THE TITZ METRIC ON THE BOUNDARIES OF FOCAL POINTS FREE MANIFOLDS AND ITS APPLICATION

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Recently, Riemannian manifolds with nonpositive sectional curvature have been investigated by many people and many properties of these manifolds became to be known. Besides, several people have investigated Riemannian manifolds without focal points. Still, there are few properties which distinguish the former from the latter notwithstanding the Gulliver's examples (cf. [5]); Riemannian manifolds without focal point whose sectional curvatures change sign. We want to know whether there are any properties which distinguish the former from the latter or not.

In ([2]), M. Gromov defined Titz metric on the boundaries of Hadamard manifolds. This metric go far towards founding flats in Hadamard manifolds. This metric is related with the growth rates of lengths of Jacobi fields which vanish at a point and also related with the growth rate of distances of a pair of geodesic rays with a common starting point. Using this metric one can find easily some conditions for Hadamard manifolds to satisfy the visibility axiom.

M. Gromov said that the Gap Theorem would be proved using the metric as well.

Many good properties of Tits metric on boundaries of Hadamard manifolds are based on the following three facts:

- (1) For a pair of unit speed geodesic rays γ and τ , the function; $t \rightarrow \text{dist.}(\gamma(t), \tau(t))$ is convex.
- (2) the sum of three inner angles of a geodesic triangle is not greater than 2π .
- (3) The law of cosine holds.

As it is, we can not expect these facts would be true for Riemannian manifolds without focal point. However using the fact that the distance functions from every points are convex, we can define Tits metric on boundaries of Riemannian

manifolds without focal point, and we can find some relations between Tits metric and the growth rates of length of Jacobi fields which vanish at a point and the growth rate of distances of two points with the same parameter on a pair of geodesic rays with a common starting point.

In this paper, we prove mainly the following two theorems.

THEOREM 1. Let X be a complete, connected, simply connected Riemannian manifold without focal point. Then the following three conditions are equivalent.

- (1) X satisfies the visibility axiom, that is, any pair of points in the boundary $X(\infty)$ can be joined by a geodesic of X .
- (2) $\kappa(z, w) = \pi$ for any different points z and w in $X(\infty)$.
- (3) $\tilde{d}_i(z, w) = +\infty$ for any different points z and w in $X(\infty)$.

Concerning the metrics κ and $\tilde{d}_i$, see §1. We remark that the metric $\tilde{d}_i$ is closely related with the growth rates of lengths of Jacobi fields which vanish at a point.

THEOREM 2. Let X be a complete connected simply connected Riemannian manifold without focal point. Suppose X satisfies the following two conditions (H.1) and (H.2), and the dimension of X is greater than or equal to 3. Then X is isometric to the Euclidean space of the same dimension.

(H.1) There exists a point 0 in X and a monotonely nonincreasing function $k: [0, \infty) \rightarrow [0, \infty)$ satisfying the following:

- (1) $\int_0^\infty tk(t)dt < +\infty$.
- (2) For any point p in X , the sectional curvatures at p are not less than $-k(d(0, p))$, where d is the Riemannian distance of X .

(H.2) For $K(t) := \sup \{ \text{a sectional curvature of } X \text{ at } p \text{ with } d(0, p) \geq t \}$, $\lim_{t \rightarrow \infty} t^2 K(t) < \infty$.

We remark that Gap Theorems which have been proved so far are about Riemannian manifolds whose sectional curvatures or Ricci curvatures do not change sign.

In this paper, a geodesic, a geodesic ray and a geodesic segment mean geodesics whose domains are $(-\infty, +\infty)$, $[0, +\infty)$ and a closed interval, respectively.

§1. Riemannian manifold without focal point

A complete Riemannian manifold is said to have no focal point if its any

maximal geodesic has no focal points along any perpendicular geodesics. In this paper X always denotes a complete connected simply connected Riemannian manifold without focal point and d denotes its Riemannian distance. Two geodesic rays γ and σ of X are asymptotic, by definition, if, for some constant c , $d(\gamma(t), \sigma(t)) \leq c$ for every $t \geq 0$. The asymptotic relation is an equivalence relation and the set of all the asymptotic classes of geodesic rays of X is called the boundary of X and denoted by $X(\infty)$. The Busemann function f_γ for a geodesic ray γ of X is, by definition, a function $f_\gamma: p \rightarrow \lim_{t \rightarrow \infty} [d(p, \gamma(t)) - t]$. We relate some basic facts about Riemannian manifolds without focal points (cf. [6]).

FACT 1. With the cone topology, $X \cup X(\infty)$ is homeomorphic to the unit disc in the Euclidian space of the same dimension.

FACT 2. The distance functions from each points in X are convex. Therefore Busemann functions are convex.

FACT 3. If geodesic rays γ and σ are asymptotic to each other, then the difference of the Busemann functions $f_\gamma - f_\sigma$ is equal to a constant.

FACT 4. Let γ and σ be geodesic rays starting from a point p . Then we have,

$$2 \sin^2 \{ \angle(\dot{\gamma}(0), \dot{\sigma}(0))/2 \} \leq d(\gamma(t), \sigma(t))/t,$$

where $\angle(\dot{\gamma}(0), \dot{\sigma}(0))$ means the angle subtended by two vectors $\dot{\gamma}(0)$ and $\dot{\sigma}(0)$.

For each point p in X and each point z in $X(\infty)$, there exists only one geodesic ray in the class z which start from p . We say that the geodesic ray joins p and z or that the geodesic ray is from p to z , and we denote it by γ_z . Let γ_z and γ_w be two geodesic rays from p to z and from p to w , respectively. We denote the angle subtended by $\dot{\gamma}_z(0)$ and $\dot{\gamma}_w(0)$ by $\angle_p(z, w)$. Since Busemann functions are convex, we have the following.

FACT 5. Let z and w be points in $X(\infty)$, p a point in X and γ the geodesic ray from p to z . Then $\angle_{\gamma(t)}(z, w)$ is monotonely nondecreasing in t .

By Fact 5, we have the following lemma.

LEMMA 1. Let Δpqz be a triangle in X where p and q are points in X and z is a point in $X(\infty)$. Then the sum of the three inner angles (the inner angle at z is equal to zero) is not greater than π .

FACT 6. Busemann functions are C^2 (cf. [3]).

§ 2. Some metrics on boundaries

DEFINITION 1. For z and w in $X(\infty)$, we define the angle $\angle(z, w)$ subtended by z and w as follows: $\angle(z, w) := \sup_{p \in X} \angle_p(z, w)$.

It is easy to prove that the angle is a metric on $X(\infty)$ and that $(X(\infty), \angle)$ becomes a complete metric space.

Since Busemann functions are convex, for geodesic rays γ and σ , the numbers $f_{\gamma \circ \sigma}(t)/t$ converges to a real number as t tends to infinity. Therefore, for z and w in $X(\infty)$, the limit of $f_{\sigma \circ \gamma_z}(t)/t$ exists as t tends to infinity where σ is a geodesic ray in the class w , and, using Fact 3, the limit is independent on a choice of geodesic rays σ and γ_z in the classes w and z , respectively. We have the following lemma.

LEMMA 2. For z and w in $X(\infty)$, we have

$$\cos \angle(z, w) = -\lim_{t \rightarrow \infty} f_w \circ \gamma_z(t)/t,$$

where f_w is a Busemann function for a geodesic ray in the class w .

PROOF. Let f and γ be a Busemann function for w and a geodesic ray in the class z with $f \circ \gamma(0) = 0$, respectively. Since $f \circ \gamma(0) = 0$ and $f \circ \gamma(t)$ is convex in t , we have

$$-d(f \circ \gamma(t))/d\tau|_{\tau=t} \leq -f \circ \gamma(t)/t \leq -d(f \circ \gamma(\tau))/d\tau|_{\tau=0}.$$

Hence, $\cos \angle_{\gamma(t)}(z, w) \leq -f \circ \gamma(t)/t \leq \cos \angle_{\gamma(0)}(z, w)$. Thus we have,

$$\cos \angle(z, w) = -\lim_{t \rightarrow \infty} f \circ \gamma(t)/t, \quad \text{and} \quad \cos \angle(z, w) = \lim_{t \rightarrow \infty} \cos \angle_{\gamma(t)}(z, w). \quad \square$$

For a metric δ , Gromov defined the interior metric δ_i of the metric (cf. [2]). According to M. Gromov, we define the Tits metric as follows.

DEFINITION 2. $Td := \angle_i$.

Pick a point p in X . Let γ_z and γ_w be geodesic rays from p to z and from p to w , respectively. Since $d(\gamma_z(t), \gamma_w(t))/t \leq 2$ holds, there exists the superior limit of $d(\gamma_z(t), \gamma_w(t))/t$ as t tends to infinity.

DEFINITION 3. For z and w in $X(\infty)$,

$$l(z, w) := \limsup_{t \rightarrow \infty} d(\gamma_z(t), \gamma_w(t))/t.$$

Using Fact 4, we can prove that l is a metric on $X(\infty)$. Also we can prove that l is independent on a choice of a point p .

LEMMA 3. We have the following inequality,

$$2 \sin^2 \{ \angle(z, w)/2 \} \leq l(z, w) \leq 2 \sin \{ \angle(z, w)/2 \}.$$

PROOF. The first inequality is obtained by Fact 4, because l is independent on a choice of a point p . We prove the second inequality. Let γ and σ be the geodesic rays from p to z and from p to w , respectively, where z, w are points in $X(\infty)$ and p is a point in X . Let $\alpha_t := \angle_{\gamma(t)}(p, \sigma(t))$ and $\beta_t := \angle_{\sigma(t)}(p, \gamma(t))$ (see Figure 1). By the first variational formula for arc length, we have

Figure. 1

$$d\{d(\gamma(\tau), \sigma(\tau))\}/d\tau|_{\tau=t} = \cos \alpha_t + \cos \beta_t.$$

Using $\cos \alpha_t + \cos \beta_t \leq 2 \cos \{(\alpha_t + \beta_t)/2\}$, we have

$$d\{d(\gamma(\tau), \sigma(\tau))\}/d\tau|_{\tau=t} \leq 2 \sin \{(\pi - \alpha_t - \beta_t)/2\}.$$

By Lemma 1, $\angle_{\gamma(t)}(\sigma(t), w) + \pi - \beta_t \leq \pi$ holds. Thus, using the inequality; $\angle_{\gamma(t)}(z, w) + \angle_{\gamma(t)}(w, \sigma(t)) + \alpha_t \geq \pi$, we have $\angle_{\gamma(t)}(z, w) \leq \pi - \alpha_t - \beta_t$. The left hand side one is non negative because X has no focal point. Hence

$$d\{d(\gamma(\tau), \sigma(\tau))\}/d\tau|_{\tau=t} \leq 2 \sin \{ \angle(z, w)/2 \}.$$

Therefore, $d(\gamma(t), \sigma(t))/t \leq 2 \sin \{ \angle(z, w)/2 \}$, that is

$$l(z, w) \leq 2 \sin \{ \star(z, w)/2 \}.$$

□

We state about the third metric on $X(\infty)$. Choose a point p in X . For z and w in $X(\infty)$, let γ and σ be the geodesic rays from p to z and from p to w , respectively. Let $d_{p,t}^S$ be the Riemannian distance of the geodesic sphere centered at p with radius t .

DEFINITION 4. The metric $\tilde{d}_p$ on $X(\infty)$ is defined as follows.

$$\tilde{d}_p(z, w) := \limsup_{t \rightarrow \infty} d_{p,t}^S(\gamma(t), \sigma(t))/t.$$

Then the following inequality is obvious: $\tilde{d}_p(z, w) \geq l(z, w)$ for every z and w in $X(\infty)$. We state about the reverse inequality. Let $\rho(t)$ be the distance between a point p and the geodesic segment which joins $\gamma(t)$ and $\sigma(t)$. Since the length of a Jacobi field of X which vanish at a point monotonely increase, we have

$$d(\gamma(t), \sigma(t)) \geq d_{p,\rho(t)}^S(\gamma(\rho(t)), \sigma(\rho(t))).$$

Using the fact that geodesic sphere of X are convex, we have

$$2(t - \rho(t)) \leq d(\gamma(t), \sigma(t)).$$

Thus, if there is no geodesic of X which joins z and w , then

$$l(z, w) \geq \tilde{d}_p(z, w)(1 - l(z, w)).$$

Therefore, $l_i = (\tilde{d}_p)_i$ holds. And the metric $(\tilde{d}_p)_i$ does not depend on a choice of a point p . We denote this metric by $\tilde{d}_i$.

PROPOSITION 1. Let X be a m -dimensional complete connected simply connected Riemannian manifold without focal point. Suppose $(X(\infty), Td)$ is isometric to (S, g_0) , where (S, g_0) is the $(m-1)$ -dimensional sphere with the standard metric. Then X is isometric to the m dimensional Euclidian space.

PROOF. Since the metric space $(X(\infty), Td)$ is isometric to the $(m-1)$ -dimensional standard sphere, there exist points S and N in X and a closed subset A of $(X(\infty), Td)$ such that, for every point z in A , the following equality holds; $Td(S, z) = Td(N, z) = \pi/2$. Choose a geodesic γ with $\gamma(\infty) = N$. Then using the fact that $Td(\gamma(\infty), \gamma(-\infty)) \geq \star(\gamma(\infty), \gamma(-\infty)) \geq \pi$, and that S is the unique point which satisfy the inequality $Td(N, S) \geq \pi$, $\gamma(-\infty) = S$ holds.

Set; $A := \{z \in X(\infty) : Td(S, z) = Td(N, z) = \pi/2\}$,

$\Gamma_i := \{\sigma : \sigma \text{ is a geodesic ray from } \gamma(t) \text{ to a point } z \text{ in } A\}$

and

$$Y_t := \{\sigma(s) : \sigma \in \Gamma_t \text{ and } s \geq 0\}.$$

Since $Td(N, z) = Td(S, z) = \pi/2$, for a point z in A ,

$$\star_{\gamma(0)}(N, z) \leq \pi/2 \quad \text{and} \quad \star_{\gamma(0)}(S, z) \leq \pi/2.$$

Using the triangular inequality; $\star_{\gamma(0)}(N, z) + \star_{\gamma(0)}(z, S) \geq \star_{\gamma(0)}(N, S) = \pi$, we have $\star_{\gamma(0)}(z, N) = \star_{\gamma(0)}(z, S) = \pi/2$. Hence $\star_{\sigma(s)}(\sigma(\infty), N) = \star_{\sigma(s)}(\sigma(\infty), 0) = \pi/2$ for every σ in Γ_t . Thus we have $f_N^{-1}(0) = f_S^{-1}(0) = Y_0$, where f_N and f_S are Busemann functions for N and S with $f_N^{-1}(0) = f_S^{-1}(0) = 0$, respectively. Similarly we have $f_N^{-1}(t) = f_S^{-1}(t) = Y_t$, for every real number t . When we prove that Y_t is isometric to Y_0 for every real number t and that X is isometric to $\mathbf{R} \times Y_0$, the proposition can be proved inductively. Therefore we will prove that Y_t is isometric to Y_0 for every real number t and that X is isometric to $\mathbf{R} \times Y_0$. Let $\{\varphi(t)\}$ be the one parameter group of diffeomorphisms of X generated by the gradient vector field of f_N . Then because of $f_N^{-1}(t) = f_S^{-1}(-t)$, $\{\psi_t := \varphi_t^{-1}\}$ is the one parameter group of diffeomorphisms of X generated by the gradient vector field of f_S . Using the fact that the length of a stable Jacobi field of X is monotonically nonincreasing (cf. [3]), φ_t 's are isometries. Thus Y_t is isometric to Y_0 and X is isometric to $\mathbf{R} \times Y_0$. $\square$

§3. Proof of Theorem 1

DEFINITION 5. A Riemannian manifold X without focal point satisfies the visibility axiom if, for each $p \in X$ and each $\varepsilon > 0$, there exists a number $r = r(p, \varepsilon)$ with the property; if $\gamma : [a, b] \rightarrow X$ is a geodesic segment such that $d(p, \gamma) \geq r$, then $\star_p(\sigma(a), \sigma(b)) \leq \varepsilon$.

In the same way as for an Hadamard manifold, it can be easily proved that the visibility axiom for X is equivalent to the property; for every pair of different points z and w in $X(\infty)$, there exists a geodesic γ of X such that $\gamma(-\infty) = z$ and $\gamma(\infty) = w$.

LEMMA 4. Let z and w be points in $X(\infty)$ such that $\star(z, w) = \pi$. Suppose there is no geodesic of X which joins z and w . Then there is a point such that the following equality holds,

$$\star(z, u) = \star(w, u) = \star(z, w)/2.$$

PROOF. Choose a point p in X . Let γ and σ be the geodesic rays from p to z and from p to w , respectively. Let r_t be the nearest point on the geodesic segment which joins $\gamma(t)$ and $\sigma(t)$ to the point p . Since there is no geodesic of X which joins z and w , there is a divergent sequence $\{t_n\}$ of positive real numbers such that r_{t_n} converges to a point in $X(\infty)$. When we prove that $\angle_q(z, u) \leq \pi/2$ and $\angle_q(w, u) \leq \pi/2$ for each q in X , then, using the triangular inequality, we have $\angle(z, u) = \pi/2$ and $\angle(w, u) = \pi/2$. We will prove $\angle_q(z, u) \leq \pi/2$ and $\angle_q(w, u) \leq \pi/2$ for each q in X .

Set $a_n := \gamma(t_n)$, $b_n := \sigma(t_n)$ and $r_n := r_{t_n}$. For $\varepsilon > 0$, there exists a natural number N such that $\angle_{r_n}(p, q) < \varepsilon$, for $n \geq N$, because X has no focal point. Thus $\angle_{r_n}(q, a_n) > \pi/2 - \varepsilon$ and $\angle_{r_n}(q, b_n) > \pi/2 - \varepsilon$. Since the distance function from a_n is strictly convex, $\angle_{r_n}(a_n, q) + \angle_q(a_n, r_n) < \pi$. Thus we have $\angle_q(r_n, a_n) < \pi/2 - \varepsilon$. Similarly, $\angle_q(r_n, b_n) < \pi/2 - \varepsilon$ holds. Hence we have $\angle_q(z, w) \leq \pi/2$ and $\angle_q(w, z) \leq \pi/2$. $\square$

LEMMA 5. The following three conditions are equivalent.

- (1) X satisfies the visibility axiom.
- (2) $\angle(z, w) = \pi$ for every pair of different points z and w in $X(\infty)$.
- (3) $l(z, w) = 2$ for every pair of different points z and w in $X(\infty)$.

PROOF. From (1) to (2) is obvious, and (2) and (3) are equivalent because of Lemma 3. And so it is sufficient to prove that (2) implies (1). Suppose there is a points z and w in $X(\infty)$ such that $\angle(z, w) = \pi$ and there is no geodesic which joins z and w . Then by Lemma 4, there is a point u in $X(\infty)$ with $\angle(z, w) = \pi/2$. This proves that (2) implies (1). $\square$

LEMMA 6. Suppose there is a pair of different two points z and w in $X(\infty)$ such that z and w can not be joined by a geodesic of X . Then for any $\theta > 0$, there exist a point u in $X(\infty)$ such that $\angle(w, u) \leq \theta$.

PROOF. Choose a point p in X . Let γ be the geodesic ray from p to z and τ_t the geodesic ray from $\gamma(t)$ to w . Let $x_{t,s}$ be the asymptotic class containing the geodesic ray from p to $\tau_t(s)$. Then the angle $\angle_{\tau_t(s)}(x_{t,s}, w)$ converges to zero as s tends to infinity by Proposition 8 in [6]. Thus there is a point r_t on τ_t such that $\angle_{r_t}(x_{t,s}, w) \leq \theta$. Since there is no geodesic of X which joins z and w , there is a divergent sequence $\{t_n\}$ such that $\{r_{t_n}\}$ converges to a point u in $X(\infty)$. We prove that $\angle_q(u, w) \leq \theta$ for every q in X . Set $r_n := r_{t_n}$. Take a point q in X . For any $\varepsilon > 0$, there is a natural number n such that $\angle_{r_n}(p, q) < \varepsilon$. Since $\angle_{r_n}(p, q)$

$\geq \pi - \theta$, $\angle_{r_n}(q, w) \geq \pi - \theta - \varepsilon$ holds. Hence $\angle_q(r_n, w) \leq \pi - (\pi - \theta - \varepsilon) = \theta + \varepsilon$. Thus we have $\angle_q(u, w) \leq \theta$. $\square$

For the proof of Theorem 1, it is sufficient to prove the following claim.

CLAIM. The following two conditions are equivalent.

- (a) $\tilde{d}_p(z, w) = +\infty$ for every pair of different z and w in $X(\infty)$.
- (b) $l(z, w) = 2$ for every pair of different z and w in $X(\infty)$.

PROOF OF CLAIM. Suppose there is a pair of different z and w such that $l(z, w) < 2$. Then z and w can not be joined by any geodesic of X . Hence by Lemma 6, there is a point u in $X(\infty)$ with $2 \sin \{ \angle(w, u)/2 \} \leq l/2$. Hence $\tilde{d}_p(w, u) \leq \frac{l(w, u)}{1 - l(w, u)/2} \leq 2/3$. Next we prove that (b) implies (a). Choose a pair of different points z and w . Let $c: [0, 1] \rightarrow (X(\infty), \text{sph.})$ be any continuous curve, where sph. is the sphere topology induced by the cone topology on $X \cup X(\infty)$. Let $0 = s_0 < s_1 < s_2 < \dots < s_k = 1$ be k points in $[0, 1]$. Because $l(c(s_i), c(s_{i+1})) = 2$, for $\varepsilon > 0$ there is a positive t such that $d(\gamma_i(t), \gamma_{i+1}(t))/t \geq 2 - \varepsilon/k$. Thus $d_{p, \varepsilon}^S(\gamma_i(t), \gamma_{i+1}(t))/t \geq 2 - \varepsilon/k$.

Hence $\text{length}(c_t)/t \geq 2k - \varepsilon$, where c_t is the curve in $S_\varepsilon(p)$ naturally corresponding to c . $\square$

§4. Proof of Theorem 2

DEFINITION 5 (cf. [1]). A complete Riemannian manifold M with a base point 0 satisfying the following condition (H.1) is said to be asymptotically nonnegatively curved;

(H.1) There is a monotonly nonincreasing function $k: [0, \infty) \rightarrow [0, \infty)$ such that

$$(1) \int_0^\infty tk(t)dt < \infty,$$

(2) the sectional curvatures at $p \geq -k(d(0, p))$, for each point p in M , where d is the Riemannian distance of M .

Let M be an asymptotically nonnegatively curved manifold with a base point 0 . Set

$$K(t) := \sup \{ \text{a sectional curvature of } M \text{ at } p \text{ with } d(0, p) \geq t \}.$$

Let (H.2) be the following condition;

$$(H.2) \quad \kappa_M := \limsup_{t \rightarrow \infty} t^2 K(t) < \infty.$$

For the proof of Theorem 2, we refer to A. Kasue's results (although he stated

them in more general case, we need them only in the following simple case).

Let $\theta(z, w) := 2 \arcsin \{l(z, w)/2\}$ for z and w in $X(\infty)$, where as we stated in §1, X denotes always a complete connected simply connected Riemannian manifold without focal point and $X(\infty)$ is its boundary.

PROPOSITION A (cf. [7]; Proposition 2.4). Let X be an asymptotically non-negatively curved m -dimensional manifold without focal point with a base point 0. Then $\theta(z, w) = \min \{\tilde{d}_0(z, w), \pi\}$ holds and θ is a metric on $X(\infty)$.

PROPOSITION B (cf. [7]; Theorem 2.10 and Corollary 2.11). Let X be as in Proposition A satisfying also (H.2). Then $(X(\infty), \theta)$ is a Riemannian manifold of dimension $m-1$ with $C^{1,\beta}$ -metric ($0 < \beta < 1$) and it is the Lipschitz limit of a sequence $\{M_i\}$ of C^∞ -Riemannian manifolds with the following properties:

- (1) M_i is diffeomorphic to $(m-1)$ -dimensional sphere.
- (2) The sectional curvature K_i of M_i satisfies the following inequality; $1 - \varepsilon_i \leq K_i \leq 1 + \kappa_i + \varepsilon_i$, where ε_i tends to zero as i tends to infinity and κ_i is the real number for M_i defined in (H.2).

A. Kasue did not state exactly as above. However Proposition A is obvious by Proposition 2.4 in [7] and Proposition B is proved as follows: Let $\tilde{S}_t$ be a hypersurface of X as in Theorem 2.10 in [7], the Riemannian manifold $(\tilde{S}_t, (1/t^2)g_t)$ is diffeomorphic to $(m-1)$ -dimensional sphere and their curvatures are smaller than some positive constant, where g_t is the Riemannian metric of the hypersurface $\tilde{S}_t$. Thus we have,

$$\text{Vol}_{m-1} \left(\tilde{S}_t, \frac{1}{t^2} g_t \right) = \frac{\text{Vol}_{m-1} \tilde{S}_t}{t^{m-1}} > c,$$

for some positive constant c independent on t . Thus, again by Theorem 2.10 and Corollary 2.11 in [7], we have Proposition B.

Let sph. denotes the topology on $X(\infty)$ induced by the cone topology on $X \cup X(\infty)$.

PROPOSITION 2. Suppose X satisfies (H.1). Then the metric space $(X(\infty), Td)$ and $(X(\infty), \text{sph.})$ are homeomorphic to $(X(\infty), \text{sph.})$.

First we state several lemmas to prove the proposition.

For the function k in Condition (H.1), let H_k be the model for k , that is, H_k is the complete connected simply connected two dimensional Riemannian manifold with a base point $\bar{o}$ such that the Gaussian curvature at $\bar{p}$ is equal to $-k(\bar{d}(\bar{o}, \bar{p}))$ for every $\bar{p}$ in H_k , where $\bar{d}$ is the Riemannian distance of H_k . We begin by calcu-

lating the Tits metric on $H_k(\infty)$. Let y be the solution of a Jacobi equation: $y'' = ky$ with $y(0)=0$, $y'(0)=1$. Since H_k is an Hadamard manifold $Td = \tilde{d}_o$ (cf. [2]). Thus we have,

$$Td(\bar{z}, \bar{w}) = \lim_{r \rightarrow \infty} \angle_o(\bar{z}, \bar{w}) \{y(r)/r\} = \lim_{r \rightarrow \infty} \angle_o(\bar{z}, \bar{w}) y'(r), \quad \text{for } \bar{z}, \bar{w} \in H_k(\infty).$$

Because of $y'(r) = \int_0^r y''(t)dt + 1 = \int_0^r k(t)y(t)dt + 1 \leq \int_0^r tk(t)dt + 1$, $\lim_{r \rightarrow \infty} y'(r)$ is finite. Set $c := \lim_{r \rightarrow \infty} y'(r)$. Then we have the following.

LEMMA 7. Suppose $c \angle_o(\bar{z}, \bar{w}) < \pi$. Then

$$\angle(\bar{z}, \bar{w}) = Td(\bar{z}, \bar{w}) = c \angle_o(\bar{z}, \bar{w}).$$

Next we compare angles of a manifold X with (H.1) and the model H_k . Before doing that, we state a lemma which is used in the proof of Lemma 9.

LEMMA 8. Let M be a complete connected simply connected two dimensional Riemannian manifold without conjugate point. Let c be a geodesic sphere (circle) around a point o in M . Choose any point p in M . Let γ be a geodesic through o and p , and q_0 and q_1 intersedtion points of γ and c with $d(p, q_0) < d(p, q_1)$. Assume c is parametrised on $[0, 2]$ with $c(0)=c(2)=q_0$, $c(1)=q_1$ and $\|\dot{c}(t)\| \neq 0$ for every $t \in [0, 2]$. Then the function: $t \rightarrow d(p, c(t))$ is monotonely increasing on $(0, 1)$ and monotonely decreasing on $(1, 2)$ (see Figure 2).

The proof of the lemma is easy.

Figure. 2

LEMMA 9. Let X be as in Proposition A with a base point 0 and a function k . Let H_k be the model for k with a base point $\bar{o}$. Then for z and w in $X(\infty)$ with $\angle(z, w) < \pi$, we have $\angle(\bar{z}, \bar{w}) \geq \angle(z, w)$, where $\bar{z}$ and $\bar{w}$ are points in $H_k(\infty)$ with

$\star_{\bar{o}}(\bar{z}, \bar{w}) = \star_o(z, w)$. Thus $\star(z, w) \leq c \star_o(z, w)$ for z and w in $X(\infty)$ with $\star_o(z, w) \leq \pi/c$, and $Td(z, w) \leq c \star_o(z, w)$ for any z and w in $X(\infty)$, where c is the constant as in Lemma 7.

PROOF. Let $\gamma, \sigma, \bar{\gamma}$ and $\bar{\sigma}$ be geodesic rays from o to z , from o to w , from $\bar{o}$ to $\bar{z}$ and from $\bar{o}$ to $\bar{w}$, respectively. Choose points $p \in \gamma, \bar{p} \in \bar{\gamma}, q \in \sigma$ and $\bar{q} \in \bar{\sigma}$ such that $d(o, p) = d(\bar{o}, \bar{q}) < d(o, q) = d(\bar{o}, \bar{q}), \star_{\bar{p}}(\bar{z}, \bar{q}) < \pi/2$ and $\star_p(z, q) < \pi/2$. Since the distance functions from $\bar{o}$ and o are convex, using the above inequality, we have $\star_{\bar{q}}(\bar{p}, \bar{q}) > \pi/2$ and $\star_q(p, q) > \pi/2$. By Rauch's comparison theorem $d(\bar{p}, \bar{q}) \geq d(p, q)$, because of $\star_o(\bar{z}, \bar{w}) = \star_o(z, w)$. Thus, using Lemma 8, there is a point $\tilde{q}$ in the acute-angled domain bounded by two geodesic rays $\bar{\gamma}$ and $\bar{\sigma}$ such that $d(\bar{o}, \tilde{q}) = d(o, q)$ and $d(\bar{p}, \tilde{q}) = d(p, q)$. Since $\tilde{q}$ is in the outside of the geodesic triangle $\Delta \bar{o} \bar{p} \bar{q}$, $\star_{\bar{p}}(\bar{o}, \tilde{q}) \geq \star_{\bar{p}}(\bar{o}, \bar{q})$. By Theorem 3.2 in [1], we have the inequality $\star_{\bar{p}}(\bar{o}, \tilde{q}) \geq \star_p(o, p)$. Thus $\star_{\bar{p}}(\bar{o}, \bar{q}) \leq \star_p(o, q)$. This proves the lemma. $\square$

Thus we have proved Proposition 2.

PROPOSITION 3. Suppose X satisfies (H.1). Then $\theta(z, w) = \star(z, w)$ for every z and w in $X(\infty)$.

Before proving Proposition 3, we prove the following.

LEMMA 10. Suppose $(X(\infty), \star)$ and $(X(\infty), \text{sph.})$ are homeomorphic to each other. Then, for z and w in $X(\infty)$ with $\star(z, w) < \pi$, there is a point u in $X(\infty)$ such that $\star(z, w) + \star(w, u) = \star(z, u) = \pi$.

PROOF. Let γ be the geodesic ray from o to w , σ_n the geodesic with $\sigma_n(0) = \gamma(n)$ and $\sigma_n(\infty) = z$, and $u_n := \gamma(-\infty)$, where n 's are positive integers. Since $(X(\infty), \star)$ is compact, there is a subsequence of $\{u_n\}$ (which we also denote by $\{u_n\}$) such that u_n converges to u in $(X(\infty), \star)$ as n tends to infinity. For any $\varepsilon > 0$, there is a positive number N such that $\star(u_n, u) < \varepsilon, \star(z, w) < \star_{\gamma(n)}(z, w) + \varepsilon$ and $\star(w, u) < \star_{\gamma(n)}(w, u) + \varepsilon$, for $n \geq N$. By $\star(z, w) \geq \star_{\gamma(n)}(z, u_n) = \pi$, we have $\star(z, w) \geq \star(z, u_n) - \star(u_n, u) \geq \pi - \varepsilon$. Therefore $\star(z, u) = \pi$. On the other hand, $\star(z, w) + \star(w, u) < \star_{\gamma(n)}(z, w) + \star_{\gamma(n)}(w, u) + 2\varepsilon \leq \star_{\gamma(n)}(z, w) + \star_{\gamma(n)}(w, u_n) + \star_{\gamma(n)}(u_n, u) + 2\varepsilon \leq \pi + 3\varepsilon$. Hence $\star(z, w) + \star(w, u) = \star(z, u) = \pi$. $\square$

By Lemma 3, we have; for z and w in $X(\infty)$,

(a) $\theta(z, w) \leq \star(z, w)$, and

(b) if $\angle(z, w) = \pi$, then $\theta(z, w) = \pi$.

PROOF OF PROPOSITION 3. Choose two points z and w in $X(\infty)$.

Assume $\angle(z, w) < \pi$. By Lemma 10, there is a point u in $X(\infty)$ with $\angle(z, w) + \angle(w, u) = \angle(z, u) = \pi$. Since θ is a metric, by (b), we have $\theta(z, w) + \theta(w, u) \geq \theta(z, u) = \pi$. Thus, according to (a), we have $\angle(z, w) = \theta(z, w)$. This proves Proposition 3.

LEMMA 11. Let S be the standard C^∞ -sphere of dimension $m-1$ and θ_0 the standard Riemannian distance on S . Let θ and $\{d_n\}_{n=1}^\infty$ be Riemannian distances on S of C^0 -Riemannian metric and C^∞ -Riemannian metrics, respectively. Suppose the following.

- (1) $\theta(z, w) \geq \theta_0(z, w)$ for every z and w in S .
- (2) The sectional curvatures of $d_n \geq 1 - \varepsilon(n)$, where $\varepsilon(n)$ converges to zero as n tends to infinity.
- (3) $\theta(z, w) \leq d_n(z, w)\Lambda(n)$ for every z and w in S , where $\Lambda(n)$ converges to 1 as n tends to infinity.

Then $\theta(z, w) = \theta_0(z, w)$ for every z and w in S .

PROOF. First we prove the following claim.

CLAIM. Let g and h be C^0 -Riemannian metric on a compact oriented C^1 -manifold M , and let d and l be Riemannian distances of g and h , respectively. Suppose $d(p, q) \geq l(p, q)$ for every p and q in M . Then $\text{Vol}(M, g) \geq \text{Vol}(M, h)$ holds.

Moreover, if there are points p and q in M with $d(p, q) > l(p, q)$, then $\text{Vol}(M, g) > \text{Vol}(M, h)$ holds.

PROOF OF CLAIM. Let $(e_1^{(\alpha)}, e_2^{(\alpha)}, \dots, e_m^{(\alpha)})$ be positive continuous local orthonormal frame field on an open set U_α of (M, h) , and $(\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \dots, \omega_m^{(\alpha)})$ the dual one forms on U_α . Let $\{U_\alpha\}_{\alpha \in A}$ be a finite open covering such that on each U_α ($\alpha \in A$), there are orthonormal frame field and the dual one-forms as above. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in A}$. Then $\text{Vol}(M, h) = \int_M \sum_{\alpha \in A} \varphi_\alpha \omega_1^{(\alpha)} \wedge \dots \wedge \omega_m^{(\alpha)}$. Since $d \geq l$, $g \geq h$ holds as bilinear forms. Hence $\det(g_{ij}^{(\alpha)}) \geq 1$, where $g_{ij}^{(\alpha)} = g(e_i^{(\alpha)}, e_j^{(\alpha)})$. Thus we have $\text{Vol}(M, g) \geq \text{Vol}(M, h)$ because of

$$\text{Vol}(M, g) = \int_M \sum_{\alpha \in A} \varphi_\alpha \det(g_{ij}^{(\alpha)}) \omega_1^{(\alpha)} \wedge \dots \wedge \omega_m^{(\alpha)}.$$

Suppose there are points p and q in M with $d(p, q) > l(p, q)$. Set $2\delta := d(p, q) - l(p, q)$. There is a broken C^1 -curve $c: [0, 1] \rightarrow M$ with $c(0) = p$, $c(1) = q$ and $\text{length}_h(c) < l(p, q) + \delta$, where $\text{length}_h(c)$ is the length of c with respect to the metric h . Since $d(p, q) = l(p, q) + 2\delta$, $\text{length}_g(c) \geq l(p, q) + 2\delta$, where $\text{length}_g(c)$ is the length of c with respect to the metric g . Hence

$$\int_0^1 (\|\dot{c}(t)\|_g - \|\dot{c}(t)\|_h) dt > \delta,$$

where $\|\cdot\|_g$ and $\|\cdot\|_h$ are length with respect to g and h , respectively. Thus, there is a number $t_0 \in [0, 1]$ such that $\|\dot{c}(t_0)\|_g - \|\dot{c}(t_0)\|_h > 0$.

Therefore $\det(g_{ij})(t_0) > \det(h_{ij})(t_0)$ for a coordinate system at $c(t_0)$. Thus $\text{Vol}(M, g) > \text{Vol}(M, h)$ holds. This proves Claim.

Thus, according to (2), it is sufficient to prove that $\text{Vol}(S, \theta) = \text{Vol}(S, \theta_0)$. Set $\theta_n := \theta_0 / \Lambda(n)$. Then by (2) and (4), we have $\theta_n \leq d_n$. Therefore by Claim, we have,

$$\text{Vol}(S, \theta_0) / \{\Lambda(n)\}^{m-1} = \text{Vol}(S, \theta_n) \leq \text{Vol}(S, d_n).$$

On the other hand, using (3) we have

$$\text{Vol}(S, d_n) \leq (1 - \varepsilon(n))^{-(m-1)/2} \text{Vol}(S, \theta_0).$$

Hence $\text{Vol}(S, \theta) = \text{Vol}(S, \theta_0)$. Again by Claim, we have $\theta = \theta_0$ on S . $\square$

By Proposition 3, we have $\theta(z, w) \geq \varkappa_0(z, w)$. Thus using Proposition B and Lemma 11, we have $\theta(z, w) = \varkappa_0(z, w)$. Therefore by Proposition 1, X is isometric to the Euclidian space of the same dimension. This proves Theorem 2.

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