A NOTE ON BUILT-UPNESS

By

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(Received March 10, 1988)

1. Introduction.

The purpose of this note is to extend the concept of built-up systems of fundamental sequences by D. Schmidt [4]. We will introduce (n) - k -diagonal-built-up systems of fundamental sequences and show some results corresponding to Theorem 1 of [4]. This note is an extension of our previous paper [1].

The concept “built-up” was introduced in [4], and extended “ (n) -built-up” in [1]. “(0)-built-up” agrees with “built-up”. The canonical system of fundamental sequences used in Ketonen-Solovey [2] is (1)-built-up but not (0)-built-up (cf. [1]). However the system of fundamental sequences used in Löb-Wainer [3] is not (n) -built-up for any $n < \omega$. Löb-Wainer’s system is (1)-0-diagonal-built-up and (0)-1-diagonal-built-up. Moreover, Ketonen-Solovey’s (1)-built-up system is also (1)-0-diagonal-built-up and (0)-1-diagonal-built-up.

Authors would like to express their deep thanks to Prof. T. Uesu and Prof. H. Ono for many useful advice and helpful comments.

2. Preliminaries.

Let Δ be an initial segment of the second number class. We will use Greek letters $\alpha, \beta, \gamma, \dots$, for ordinal numbers in Δ . We denote the set of limit ordinals in Δ by $\text{Lim}(\Delta)$.

DEFINITION 2.1.

Let $\alpha \in \text{Lim}(\Delta)$, and $\alpha_i \in \Delta$ for all $i < \omega$.

- (1) The sequence $\langle \alpha_i \rangle_{i \in \omega}$ is a *quasi-fundamental sequence* for α if $\alpha_i < \alpha$ for all $i < \omega$.
- (2) The sequence $\langle \alpha_i \rangle_{i \in \omega}$ is a *fundamental sequence* for α if $\alpha_i < \alpha_{i+1} < \alpha$ for all $i < \omega$ and $\lim_{i \in \omega} \alpha_i = \alpha$.

DEFINITION 2.2.

Assume that, for each $\alpha \in \text{Lim}(\mathcal{A})$, a (quasi-) fundamental sequence for α is given. We call $P: \mathcal{A} \rightarrow \mathcal{A}^\omega$ the system of (quasi-) fundamental sequences if

$$P(\beta) = \begin{cases} \lambda x. \beta_x & \text{if } \beta \in \text{Lim}(\mathcal{A}), \text{ where } \langle \beta_i \rangle_{i \in \omega} \text{ is the} \\ & \text{(quasi-) fundamental sequence for } \beta, \\ \lambda x. \gamma & \text{if } \beta = \gamma + 1, \\ \lambda x. 0 & \text{if } \beta = 0. \end{cases}$$

We write $\alpha[i]$ for $(P(\alpha))(i)$ whenever $\alpha \in \mathcal{A}$.

DEFINITION 2.3.

Let P be a system of quasi-fundamental sequences. For each $n < \omega$, $\xrightarrow[n]{1}$, $\xRightarrow[n]{1}$ are defined as follows;

- (1) $\alpha \xrightarrow[n]{1} \beta$ if and only if $0 < \alpha$ and $\alpha[n] = \beta$,
- (2) $\alpha \xrightarrow[n]{1} \beta$ if and only if there is a sequence $\gamma_0, \dots, \gamma_j$ ($0 < j < \omega$) such that

$$\gamma_0 = \alpha, \gamma_j = \beta \text{ and } \gamma_i \xrightarrow[n]{1} \gamma_{i+1} \text{ (} 0 \leq i < j \text{),}$$

- (3) $\alpha \xRightarrow[n]{1} \beta$ if and only if $\alpha \xrightarrow[n]{1} \beta$ or $\alpha = \beta$.

3. Diagonal-built-up systems.

DEFINITION 3.1.

Let $k=0$ or 1 . For each $n < \omega$ and for a system of quasi-fundamental sequences P , P is (n) - k -diagonal-built-up if

$$\alpha[i+1] \xRightarrow[n+k]{1} \alpha[i]$$

for all i with $n \leq i < \omega$ and for all $\alpha \in \text{Lim}(\mathcal{A})$.

P is (n) -built-up if $\alpha[i+1] \xRightarrow[n]{1} \alpha[i]$ for all $i < \omega$ and for all $\alpha \in \text{Lim}(\mathcal{A})$ (cf. [1]). By Proposition 1.1 of [1] and the above definition, if P is $(n+1)$ -built-up then P is $(n+1)$ -0-diagonal-built-up and (n) -1-diagonal-built-up, and if P is (0) -built-up then P is (0) -0-diagonal-built-up and (0) -1-diagonal-built-up. It is clear that if

P is (n) - k -diagonal-built-up and $n < m$ then P is (m) - k -diagonal-built-up, where $k=0$ or 1 .

PROPOSITION 3.2.

Let $k=0$ or 1 , and P be (n) - k -diagonal-built-up. If $n \leq m < s$ and $\alpha \xrightarrow{m} \beta$, then $\alpha \xrightarrow{s} \beta$.

PROOF.

By induction on α . If $\alpha=0$ then the above assertion is trivial. If $\alpha=\gamma+1$ and $\alpha \xrightarrow{m} \beta$, then $\alpha \xrightarrow{s} \alpha[s] = \gamma = \alpha[m] \xrightarrow{m} \beta$. So $\alpha \xrightarrow{s} \beta$ since $\alpha[m] \xrightarrow{s} \beta$ by the induction hypothesis. Assume that α is limit and $\alpha \xrightarrow{m} \beta$. Since $\alpha[m] \xrightarrow{m} \beta$ and the induction hypothesis, we have $\alpha[m] \xrightarrow{s} \beta$. On one hand, since P is (n) - k -diagonal-built-up and $n \leq m < s$, we have

$$\alpha[s] \xrightarrow{(s-1)+k} \alpha[s-1] \cdots \xrightarrow{m+k} \alpha[m].$$

Using the induction hypothesis, respectively,

$$\alpha[s] \xrightarrow{s} \alpha[s-1] \cdots \xrightarrow{s} \alpha[m],$$

because $k=0$ or 1 . That is to say, $\alpha[s] \xrightarrow{s} \alpha[m]$. So $\alpha \xrightarrow{s} \beta$ since $\alpha \xrightarrow{s} \alpha[s]$.

COROLLARY 3.3.

If P is (n) -0-diagonal-built-up then P is (n) -1-diagonal-built-up.

COROLLARY 3.4.

Let $k=0$ or 1 , and P be (n) - k -diagonal-built-up. If $n \leq m < s$, then $\alpha[s] \xrightarrow{(s-1)+k} \alpha[m]$.

PROPOSITION 3.5.

Let $k=0$ or 1 , and a system of fundamental sequences P be given. If P is (n) - k -diagonal-built-up and $\beta < \alpha$, then there exists an m with $n \leq m$ such that $\alpha \xrightarrow{m} \beta$.

PROOF.

By induction on α . If $\alpha=0$ then this proposition is trivial. If $\alpha=\gamma+1$ and $\beta < \alpha$, then $\beta \leq \gamma$. By the induction hypothesis, there exists an m with $n \leq m$ such that $\gamma \xrightarrow{m} \beta$. So $\alpha \xrightarrow{m} \beta$ since $\alpha \xrightarrow{m} \gamma$. Assume that α is limit and $\beta < \alpha$. There exists an i with $n \leq i$ such that $\beta < \alpha[i]$ because $\lim_{i \in \omega} \alpha_i = \alpha$. By the induction

hypothesis, there exists a j with $n \leq j$ such that $\alpha[i] \xrightarrow{j} \beta$. Let $m = \max(i, j)$. By Proposition 3.2 and 3.4, we get $\alpha[m] \xrightarrow{m} \alpha[i]$ and $\alpha[i] \xrightarrow{m} \beta$. So $\alpha \xrightarrow{m} \beta$ since $\alpha \xrightarrow{1} \alpha[m]$.

4. Main results.

DEFINITION 4.1.

A function $f: \omega \rightarrow \omega$ is *strictly increasing after n* if $f(x) < f(x+1)$ for $n \leq x < \omega$.

DEFINITION 4.2.

Let a sequence of functions $\langle f_\delta \rangle_{\delta \in \text{Lim}(A)}$ and n and a system of quasi-fundamental sequences P be given. If a sequence of functions $\langle F_\delta \rangle_{\delta \in A}$ satisfies the following conditions (a)–(c), then we say that $\langle F_\delta \rangle_{\delta \in A}$ satisfies $GH(n, P, \langle f_\delta \rangle_{\delta \in \text{Lim}(A)})$.

- (a) F_0 is strictly increasing after n .
- (b) If F_δ is strictly increasing after n , so is $F_{\delta+1}$.
 $F_\delta(n) \leq F_{\delta+1}(n)$ and $F_\delta(x) < F_{\delta+1}(x)$ for $n < x < \omega$.
- (c) $F_\delta(x) = F_{\delta[x]}(f_\delta(x))$ for $n \leq x < \omega$ if δ is limit, where $\delta[x]$ is $(P(\delta))(x)$.

DEFINITION 4.3.

- (1) A sequence of functions $\langle f_\delta \rangle_{\delta \in \text{Lim}(A)}$ is *suitable* as auxiliary functions in $GH(n, P, \langle f_\alpha \rangle_{\alpha \in \text{Lim}(A)})$ if $x \leq f_\delta(x)$ for $n \leq x < \omega$ and f_δ is strictly increasing after n for all $\delta \in \text{Lim}(A)$.
- (2) A sequence of functions $\langle f_\delta \rangle_{\delta \in \text{Lim}(A)}$ is *quite suitable* as auxiliary functions in $GH(n, P, \langle f_\delta \rangle_{\delta \in \text{Lim}(A)})$ if $x < f_\delta(x)$ for $n \leq x < \omega$ and f_δ is strictly increasing after n for all $\delta \in \text{Lim}(A)$.

THEOREM 4.4.

Let $\langle f_\delta \rangle_{\delta \in \text{Lim}(A)}$ and a system of fundamental sequences P be given. If $\langle F_\delta \rangle_{\delta \in A}$ satisfies $GH(n, P, \langle f_\delta \rangle_{\delta \in \text{Lim}(A)})$ and $\langle f_\delta \rangle_{\delta \in \text{Lim}(A)}$ is suitable and P is $(n+1)$ -0-diagonal-built-up and (n) -1-diagonal-built-up, then the following hold for each $\alpha, \beta \in A$.

- (1) F_α is strictly increasing after n .
- (2) If $n < m$ and $\alpha \xrightarrow{m} \beta$, then $F_\beta(m) \leq F_\alpha(m)$ and $F_\beta(x) < F_\alpha(x)$ for $m < x < \omega$.

PROOF.

By induction on α .

Case 1. $\alpha = 0$.

(1) is (a), and (2) is trivial in this case.

Case 2. $\alpha = \gamma + 1$.

- (1) F_γ is strictly increasing after n by the induction hypothesis, so is F_α by (b).
 (2) If $\alpha \xrightarrow{m} \beta$ then $\gamma \xrightarrow{m} \beta$. By the induction hypothesis and (b), we can see $F_\beta(m) \leq F_\gamma(m) \leq F_\alpha(m)$ and $F_\beta(x) < F_\gamma(x) < F_\alpha(x)$ for $m < x$.

Case 3. α is limit.

- (1) Let $n \leq x$. Since f_α is strictly increasing after n , $f_\alpha(x) < f_\alpha(x+1)$. So $F_{\alpha[x]}(f_\alpha(x)) < F_{\alpha[x]}(f_\alpha(x+1))$ by (1) of the induction hypothesis. P is (n) -1-diagonal-built-up and $n \leq x$, then $\alpha[x+1] \xrightarrow{x+1} \alpha[x]$. Since $n < x+1 \leq f_\alpha(x+1)$, $F_{\alpha[x]}(f_\alpha(x+1)) \leq F_{\alpha[x+1]}(f_\alpha(x+1))$ by (2) of the induction hypothesis. Hence $F_{\alpha[x]}(f_\alpha(x)) < F_{\alpha[x+1]}(f_\alpha(x+1))$. That is to say, $F_\alpha(x) < F_\alpha(x+1)$.
 (2) Let $n < m$ and $\alpha \xrightarrow{m} \beta$. Then $\alpha[m] \xrightarrow{m} \beta$. By $m \leq f_\alpha(m)$ and the induction hypothesis, $F_\beta(m) \leq F_{\alpha[m]}(m) \leq F_{\alpha[m]}(f_\alpha(m)) = F_\alpha(m)$. Let $m < x$. By (2) of the induction hypothesis, $F_\beta(x) \leq F_{\alpha[m]}(x)$. P is $(n+1)$ -0-diagonal-built-up and $n < m < x$, then $\alpha[x] \xrightarrow{x-1} \alpha[m]$ by Corollary 3.4. Hence $F_{\alpha[m]}(x) < F_{\alpha[x]}(x)$ by $n < x-1$ and (2) of the induction hypothesis. By (1) of the induction hypothesis and $x \leq f_\alpha(x)$, $F_{\alpha[x]}(x) \leq F_{\alpha[x]}(f_\alpha(x))$. Hence, $F_\beta(x) < F_{\alpha[x]}(f_\alpha(x))$. Namely, $F_\beta(x) < F_\alpha(x)$.

COROLLARY 4.5.

Under the same assumptions in Theorem 4.4, if $\alpha < \beta$ then there exists an m such that $F_\alpha(x) < F_\beta(x)$ for $m \leq x$.

This corollary is an immediate consequence of Theorem 4.4 and Proposition 3.5.

THEOREM 4.6.

Let $\langle f_\delta \rangle_{\delta \in \text{Lim}(\Delta)}$ and a system of quasi-fundamental sequences P be given. If $\langle F_\delta \rangle_{\delta \in \Delta}$ satisfies $GH(n, P, \langle f_\delta \rangle_{\delta \in \text{Lim}(\Delta)})$ and $\langle f_\delta \rangle_{\delta \in \text{Lim}(\Delta)}$ is quite suitable and P is (n) -1-diagonal-built-up, then the following hold for each $\alpha, \beta \in \Delta$.

- (1) F_α is strictly increasing after n .
 (2) If $n < m$ and $\alpha \xrightarrow{m} \beta$, then $F_\beta(x) < F_\alpha(x)$ for $m \leq x$.

PROOF.

By induction on α . We write only a proof of (2) in the case that α is limit. Because a proof of the other cases is easy or similar to the corresponding cases of the proof of Theorem 4.4. Assume that α is limit and $n < m$ and $\alpha \xrightarrow{m} \beta$. Let $m \leq x$. Since $\alpha[m] \xrightarrow{m} \beta$, $F_\beta(x) \leq F_{\alpha[m]}(x)$ by (2) of the induction hypothesis. Since P is (n) -1-diagonal-built-up, $\alpha[x] \xrightarrow{x} \alpha[m]$. By (2) of the induction hypo-

thesis, $F_{\alpha[m]}(x) \leq F_{\alpha[x]}(x)$. By (1) of the induction hypothesis and $x < f_\alpha(x)$, $F_{\alpha[x]}(x) < F_{\alpha[x]}(f_\alpha(x))$. Hence $F_\beta(x) < F_{\alpha[x]}(f_\alpha(x))$. Namely, $F_\beta(x) < F_x(x)$.

5. Examples.

EXAMPLE 5.1 (Ketonen-Solovey [2]).

Fundamental sequences in [2] are defined as follows. For each limit ordinal $< \varepsilon_0$,

$$(\omega^\alpha + \beta)_i = \begin{cases} \omega^\alpha + \beta_i & \text{if } \beta \text{ is limit,} \\ \omega^{(\alpha_i)} & \text{if } \beta = 0 \text{ and } \alpha \text{ is limit,} \\ \omega^\gamma \cdot i & \text{if } \beta = 0 \text{ and } \alpha = \gamma + 1. \end{cases}$$

The system of fundamental sequences P_{K-S} determined by the above is (1)-built-up by Theorem 2.4 of [2]. Then P_{K-S} is (1)-0-diagonal-built-up and (0)-1-diagonal-built-up by the remark following Definition 3.1.

The fast growing hierarchy $\langle F_\delta \rangle_{\delta \in \varepsilon_0}$ in [2] is defined as follows.

$$\begin{cases} F_0(x) = x + 1, \\ F_{\alpha+1}(x) = F_\alpha^{x+1}(x), \\ F_\alpha(x) = F_{\alpha[x]}(x) \text{ if } \alpha \text{ is limit,} \end{cases}$$

where $\alpha[x]$ is $(P_{K-S}(\alpha))(x)$. We can easily see that $\langle F_\delta \rangle_{\delta \in \varepsilon_0}$ satisfies $GH(0, P_{K-S}, \langle \lambda x \cdot x \rangle_{\delta \in \text{Lim}(\varepsilon_0)})$. Of course $\langle \lambda x \cdot x \rangle_{\delta \in \text{Lim}(\varepsilon_0)}$ is suitable as auxiliary functions.

EXAMPLE 5.2 (Löb-Wainer [3]).

The system of fundamental sequences P_{L-W} in [3] is determined by the following. For each limit ordinal $< \varepsilon_0$,

$$(\omega^\alpha + \beta)_i = \begin{cases} \omega^\alpha + \beta & \text{if } \beta \text{ is limit,} \\ \omega^{(\alpha_i)} & \text{if } \beta = 0 \text{ and } \alpha \text{ is limit,} \\ \omega^\gamma \cdot i & \text{if } \beta = 0 \text{ and } \alpha > \omega \text{ and } \alpha = \gamma + 1, \\ \omega^{\alpha-1} \cdot i + \dots + \omega^1 \cdot i + 2 \cdot i & \text{if } \beta = 0 \text{ and } \alpha < \omega. \end{cases}$$

We can see that P_{L-W} is not (n) -built-up for any n , but (1)-0-diagonal-built-up. Moreover, P_{L-W} is also (0)-1-diagonal-built-up.

The Löb-Wainer's hierarchy $\langle F_\delta^0 \rangle_{\delta \in \varepsilon_0}$ is defined as follows.

$$\begin{cases} F_0^n(x) = (n+1) \cdot (x+1), \\ F_{\alpha+1}^0(x) = F_\alpha^{x+1}(x), \\ F_\alpha^0(x) = F_{\alpha[x]}^0(\rho_\alpha(x)) \quad \text{if } \alpha \text{ is limit,} \\ F_\alpha^{n+1}(x) = F_\alpha^0(F_\alpha^n(x)) \quad \text{if } \alpha \neq 0, \end{cases}$$

where, for α a limit ordinal,

$$\begin{cases} \rho_\alpha(0) = 0, \\ \rho_\alpha(m+1) = \mu z (z > \rho_\alpha(m) \ \& \ (i) \leq m(F_{\alpha[m+1]}^0(x) > F_{\alpha[i]}^0(z))), \end{cases}$$

and where $\alpha[i]$ is $(P_{L-W}(\alpha)) (i)$. $\langle F_\delta^0 \rangle_{\delta \in \varepsilon_0}$ satisfies $GH(0, P_{L-W}, \langle \rho_\delta \rangle_{\delta \in \text{Lim}(\varepsilon_0)})$. Moreover, $\langle \rho_\delta \rangle_{\delta \in \text{Lim}(\varepsilon_0)}$ is suitable as auxiliary functions.

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