TESTING THE NULL HYPOTHESIS THAT A NORMAL MEAN VECTOR LIES IN THE POSITIVE ORTHANT

By

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§1. Introduction

Suppose that observations are drawn from a p -variate normal population $N_p(\theta, \sigma^2 A)$, where $\theta = (\theta_1, \dots, \theta_p)'$, A is a known non-singular matrix and σ^2 is a known or unknown scale factor. The likelihood ratio (LR) test for testing the null hypothesis

$$H_0: \theta_i = 0 \quad (i = 1, \dots, p) \quad (1.1)$$

against the alternative $H_1 - H_0$ with

$$H_1: \theta_i \geq 0 \quad (i = 1, \dots, p) \quad (1.2)$$

was first studied by Kudô [5] and independently by Nüesch [7]. It was generalized by Kudô and Choi [6], Shapiro [13] and Hillier [4] to that A is singular. In this paper we consider the problem of testing the hypothesis H_1 against $H_2 - H_1$, where H_2 places no restriction on θ .

A related problem has been considered by Robertson and Wegman [11] as follows. Let $z_1, \dots, z_{p+1}$ be independent observations, where z_i is distributed as $N(v_i, 1)$. They derived the LR test for testing an order restriction of the means against all alternatives and treated the case of a simple order restriction: the means satisfy $v_1 \leq \dots \leq v_{p+1}$. If the transformations $x_i = z_{i+1} - z_i$ and $\theta_i = v_{i+1} - v_i$ ($i = 1, \dots, p$) are made, the null and alternative hypotheses are seen to be equivalent to H_1 and $H_2 - H_1$ respectively, and hence it is a special case of our problem. Some other related papers were given by Robertson [10] and Raubertras and Lee [9].

We employ the LR criterion for our problem. It is well known that the usual procedure for deriving the LR statistic is to consider the projection of a vector onto

a polyhedral cone determined by some linear inequalities. This paper approaches the issue somewhat differently and considers the projection of a vector onto a polyhedral cone determined by the edge vectors. It can be seen that the latter method is more favorable for the present case.

A projection theorem about a polyhedral cone determined by the edge vectors is given in Section 2 and it plays a fundamental role in the subsequent arguments. In Section 3, the LR statistic is derived and its critical point (the upper α point) is determined. An example of the application of our test is shown in Section 4. In Section 5, the monotonicity of the power function is discussed when $p=2$. Finally, the problem in the case of an unknown σ^2 is treated in Section 6.

§ 2. A projection theorem

For a given $p \times p$ non-singular matrix $A=(a_1, \dots, a_p)$, let

$$A^{-1} = B' = \begin{pmatrix} b'_1 \\ \vdots \\ b'_p \end{pmatrix}. \quad (2.1)$$

By S we denote a convex cone with respect to A , that is,

$$\begin{aligned} S &= \{y \in R^p; a'_i y \geq 0, \quad i=1, \dots, p\} \\ &= \{y \in R^p; y = \sum_i \gamma_i b_i, \quad \gamma_i \geq 0, \quad i=1, \dots, p\}. \end{aligned} \quad (2.2)$$

The dual cone of S is denoted by S^* and defined by

$$\begin{aligned} S^* &= \{z \in R^p; z' y \leq 0, \quad y \in S\} \\ &= \{z \in R^p; z = \sum_i \lambda_i a_i, \quad \lambda_i \leq 0, \quad i=1, \dots, p\}. \end{aligned} \quad (2.3)$$

Throughout this paper, $\hat{y}$ and $\tilde{y}$ denote the projections of a vector y onto S and S^* , respectively. The following lemma is well known.

LEMMA 2.1. *For any vector y in R^p , there uniquely exist $\hat{y} \in S$ and $\tilde{y} \in S^*$ such that*

$$y = \hat{y} + \tilde{y}, \quad \hat{y}' \tilde{y} = 0. \quad (2.4)$$

Let M be a subset of $P=\{1, \dots, p\}$ and let $M'=P-M$. Then a face S_M^* of S^* is defined by

$$S_M^* = \{z \in R^p; z = \sum_{i \in M} \lambda_i a_i, \quad \lambda_i < 0, \quad i \in M\}. \quad (2.5)$$

When $M = \phi$, we define $S_\phi^* = \{0\}$. It is easy to see that

$$\begin{aligned} S^* &= \bigcup_{\phi \subseteq M \subseteq P} S_M^*, \\ S_M^* \cap S_N^* &= \phi, \quad \text{if } M \neq N. \end{aligned} \quad (2.6)$$

We denote the number of elements in M by $n(M)$. Let A_M and $B_{M'}$ be two submatrices of A and B with column vectors a_i for $i \in M$ and b_i for $i \in M'$, respectively, and let $[A_M]$ denote the range space of A_M . Then the projection matrices onto $[A_M]$ and $[B_{M'}]$ are

$$A_M(A_M' A_M)^{-1} A_M' \quad \text{and} \quad B_{M'}(B_{M'}' B_{M'})^{-1} B_{M'}'$$

with rank $n(M)$ and $n(M')$ respectively. When $M = \phi$, we define $n(M) = 0$, $A_M = 0$, the p -dimensional zero vector, and $(A_M' A_M)^{-1} = (0)$. From $n(M) + n(M') = p$, we can obtain the following lemma.

LEMMA 2.2. *Let M be a subset of P . For any vector y in R^p , let*

$$\begin{aligned} y_1 &= A_M(A_M' A_M)^{-1} A_M' y \\ y_2 &= B_{M'}(B_{M'}' B_{M'})^{-1} B_{M'}' y. \end{aligned} \quad (2.7)$$

Then we have

$$y = y_1 + y_2, \quad y_1' y_2 = 0,$$

and this is the unique decomposition of y into the elements of $[A_M]$ and $[B_{M'}]$.

For any given vector y , the relation (2.6) implies that there exists a unique subset $M \subseteq P$ such that $\tilde{y} \in S_M^*$. Next theorem is the basis of our present work.

THEOREM 2.1. *Let M be a subset of $P = \{1, \dots, p\}$ and let y be a vector in R^p . Then a necessary and sufficient condition for $\tilde{y} \in S_M^*$ is that*

$$(A_M' A_M)^{-1} A_M' y < 0 \quad (2.8)$$

$$(B_{M'}' B_{M'})^{-1} B_{M'}' y \geq 0. \quad (2.9)$$

If $\tilde{y} \in S_M^*$, then

$$\tilde{y} = A_M(A_M' A_M)^{-1} A_M' y. \quad (2.10)$$

PROOF. Necessity. If $\tilde{y} \in S_M^*$, then

$$y = \tilde{y} + \hat{y} = \sum_{i \in M} \lambda_i a_i + \sum_{j \in P} \mu_j b_j$$

with $\lambda_i < 0$ and $\mu_j \geq 0$. The relation $\tilde{y}'\hat{y} = 0$ implies $\sum_{i \in M} \lambda_i \mu_i = 0$, and hence $\mu_j = 0$ for any $j \in M$. Thus we have

$$y = \sum_{i \in M} \lambda_i a_i + \sum_{j \in M'} \mu_j b_j.$$

From Lemma 2.2, we have $y_1 = \tilde{y}$ and $y_2 = \hat{y}$, and it follows that the conditions (2.8) and (2.9) are satisfied, and now $\tilde{y}$ is of the form (2.10).

Sufficiency. Let y_1 and y_2 be the vectors of the form given in (2.7), the projection of y onto $[A_M]$ and $[B_{M'}]$ respectively. The condition (2.8) implies $y_1 \in S_M^*$. Since $S_M^* \subset S^*$, we have $\tilde{y} = y_1$ and this completes the proof.

§3 The likelihood ratio statistic

We consider the testing problem in the case that σ^2 is known until Section 5, without loss of generality, assume that $\sigma^2 = 1$.

Let X be a sample from a normal population $N_p(\theta, A)$, where A is known. Our problem is testing $H_1: \theta_i \geq 0$ ($i = 1, \dots, p$) against $H_2 - H_1$, where H_2 places no restriction on θ .

Let A be the $p \times p$ matrix satisfying $A'A = A$ and let $B' = A^{-1}$ be of the form (2.1). Then $Y = BX$ is distributed as $N_p(\mu, I_p)$ with $\mu = B\theta$.

Let S be the convex cone of the form (2.2) and S^* be the dual cone of the form (2.3). Our problem is transformed into

$$H_1: \mu \in S \text{ against } H_2 - H_1: \mu \notin S.$$

The likelihood ratio test is based on

$$\exp \left\{ - \inf_{\mu \in S} (Y - \mu)'(Y - \mu)/2 \right\} = \exp \left\{ - \tilde{Y}'\tilde{Y}/2 \right\}.$$

Then the critical region can be written as

$$\tilde{\chi}^2 = \tilde{Y}'\tilde{Y} \geq c_\alpha^2, \quad (3.1)$$

where c_α^2 is the upper α point for $0 < \alpha < 1$ and can be calculated using the equation (3.9) in Theorem 3.3.

From Theorem 2.1, there is a unique subset M of $P = \{1, \dots, p\}$ such that the conditions (2.8) and (2.9) are satisfied, and

$$\tilde{\chi}^2 = Y'A_M(A'_MA_M)^{-1}A'_MY. \quad (3.2)$$

For a given subset M , let X_M be a subvector of X , consisting of the elements X_i

for $i \in M$. The following theorem gives a description of the critical region in terms of X .

THEOREM 3.1. *There exists a subset M of P such that*

$$(A'_M A_M)^{-1} X_M < 0 \quad (3.3)$$

$$X_{M'} - A'_{M'} A_M (A'_M A_M)^{-1} X_M \geq 0, \quad (3.4)$$

and the critical region of the LR test is

$$\tilde{\chi}^2 = X'_M (A'_M A_M)^{-1} X_M \geq c_\alpha^2. \quad (3.5)$$

For the computation of the upper α point c_α^2 , we need the following lemma which follows directly from Theorem 2.1 of Robertson and Wegman [11].

LEMMA 3.1. *If $y \in R^p$ and $\mu \in S$, then*

$$\tilde{y}' \tilde{y} \leq (\widetilde{y - \mu})' (\widetilde{y - \mu}).$$

THEOREM 3.2. *For given $0 < \alpha < 1$, if c_α^2 is a constant such that*

$$\sum_{\phi \subseteq M \subseteq P} Pr\{Y' A_M (A'_M A_M)^{-1} A'_M Y \geq c_\alpha^2, (A'_M A_M)^{-1} A'_M Y < 0, \\ (B'_{M'} B_{M'})^{-1} B'_{M'} Y \geq 0\} = \alpha, \quad (3.6)$$

where the probability is computed under H_0 , i.e., the mean vector $\theta = \mu = \mathbf{0}$, the p -dimensional zero vector, then the level of the LR test with the form (3.1) is α .

PROOF. By $\{^3M \subseteq P\}$ we denote the event that there exists a subset $M \subseteq P$ such that the conditions (2.8) and (2.9) in Theorem 2.1 are satisfied. Thus when the normal mean $\mu \in S$ is given, the probability of the critical region can be written as

$$Pr\{\tilde{Y}' \tilde{Y} \geq c_\alpha^2, ^3M \subseteq P | \mu\}.$$

By lemma 3.1, we have

$$\begin{aligned} & Pr\{\tilde{Y}' \tilde{Y} \geq c_\alpha^2, ^3M \subseteq P | \mu\} \\ & \leq Pr\{(\widetilde{Y - \mu})' (\widetilde{Y - \mu}) \geq c_\alpha^2, ^3M \subseteq P | \mu\} \\ & = Pr\{\tilde{Y}' \tilde{Y} \geq c_\alpha^2, ^3M \subseteq P | \mathbf{0}\}. \end{aligned} \quad (3.7)$$

The last term in (3.7) is the probability of the critical region under $\mu=0$ and the proof is completed.

Let $Z_M = (A'_M A_M)^{-1} A'_M Y$ and $Z_{M'} = (B'_{M'} B_{M'})^{-1} B'_{M'} Y$. Then Z_M and $Z_{M'}$ are independently distributed as $N_{n(M)}(0, A_M^{-1})$ and $N_{n(M')}(0, A_{M'}^{-1})$ respectively, where $A_M = A'_M A_M$ and $A_{M'} = B'_{M'} B_{M'}$. Thus $(A_M Z_M)'(A_M Z_M) = Y' A_M (A'_M A_M)^{-1} A'_M Y$ has the chi-square distribution with $n(M)$ degrees of freedom (c.f. Rao, p. 187, [8]) and we have

$$\begin{aligned} & Pr\{Y' A_M (A'_M A_M)^{-1} A'_M Y \geq c_\alpha^2, (A'_M A_M)^{-1} A'_M Y < 0, (B'_{M'} B_{M'})^{-1} B'_{M'} Y \geq 0\} \\ &= Pr\{\chi_{n(M)}^2 \geq c_\alpha^2, Z_M < 0\} Pr\{Z_{M'} \geq 0\} \\ &= Pr\{\chi_{n(M)}^2 \geq c_\alpha^2\} Pr\{Z_M < 0\} Pr\{Z_{M'} \geq 0\}. \end{aligned}$$

The last equality follows from Lemma 3.2 of Kudô [5].

For $i=0, 1, \dots, p$, let

$$P(i, A) = \sum_{\substack{\phi \subseteq M \subseteq P \\ n(M)=i}} Pr\{Z_M < 0\} Pr\{Z_{M'} \geq 0\}, \quad (3.8)$$

where $Pr\{Z_M < 0\} = 1$ if $M = \phi$ and $Pr\{Z_{M'} \geq 0\} = 1$ if $M = P$. Note that $Z_{M'}$ is distributed as the conditional distribution of $X_{M'}$ under the condition $X_i = 0$, $i \in M$, and hence the $Pr\{Z_{M'} \geq 0\}$ is the conditional probability under the condition $X_i = 0$, $i \in M$, for given M . From Theorem 3.2, we have the next theorem.

THEOREM 3.3. *For given $0 < \alpha < 1$, the upper α point c_α^2 is a constant satisfying the following equation*

$$\sum_{i=1}^p P(i, A) Pr\{\chi_i^2 \geq c_\alpha^2\} = \alpha, \quad (3.9)$$

where the weights $P(i, A)$ are given in (3.8).

LEMMA 3.1. *From the relation (2.6) and Theorem 2.1, we can prove the following relation*

$$P(0, A) = 1 - \sum_{i=1}^p P(i, A).$$

Then if $\alpha = 1 - P(0, A)$ the upper α point $c_\alpha^2 = 0$, and if $\alpha > 1 - P(0, A)$, there does not exist a solution satisfying the equation (3.9).

THEOREM 3.2. *The weights $P(i, A)$, given in (3.8), can be computed exactly*

for $p \leq 3$ (see Kudô [5] and Shapiro [13]). When $p \geq 4$, the methods proposed by Abrahamson [1] and Childs [3] may be applied for some special correlation matrices. When $p \leq 7$, an algorithm given by Schervish [12] is useful.

When $p=2$, it is easy to obtain the weights as follows:

$$P(0, A) = \frac{1}{2\pi} (\pi - \cos^{-1} \rho), P(1, A) = \frac{1}{2}, P(2, A) = \frac{1}{2\pi} \cos^{-1} \rho,$$

where ρ is the correlation. The upper α point c_α^2 can be computed from

$$\frac{1}{2} Pr\{\chi_1^2 \geq c_\alpha^2\} + \frac{1}{2\pi} \cos^{-1} \rho Pr\{\chi_2^2 \geq c_\alpha^2\} = \alpha.$$

Some values of c_α , the square root of the upper α point c_α^2 , are listed in the following Table I for $\alpha=0.05, 0.025$ and 0.01 , and $\rho = -1.0(0.1)1.0$.

TABLE I. The values of $c_{2\alpha}$

ρ	$\alpha=0.05$	$\alpha=0.025$	$\alpha=0.01$	ρ	$\alpha=0.05$	$\alpha=0.025$	$\alpha=0.01$
-1.0	2.2668	2.5461	2.8763	0.1	2.0390	2.3366	2.6847
-0.9	2.2169	2.5004	2.8346	0.2	2.0202	2.3191	2.6689
-0.8	2.1935	2.4789	2.8151	0.3	2.0002	2.3005	2.6515
-0.7	2.1740	2.4610	2.7987	0.4	1.9785	2.2803	2.6328
-0.6	2.1563	2.4447	2.7839	0.5	1.9545	2.2579	2.6120
-0.5	2.1395	2.4293	2.7697	0.6	1.9274	2.2325	2.5883
-0.4	2.1231	2.4142	2.7560	0.7	1.8956	2.2027	2.5605
-0.3	2.1069	2.3993	2.7423	0.8	1.8562	2.1654	2.5254
-0.2	2.0906	2.3842	2.7285	0.9	1.8013	2.1130	2.4757
-0.1	2.0739	2.3689	2.7144	1.0	1.6449	1.9599	2.3263
0.0	2.0568	2.3531	2.6999				

REMARK 3.3. For the problem of testing H_0 against $H_1 - H_0$, the LR test rejects H_0 if $\bar{\chi}^2 = Y'Y - \tilde{Y}'\tilde{Y}$ is too large. By using Theorem 2.1, there exists a subset $M \subseteq P$ such that the conditions (3.3) and (3.4) are satisfied, and the LR test statistic is expressed as

$$\bar{\chi}^2 = X' A^{-1} X - X'_M (A'_M A_M)^{-1} X_M.$$

From arguments parallel to those used for establishing Theorem 3.3, we have that under H_0 , $c > 0$

$$Pr\{\bar{\chi}^2 \geq c\} = \sum_{i=0}^{p-1} P(i, A) Pr\{\chi_{p-i}^2 \geq c\}.$$

Thus we have obtained the same result given by Kudô [5] and Kudô and Choi [7].

§ 4. An example

In order to demonstrate our method, a data on the intake of water of genetically obese (fat: fa/fa) and lean (non-obese: Fa/-) Zucker rats (c.f. Zucker and Zucker [15]) is used, which is from the Laboratory of the First Department of Internal Medicine of Kyushu University (see Shi et al., [14]).

Amount of the intake of water was measured by the number of drops of water during the light period (8 a.m.–8 p.m.), denoted by A , and the dark period (8 p.m.–8 a.m.), denoted by B , at four different ages, namely, 12, 23, 30 and 43 weeks, which correspond roughly to 18, 30, 40 and 60 human ages in years, respectively. It was presupposed that for both obese and lean rats the ratio $A/(A+B)$ are increasing with the aging effect. The problem is to test that the increasing trend of obese rats is not lower than that of lean rats.

The random variable A is distributed as Binomial distribution $B(p, n)$, where $n = A + B$. We take the arcsin transformation

$$Y = \sin^{-1}(\sqrt{A/n}).$$

It is well known that Y is asymptotically normal with mean $\sin^{-1}(\sqrt{p})$ and variance $1/(4n)$.

For four different ages, let Y_i ($i=1, 2, 3, 4$) denote the arcsin transformed random variables, distributed asymptotically as $N(\mu_i, \sigma_i^2)$, of the intake of water of obese rats and let Z_i ($i=1, 2, 3, 4$), distributed asymptotically as $N(v_i, \omega_i^2)$, denote that of lean rats. Let $X_i = (Y_{i+1} - Y_i) - (Z_{i+1} - Z_i)$ and let $\theta_i = (\mu_{i+1} - \mu_i) - (v_{i+1} - v_i)$ for $i=1, 2, 3$. Then the problem is testing the null hypothesis

$$H_1: \theta_i \geq 0 \quad (i=1, 2, 3)$$

against the alternative $H_2 - H_1$, where

$$H_2: \theta_i \in (-\infty, \infty) \quad (i=1, 2, 3).$$

From the data, we obtain the sample mean

$$X = (-0.04048, -0.01588, 0.56822)'$$

and the covariance matrix

$$A = \begin{pmatrix} 0.0234 & -0.0112 & 0.0000 \\ -0.0112 & 0.0220 & -0.0108 \\ 0.0000 & -0.0108 & 0.0218 \end{pmatrix}.$$

The conditions (3.3) and (3.4) in Theorem 3.1 are satisfied for $M = \{1, 2\}$, namely, setting $X_{(1)} = (X_1, X_2)'$ and $X_{(2)} = X_3$,

$$A_{11}^{-1} X_{(1)} = (-2.74403, -2.11878)' < 0$$

$$X_{(2)} - A_{12} A_{11}^{-1} X_{(1)} = 0.59110 > 0,$$

and the likelihood ratio statistic $\tilde{\chi}^2 = 0.14472$.

In our case, the weights given in (3.8) are

$$P(1, A) = \frac{1}{4\pi} [3\pi - \cos^{-1} \rho_{12:3} - \cos^{-1} \rho_{13:2} - \cos^{-1} \rho_{23:1}] = 0.2529$$

$$P(2, A) = \frac{1}{4\pi} [3\pi - \cos^{-1} (-\rho_{12}) - \cos^{-1} (-\rho_{13}) - \cos^{-1} (-\rho_{23})] = 0.4571$$

$$P(3, A) = \frac{1}{2} - P(1, A) = 0.24712,$$

where

$$\rho_{ij:k} = (\rho_{ij} - \rho_{ik}\rho_{jk}) / \sqrt{(1 - \rho_{ik}^2)(1 - \rho_{jk}^2)}$$

and ρ_{ij} is the correlation between X_i and X_j (see Shapiro [13]). The significance probability is

$$Pr\{\tilde{\chi}^2 \geq 0.14472\} = 0.8468.$$

Thus we cannot reject the null hypothesis and regard that the increasing trend of obese rats is not lower than that of lean rats.

§ 5. The power function for $p=2$

In this section, we discuss the monotonicity of the power function of the statistic $\tilde{\chi}^2$ for $p=2$. Without loss of generality, we assume that the variances are one and the correlations ρ .

Let

$$B = \begin{pmatrix} 1/\sqrt{1-\rho^2} & -\rho/\sqrt{1-\rho^2} \\ 0 & 1 \end{pmatrix},$$

$Y = BX$ and $\mu = B\theta$. Then Y is distributed as $N(\mu, I_2)$ and the null hypothesis is transformed as $H_1: B^{-1}\mu \geq 0$, that is,

$$H_1: \sqrt{1-\rho^2}\mu_1 + \rho\mu_2 \geq 0, \quad \mu_2 \geq 0.$$

The critical region can be written as

$$G = \begin{cases} y_1^2 + y_2^2 \geq c_\alpha^2, & -\rho y_1 + \sqrt{1-\rho^2} y_2 \leq 0, y_1 \leq 0 \\ -\sqrt{1-\rho^2} y_1 - \rho y_2 \geq c_\alpha, & -\rho y_1 + \sqrt{1-\rho^2} y_2 \geq 0, \sqrt{1-\rho^2} y_1 + \rho y_2 \leq 0 \\ -y_2 \geq c_\alpha, & y_1 \geq 0, y_2 \leq 0. \end{cases} \quad (5.1)$$

Let $C = \{\theta \in R^2: \theta_1 \geq 0, \theta_2 \geq 0\}$ and let $C_A = \{\theta \in R^2: \theta' A^{-1} \theta = \Delta^2, \theta \notin C\}$. The $\theta' A^{-1} \theta$ is said to be the Mahalanobis distance of the population mean vector from the origin (c.f. Anderson, pp. 206, [2]). Then $S = BC$ and we will consider the power function on the subset C_A , and equivalently on the subset $D_A = \{\mu \in R^2: \mu' \mu = \Delta^2, \mu \notin S\}$.

Fig. 1. The critical region G of the $\tilde{\chi}^2$ test for $p=2$. The vectors $a = (\sqrt{1-\rho^2}, \rho)'$, $b = (-\rho, \sqrt{1-\rho^2})'$ and the angle $\beta = \cos^{-1}(-\rho)$.

We now take the polar coordinates:

$$\begin{aligned} y_1 &= \gamma \cos \psi, & y_2 &= \gamma \sin \psi, \\ \mu_1 &= \Delta \cos \phi, & \mu_2 &= \Delta \sin \phi, \end{aligned} \quad (5.2)$$

for $0 \leq \gamma, \Delta < \infty$ and $0 \leq \psi, \phi \leq 2\pi$. Setting $\beta = \cos^{-1}(-\rho)$, then $\mu \in D_\Delta$ implies $\beta < \phi < 2\pi$. The power of $\tilde{\chi}^2$ becomes a function of $\phi = \arg(\mu)$ for $\mu \in D_\Delta$ (see Fig. 1) and found to be

$$Pr\{\tilde{\chi}^2 \geq c_\alpha^2 | \mu\} = Pr\{G | \mu\} = P_1 + P_2,$$

where G is the critical region given in (5.1) and

$$P_1 = \Phi(c_\alpha - \Delta \sin \phi) \Phi(\Delta \cos \phi) + \Phi(\Delta \cos(\phi - \beta)) \Phi(\Delta \sin(\phi - \beta) - c_\alpha)$$

$$P_2 = \frac{1}{2\pi} e^{-\Delta^2/2} \int_{c_\alpha}^{\infty} \gamma e^{-\gamma^2/2} \int_{\beta-\phi}^{\pi-\phi} e^{-\Delta \gamma \sin \psi} d\psi d\gamma.$$

THEOREM 5.1. *The power function of $\tilde{\chi}^2$ increases strictly on $\left(\beta, \pi + \frac{\beta}{2}\right]$ and decreases strictly on $\left[\pi + \frac{\beta}{2}, 2\pi\right)$.*

PROOF. Let μ_1 and μ_2 be two parameters belonging to D_Δ , where $\arg(\mu_1) = \phi_1$ and $\arg(\mu_2) = \phi_2$, and the angles satisfy

$$\beta < \phi_2 < \phi_1 \leq \pi + \beta/2. \quad (5.3)$$

Let $\eta = \phi_1 - \phi_2$ and let $G^* = \{z \in R^2: z = Hy, y \in G\}$ with $H = (h_{ij})$, $h_{11} = h_{22} = \cos \eta$, $h_{12} = \sin \eta$ and $h_{21} = -\sin \eta$. Since $\mu_1 = H\mu_2$, we have

$$\int_G f(y - \mu_2) dy = \int_{G^*} f(y - \mu_1) dy,$$

where f is the standard normal density function in R^2 , and hence

$$\begin{aligned} Pr\{G | \mu_1\} - Pr\{G | \mu_2\} &= \int_G f(y - \mu_1) dy - \int_{G^*} f(y - \mu_1) dy \\ &= \int_{G-G^*} f(y - \mu_1) dy - \int_{G^*-G} f(y - \mu_1) dy. \end{aligned}$$

As the rejection region G is symmetric with respect to the line $\{y \in R^2: \arg(y) = \beta/2\}$, $G - G^*$ is symmetric to $G^* - G$ with respect to the line $l = \{y \in R^2:$

$\arg(y) = (\beta + \eta)/2$. Let Πy be the reflection of y with respect to l . From the relation (5.3), it is easy to see that

$$(y - \mu_1)'(y - \mu_1) > (\Pi y - \mu_1)'(\Pi y - \mu_1)$$

for any $y \in G^* - G$. Thus we have

$$\begin{aligned} \int_{G^* - G} f(y - \mu_1) dy &< \int_{G^* - G} f(\Pi y - \mu_1) dy \\ &= \int_{G - G^*} f(y - \mu_1) dy, \end{aligned}$$

which implies $Pr\{G|\mu_1\} - Pr\{G|\mu_2\} > 0$.

Similarly, we can prove the case when $\phi \in \left[\pi + \frac{\beta}{2}, 2\pi\right)$ and the proof is completed.

COROLLARY 5.1. *For fixed Δ^2 , the power function of $\tilde{\chi}^2$ takes the maximum at $\phi = \pi + \beta/2$, and*

$$\inf_{\mu \in D_\Delta} Pr\{G|\mu\} = Pr\{G|v_1\} = Pr\{G|v_2\},$$

where v_1 and v_2 are two parameters satisfying $v_1'v_1 = v_2'v_2 = \Delta^2$, $\arg(v_1) = \beta$ and $\arg(v_2) = 2\pi$.

§ 6. The case of unknown σ^2

In this section, we derive the LR test in the case that the covariance matrix is of the form $\sigma^2 A$, where A is known and σ^2 is an unknown scale factor. Let $X_1, \dots, X_n$ be a sample of size n from a normal population $N_p(\theta, \sigma^2 A)$ and let $\bar{X}$ be the sample mean vector. Making the same transformation as in Section 3, let $Y_i = BX_i$ ($i = 1, \dots, n$), $\bar{Y} = B\bar{X}$ and $\mu = B\theta$, where $B' = A^{-1}$ and $A'A = A$. Then $\bar{Y}$ is distributed as $N_p(\mu, \sigma^2 I/n)$, and the null and alternative hypotheses can be written as

$$H_1: \mu \in S, \sigma^2 > 0 \quad \text{and} \quad H_2 - H_1: \mu \notin S, \sigma^2 > 0$$

respectively, where S is given in (2.2).

It is well known that the maximum likelihood estimator of σ^2 under H_2 is

$$\begin{aligned}
\hat{\sigma}^2 &= \sum_{i=1}^n (Y_i - \bar{Y})'(Y_i - \bar{Y})/np \\
&= \sum_{i=1}^n (X_i - \bar{X})' A^{-1} (X_i - \bar{X})/np.
\end{aligned} \tag{6.1}$$

After some calculations, it follows that the maximum likelihood estimator of σ^2 under H_1 is

$$\tilde{\sigma}^2 = \hat{\sigma}^2 + \tilde{Y}'\tilde{Y}/p, \tag{6.2}$$

where $\hat{\sigma}^2$ is given by (6.1) and $\tilde{Y}$ is the projection of $\bar{Y}$ onto S^* , the dual cone of S with the form (2.3).

The LR test rejects H_1 if $\hat{\sigma}^2/\tilde{\sigma}^2$ is too small and equivalently

$$\tilde{F} = \tilde{Y}'\tilde{Y}/\hat{\sigma}^2 \tag{6.3}$$

is too large. Since $\hat{\sigma}^2$ is independent of $\tilde{Y}'\tilde{Y}$ and distributed as χ^2 distribution with $(n-1)p$ degrees of freedom, from Theorem 3.1 and 3.3, we have the following

THEOREM 6.1. *There exists a unique subset M of P such that the conditions (3.3) and (3.4) are satisfied. The null hypothesis is rejected if*

$$nX'_M(A'_M A_M)^{-1}X_M/\hat{\sigma}^2 \geq d_\alpha^2.$$

The above constant d_α^2 is the upper α point computed by

$$\sum_{i=1}^p P(i, A) \Pr \left\{ \frac{i}{(n-1)p} F_{i, (n-1)p} \geq d_\alpha^2 \right\} = \alpha,$$

where $F_{a,b}$ denotes a random variable having the F distribution with parameters a and b , and $P(i, A)$ are given in (3.8).

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