-2} + \cdots + 1)$.

PROOF. Let $D(V)$ be the unit disk of V . The quotient space $\Sigma(V) = D(V)/S(V)$ is the Thom complex of the equivariant vector bundle $V \rightarrow *$ over a point. Then we have a bijection

$$\tilde{K}_G(S^1(S(V))) \xrightarrow{f^*} \tilde{K}_G(\Sigma V)$$

in the exact sequence for a G -complex pair $(D(V), S(V))$ where $f: \Sigma(V) \simeq D(V) \cup C(S(V)) \rightarrow S^1(S(V))$ is the natural map shrinking $D(V)$. Let $V = L_1 \oplus L_2 \oplus \cdots \oplus L_{s+n}$, where $L_{s+1} = \cdots = L_{s+n} = \mathbb{C}$. The canonical line bundle η over the projective space $P(V)$ is a quotient bundle $S(V) \times_G \mathbb{C} \rightarrow S(V)/G = P(V)$, where (v, z) is identified with $(\lambda v, \lambda z)$, $\lambda \in G$. According to [3] and [5], the K -group $K_G(P(V))$ is isomorphic to a free R_G -module with the basis

$$(2.3) \quad 1, (\eta_1 - 1), (\eta_1 - 1)(\eta_2 - 1), \dots, (\eta_1 - 1)(\eta_2 - 1) \cdots (\eta_{s+n-1} - 1)$$

$$\text{with } (\eta_1 - 1)(\eta_2 - 1) \cdots (\eta_{s+n} - 1) = 0, \text{ where } \eta_j = \eta \otimes L_j.$$

We also use the notation η for the canonical line bundle over $P(V \oplus \mathbb{C})$. There exists a short exact sequence:

$$0 \longrightarrow \tilde{K}_G(\Sigma(V)) \xrightarrow{j^*} K_G(P(V \oplus \mathbb{C})) \xrightarrow{i^*} K_G(P(V)) \longrightarrow 0.$$

Take an element u'' in $\tilde{K}_G(\Sigma(V))$ with $j^*(u'') = (\eta_1 - 1) \cdots (\eta_{s+n} - 1)$ and take an element u' in $\tilde{K}_G(S^1(S(V)))$ with $f^*(u') = u''$. In the group $K_G(P(V \oplus \mathbb{C}))$,

$$(\eta_1 - 1) \cdots (\eta_{s+n} - 1)(\eta - 1) = 0.$$

Since $\eta_{s+n} = \eta$,

$$j^*(u'')^2 = (\eta_1 - 1)^2 \cdots (\eta_{s+n} - 1)^2 = 0$$

and $u'^2 = 0$. Applying the Adams operation ψ^k to $K_G(P(V \oplus \mathbb{C}))$

$$\begin{aligned} & \psi^k\{(\eta_1 - 1) \cdots (\eta_{s+n} - 1)\} \\ &= \psi^k\{(\eta_1 - 1) \cdots (\eta_s - 1)(\eta - 1)^n\} \\ &= (\eta_1^k - 1) \cdots (\eta_s^k - 1)(\eta^k - 1)^n \\ &= (\eta_1 - 1) \cdots (\eta_s - 1)(\eta - 1)^n(\eta_1^{k-1} + \cdots + \eta_1 + 1) \\ & \quad \cdots (\eta_s^{k-1} + \cdots + \eta_s + 1)(\eta^{k-1} + \cdots + \eta + 1)^n. \end{aligned}$$

Noting that $(\eta_1 - 1) \cdots (\eta_s - 1)(\eta - 1)^{n+1} = 0$, we calculate

$$\begin{aligned} & \eta_j^i (\eta_1 - 1) \cdots (\eta_s - 1)(\eta - 1)^n \\ &= \eta^i L_j^i (\eta_1 - 1) \cdots (\eta_s - 1)(\eta - 1)^n \end{aligned}$$

$$\begin{aligned}
&= \{(\eta^i - 1)L_j^i + L_j^i\} (\eta_1 - 1) \cdots (\eta_s - 1) (\eta - 1)^n \\
&= L_j^i (\eta_1 - 1) \cdots (\eta_s - 1) (\eta - 1)^n.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\psi^k \{(\eta_1 - 1) \cdots (\eta_{s+n} - 1)\} \\
&= k^n (L_1^{k-1} + \cdots + L_1 + 1) \cdots (L_s^{k-1} + \cdots + L_s + 1) (\eta_1 - 1) \cdots (\eta_{s+n} - 1)
\end{aligned}$$

and the proposition follows.

Q. E. D.

PROPOSITION 2.4. *Let $V = L_1 \oplus \cdots \oplus L_s \oplus \mathbb{C}^n$, where each L_j is a nontrivial one dimensional complex G -module. Let $\varphi: S(\mathbb{C}^n) \rightarrow S(V)$ be the natural injection. Take the generators u' and $\tilde{u}'$ of $\tilde{K}_G(S^1(S(V)))$ and $\tilde{K}_G(S^1(S(\mathbb{C}^n)))$ as Proposition 2.2, then*

$$\varphi^*(u') = r\tilde{u}', \quad \text{where } r = (L_1 - 1)(L_2 - 1) \cdots (L_s - 1) \in R_G.$$

PROOF. Consider a commutative diagram for $X = S(V)$

$$\begin{array}{ccccc}
\tilde{K}_G(S^1 X) & \xrightarrow{f^*} & \tilde{K}_G(\Sigma(V)) & \xrightarrow{j^*} & K_G(P(V \oplus \mathbb{C})) \\
\downarrow \varphi^* & & \downarrow \varphi^* & & \downarrow \varphi^* \\
\tilde{K}_G(S^1 X^G) & \xrightarrow{\tilde{f}^*} & \tilde{K}_G(\Sigma(\mathbb{C}^n)) & \xrightarrow{\tilde{j}^*} & K_G(P(\mathbb{C}^{n+1}))
\end{array}$$

where X^G denotes the fixed point set of X and $\tilde{f}$ is the restriction of f . Making use of a basis (2.3) for $K_G(P(V \oplus \mathbb{C}))$ and $K_G(P(\mathbb{C}^{n+1}))$, we choose the generators u' and $\tilde{u}'$ for $\tilde{K}_G(S^1 X)$ and $\tilde{K}_G(S^1 X^G)$ as follows:

$$\begin{aligned}
j^* f^*(u') &= (\eta_1 - 1) \cdots (\eta_s - 1) (\eta - 1)^n \\
\tilde{j}^* \tilde{f}^*(\tilde{u}') &= (\eta - 1)^n.
\end{aligned}$$

Noting that $(\eta - 1)^{n+1} = 0$ in $K_G(P(\mathbb{C}^{n+1}))$ and

$$\begin{aligned}
&(\eta_j - 1)(\eta - 1)^n \\
&= (\eta L_j - 1)(\eta - 1)^n \\
&= \{(\eta - 1)L_j + (L_j - 1)\} (\eta - 1)^n \\
&= (L_j - 1)(\eta - 1)^n,
\end{aligned}$$

we have

$$\varphi^*(u') = (L_1 - 1) \cdots (L_s - 1) \tilde{u}'. \quad \text{Q. E. D.}$$

3. K_G -group of the projective plane for $S(V)$.

Let X be a G -space with base point x_0 whose isotropy group is G . Let G act on $X \times X$ by $g(x, x') = (gx, gx')$. X is said to be a Hopf G -space if there is a base point preserving equivariant map

$$m: X \times X \longrightarrow X$$

such that mi_1 and mi_2 are equivariantly homotopic to the identity, where $i_1(x) = (x, x_0)$ and $i_2(x) = (x_0, x)$. Here we consider a map $\mu: X * X \rightarrow S^1 X$ from the join of X 's to the suspension of X defined by

$$\mu(tx \oplus (1-t)x') = (t, m(x, x')).$$

The mapping cone $XP_2 = C_\mu$ is called the projective plane for X [6]. Suppose that $S(V)$, $V = W \oplus \mathbb{C}$, is a Hopf G -space. (2.1) implies that the reduced K_G -group $\tilde{K}_G(S(V))$ has only a single element. Noting that $S(V) * S(V)$ is equivariantly homeomorphic to $S(V \oplus V)$, we have a short exact sequence for $X = S(V)$:

$$(3.1) \quad 0 \longrightarrow \tilde{K}_G(S^1(X * X)) \xrightarrow{j^*} \tilde{K}_G(XP_2) \xrightarrow{i^*} \tilde{K}_G(S^1 X) \longrightarrow 0.$$

From the fact that $\tilde{K}_G(S^1(S(V)))$ is a free R_G -module, it follows that $\tilde{K}_G(XP_2)$, $X = S(V)$, is a free R_G -module with a basis $\{u, v\}$ such that $j^*(v') = v$, $i^*(u) = u'$, where v' is a generator of $\tilde{K}_G(S^1(X * X)) \cong \tilde{K}_G(S^1(S(V \oplus V)))$ determined by the same way as Proposition 2.2. Then we have Theorem 1.1.

PROOF OF THEOREM 1.1. Let $X = S(V)$, $V = W \oplus \mathbb{C}$. Denote by X^G the fixed point set of X . Then we have a following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{K}_G(S^1(X * X)) & \xrightarrow{j^*} & \tilde{K}_G(XP_2) & \xrightarrow{i^*} & \tilde{K}_G(S^1 X) \longrightarrow 0 \\ & & \downarrow \varphi^* & & \downarrow \varphi^* & & \downarrow \varphi^* \\ 0 & \longrightarrow & \tilde{K}_G(S^1(X^G * X^G)) & \xrightarrow{j^*} & \tilde{K}_G(X^G P_2) & \xrightarrow{i^*} & \tilde{K}_G(S^1 X^G) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & R_G \otimes \tilde{K}(S^1(X^G * X^G)) & & R_G \otimes \tilde{K}(X^G P_2) & & R_G \otimes \tilde{K}(S^1 X^G) \end{array}$$

where φ denotes the natural equivariant injection of Y^G into Y . A short exact sequence

$$0 \longrightarrow \tilde{K}(S^1(X^G * X^G)) \xrightarrow{i^*} \tilde{K}(X^G P_2) \xrightarrow{i^*} \tilde{K}(S^1 X^G) \longrightarrow 0$$

is split and $\tilde{K}(X^G P_2)$ is a free module with a basis $\{\tilde{u}, \tilde{v}\}$ such that $i^*(\tilde{u}) = \tilde{u}'$ and $j^*(\tilde{v}') = \tilde{v}$. Here $\tilde{u}'$ and $\tilde{v}'$ are chosen by the same way as Proposition 2.4 and satisfy the following relations [4]:

$$(3.2) \quad \tilde{u}^2 = \tilde{v}, \quad \tilde{u}\tilde{v} = 0 \quad \text{and} \quad \tilde{v}^2 = 0.$$

Let $\varphi^*(u) = \alpha\tilde{u} + \beta\tilde{v}$, $\alpha, \beta \in R_G$. By Proposition 2.4,

$$i^*\varphi^*(u) = i^*(\alpha\tilde{u}) + i^*(\beta\tilde{v}) = \alpha\tilde{u}',$$

$$\varphi^*i^*(u) = \varphi^*(u') = r\tilde{u}'.$$

Therefore we have

$$(3.3) \quad \varphi^*(u) = r\tilde{u} + \beta\tilde{v}, \quad \beta \in R_G.$$

Let $u^2 = cu + dv$, $c, d \in R_G$. Making use of (3.2) and the relation $\varphi^*(v') = r^2\tilde{v}'$, we calculate

$$\varphi^*(u^2) = \{\varphi^*(u)\}^2 = r^2\tilde{u}^2 = r^2\tilde{v}$$

and

$$\varphi^*(cu + dv) = cr\tilde{u} + dr^2\tilde{v}.$$

And we have that $c=0$ and $d=1$. Therefore

$$u^2 = v.$$

Since $i^*(uv) = 0$, we put $uv = \lambda v$, $\lambda \in R_G$. Applying φ^* to uv ,

$$\varphi^*(uv) = \varphi^*(u)\varphi^*(v) = (r\tilde{u} + \beta\tilde{v})r^2\tilde{v} = 0,$$

$$\varphi^*(\lambda v) = \lambda\varphi^*(v) = \lambda r^2\tilde{v}.$$

Then we have $\lambda=0$ and

$$uv = 0. \quad \text{Q. E. D.}$$

Denote by $M^{-1}K_G(X)$ a ring localized by a multiplicative set M consisting of $\{t^j - 1 \mid j = \pm 1, \pm 2, \dots\}$ and 1 where $R_G = \mathbb{Z}[t, t^{-1}]$. We identify an element u in $K_G(X)$ with $u/1$ in $M^{-1}K_G(X)$. The Adams operation

$$\psi^k: M^{-1}K_G(X) \longrightarrow M^{-1}K_G(X)$$

is defined by

$$\psi^k(a/m) = \psi^k(a)/\psi^k(m), \quad m \in M.$$

PROPOSITION 3.4. *Let $X = S(V)$, $V = L_1 \oplus \cdots \oplus L_s \oplus \mathbf{C}^n$, where L_j denotes a nontrivial one dimensional complex G -module, and $\tilde{K}_G(XP_2)$ is isomorphic to $R_G\{u, v\}$. Let $\varphi^*(u) = r\tilde{u} + \beta\tilde{v}$, $r = (L_1 - 1) \cdots (L_s - 1)$, $\beta \in R_G$, and let $\hat{u} = u - (\beta/r^2)v$. Then $M^{-1}\tilde{K}_G(XP_2)$ is isomorphic to a free $M^{-1}R_G$ -module with a basis $\{\hat{u}, v\}$ and*

$$\varphi^*(\hat{u}) = r\tilde{u}.$$

And

$$\psi^k(\hat{u}) = k^n r(k)\hat{u} + (\sigma_k r r(k)/r^2)v,$$

where $\psi^k(\tilde{u}) = k^n \tilde{u} + \sigma_k \tilde{v}$ in $\tilde{K}(X^G P_2)$ and σ_k is the integer given in [2].

PROOF. Using Proposition 2.4 and (3.3), we obtain that $\varphi^*(v) = r^2 \tilde{v}$ and $\varphi^*(\hat{u}) = r\tilde{u}$. Let

$$\psi^k(\hat{u}) = c\hat{u} + dv.$$

Applying φ^* to $\psi^k(\hat{u})$, we have

$$\varphi^*\psi^k(\hat{u}) = cr\tilde{u} + dr^2\tilde{v},$$

$$\psi^k\varphi^*(\hat{u}) = \psi^k(r\tilde{u})$$

$$= \psi^k(r)\psi^k(\tilde{u}) = rr(k)(k^n \tilde{u} + \sigma_k \tilde{v})$$

$$= k^n r r(k)\tilde{u} + \sigma_k r r(k)\tilde{v}.$$

And $c = k^n r(k)$ and $d = \sigma_k r r(k)/r^2$.

Q. E. D.

The following theorem is a direct result of Proposition 3.4.

THEOREM 3.5. *In $M^{-1}\tilde{K}_G(XP_2)$, $X = S(L_1 \oplus \cdots \oplus L_s \oplus \mathbf{C}^n)$,*

$$\psi^k(u) = k^n r(k)u + \alpha(k)v,$$

where $\alpha(k) = \{k^{2n}\psi^k(\beta) - k^n r(k)\beta + \sigma_k r r(k)\}/r^2$ is an integral element.

PROOF. Proposition 2.2 implies that $\psi^k(v) = k^{2n}r(k)^2v$ and

$$\begin{aligned}
\psi^k(u) &= \psi^k\{\hat{u} + (\beta/r^2)v\} \\
&= k^n r(k)\hat{u} + \{\sigma_k r r(k)/r^2\}v + \{\psi^k(\beta)/r^2 r(k)^2\}k^{2n} r(k)^2 v \\
&= k^n r(k)\{u - (\beta/r^2)v\} + \{(\sigma_k r r(k) + k^{2n}\psi^k(\beta))/r^2\}v \\
&= k^n r(k)u + [\{k^{2n}\psi^k(\beta) - k^n r(k)\beta + \sigma_k r r(k)\}/r^2]v
\end{aligned}$$

Q. E. D.

4. A proof of Theorem 1.2.

Suppose that $V = L^{l_1} \oplus \dots \oplus L^{l_s} \oplus C^n$ where L corresponds to t in $R_G = \mathbb{Z}[t, t^{-1}]$. Then

$$\begin{aligned}
r &= (t^{l_1} - 1) \dots (t^{l_s} - 1), \\
r(2) &= (t^{l_1} + 1) \dots (t^{l_s} + 1)
\end{aligned}$$

and

$$r(k) = (t^{l_1(k-1)} + t^{l_1(k-2)} + \dots + 1) \dots (t^{l_s(k-1)} + t^{l_s(k-2)} + \dots + 1).$$

By the Theorem 3.5, we have

$$\begin{aligned}
\psi^2 \psi^k(u) &= k^n \psi^2(r(k))\{2^n r(2)u + \alpha(2)v\} + \psi^2(\alpha(k))2^{2n} r(2)^2 v \\
&= 2^n k^n \psi^2(r(k))r(2)u \\
&\quad + \{k^n \psi^2(r(k))\alpha(2) + 2^{2n} \psi^2(\alpha(k))r(2)^2\}v, \\
\psi^k \psi^2(u) &= 2^n k^n \psi^k(r(2))r(k)u \\
&\quad + \{2^n \psi^k(r(2))\alpha(k) + k^{2n} \psi^k(\alpha(2))r(k)^2\}v.
\end{aligned}$$

Since $\psi^2 \psi^k = \psi^k \psi^2$,

$$\begin{aligned}
&k^{2n} \psi^k(\alpha(2))r(k)^2 - k^n \psi^2(r(k))\alpha(2) \\
&= 2^{2n} \psi^2(\alpha(k))r(2)^2 - 2^n \psi^k(r(2))\alpha(k).
\end{aligned}$$

Where

$$\begin{aligned}
\alpha(2) &= \{2^{2n} \psi^2(\beta) - 2^n r(2)\beta + \sigma_2 r r(2)\}/r^2, \\
\psi^k(r) &= (t^{k l_1} - 1) \dots (t^{k l_s} - 1) = r r(k),
\end{aligned}$$

$$\psi^k(r(2)) = (t^{kl_1} + 1) \cdots (t^{kl_s} + 1) = r(2k)/r(k),$$

and

$$\psi^2(r(k)) = (t^{2l_1(k-1)} + \cdots + 1) \cdots (t^{2l_s(k-1)} + \cdots + 1) = r(2k)/r(2).$$

Then we have

$$k^{2n} \psi^k(\alpha(2)) r(k)^2 - k^n \psi^2(r(k)) \alpha(2) \equiv 0 \pmod{2^n}.$$

and

$$\begin{aligned} & k^{2n} \sigma_2 \psi^k(r) \psi^k(r(2)) r(k)^2 / \psi^n(r^2) - k^n \psi^2(r(k)) \sigma_2 r r(2) / r^2 \\ &= k^n (k^n - 1) \sigma_2 r(2k) / r \\ &\equiv 0 \pmod{2^n}. \end{aligned}$$

Since σ_2 is an odd integer, if we choose k odd, we see that 2^n must divide $k^n - 1$ for each odd integer k . It follows that $n = 1, 2$ or 4 .

Next we put $V = L^a \oplus L^b \oplus L^c \oplus C$. Then

$$r = (t^a - 1)(t^b - 1)(t^c - 1)$$

and

$$r(2) = (t^a + 1)(t^b + 1)(t^c + 1).$$

It follows from Theorem 3.5 that

$$\begin{aligned} (4.1) \quad & \{(t^a - 1)(t^b - 1)(t^c - 1)\}^2 \alpha(2) \\ &= 2^2 \psi^2(\beta) - 2(t^a + 1)(t^b + 1)(t^c + 1) \beta + \sigma_2(t^{2a} - 1)(t^{2b} - 1)(t^{2c} - 1) \end{aligned}$$

in $\tilde{K}_G(XP_2)$, $X = S(V)$, where $\beta \in R_G$.

Put $x = t - 1$ and assume that

$$\beta = \{\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \cdots\} / t^j, \quad \beta_i \in \mathbf{Z}.$$

Then $\psi^2(\beta) = \{\beta_0 + \beta_1 x(x+2) + \beta_2 x^2(x+2)^2 + \cdots\} / t^{2j}$. (4.1) implies that

$$\begin{aligned} x^6 g(x) &= 4\{\beta_0 + \beta_1 x(x+2) + \beta_2 x^2(x+2)^2 + \cdots\} \\ &\quad - 2\{(x+1)^a + 1\} \{(x+1)^b + 1\} \{(x+1)^c + 1\} \\ &\quad \cdot \{\beta_0 + \beta_1 x + \beta_2 x^2 + \cdots\} (x+1)^j \\ &\quad + \sigma_2 \{(x+1)^{2a} - 1\} \{(x+1)^{2b} - 1\} \{(x+1)^{2c} - 1\} (x+1)^{2j}, \end{aligned}$$

where $g(x) \in \mathbb{Z}[t]$. And we have $\beta_0 = \beta_1 = 0$, then

$$\begin{aligned} x^4 g(x) = & 4\{\beta_2(x+2)^2 + \beta_3 x(x+2)^3 + \dots\} \\ & - 2\{(x+1)^a + 1\} \{(x+1)^b + 1\} \{(x+1)^c + 1\} \{\beta_2 + \beta_3 x + \dots\} (x+1)^j \\ & + \sigma_2 x \{(x+1)^{a-1} + \dots + 1\} \{(x+1)^{b-1} + \dots + 1\} \{(x+1)^{c-1} + \dots + 1\} \\ & \cdot \{(x+1)^a + 1\} \{(x+1)^b + 1\} \{(x+1)^c + 1\} (x+1)^{2j}. \end{aligned}$$

We calculate

$$\begin{aligned} & 4[4\beta_2 + \{4\beta_2 + 8\beta_3\}x + \{\beta_2 + 12\beta_3 + 16\beta_4\}x^2] \\ & - 2 \left\{ 2 + ax + \left(\frac{a}{2}\right)x^2 \right\} \left\{ 2 + bx + \left(\frac{b}{2}\right)x^2 \right\} \left\{ 2 + cx + \left(\frac{c}{2}\right)x^2 \right\} \\ & \quad \cdot \{\beta_2 + \beta_3 x + \beta_4 x^2\} \left\{ 1 + jx + \left(\frac{j}{2}\right)x^2 \right\} \\ & + \sigma_2 x \left\{ a + \left(\frac{a}{2}\right)x \right\} \left\{ b + \left(\frac{b}{2}\right)x \right\} \left\{ c + \left(\frac{c}{2}\right)x \right\} \\ & \quad \cdot \{2 + ax\} \{2 + bx\} \{2 + cx\} \{1 + 2jx\} \\ & \equiv 0 \quad \text{modulo } x^3 \end{aligned}$$

and

$$\begin{aligned} & 16\beta_2 + 16(\beta_2 + 2\beta_3)x + 4\{\beta_2 + 12\beta_3 + 16\beta_4\}x^2 - 16\beta_2 - 8\{(a+b+c+2j)\beta_2 \\ & + 2\beta_3\}x - \left[\left\{ 8 \left(\left(\frac{a}{2}\right) + \left(\frac{b}{2}\right) + \left(\frac{c}{2}\right) + 4(ab+bc+ca) + 8(a+b+c)j \right. \right. \right. \\ & \quad \left. \left. + 16\left(\frac{j}{2}\right) \right\} \beta_2 + 8\{a+b+c+2j\}\beta_3 + 16\beta_4 \right] x^2 \\ & + \sigma_2 x \left[8abc + \left\{ 8 \left(ab\left(\frac{c}{2}\right) + bc\left(\frac{a}{2}\right) + ca\left(\frac{b}{2}\right) \right) + 4abc(a+b+c+4j) \right\} x \right] \\ & \equiv 0 \quad \text{modulo } x^3. \end{aligned}$$

Therefore we have the coefficient of x^2 :

$$\begin{aligned} & \beta_2 + 12\beta_3 + 16\beta_4 - \left\{ 2 \left(\left(\frac{a}{2}\right) + \left(\frac{b}{2}\right) + \left(\frac{c}{2}\right) \right) + (ab+bc+ca) \right. \\ & \quad \left. + 2(a+b+c)j + 4\left(\frac{j}{2}\right) \right\} \beta_2 - 2(a+b+c+2j)\beta_3 - 4\beta_4 \end{aligned}$$

$$+ \sigma_2 \left\{ 2 \left(ab \binom{c}{2} + bc \binom{a}{2} + ca \binom{b}{2} \right) + abc(a+b+c+4j) \right\} \\ = 0.$$

Then we have

$$(4.2) \quad (1+ab+bc+ca)\beta_2 + \sigma_2 abc(a+b+c) \equiv 0 \pmod{2}.$$

If abc is odd then $1+ab+bc+ca$ is even and (4.2) implies σ_2 is even. This result is a contradiction and the theorem follows.

References

- [1] J. F. ADAMS, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. **72** (1960), 20–104.
- [2] J. F. ADAMS and M. F. ATIYAH, *K-theory and the Hopf invariant*, Quart. J. Math. Oxford (2) **17** (1966), 31–38.
- [3] M. FUJII and M. KAMATA, *On the completion of G-equivariant unitary cobordism rings of G-spaces*, Publ. RIMS, Kyoto Univ. **19** (1983), 577–600.
- [4] M. KAROUBI, *K-theory*, Springer-Verlag, 1978.
- [5] G. SEGAL, *Equivariant K-theory*, Publ. Math. I. H. E. S. **34** (1968), 129–151.
- [6] J. D. STASHEFF, *H-spaces from a homotopy point of view*, Lecture notes in Math. **161**, Springer-Verlag, 1970.

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