EXISTENCE OF THE UNSTABLE ADAMS MAP

By

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(Received September 7, 1987)

§0. Introduction

Let σ''' be a generator of the 2-component $\mathbf{Z}/2$ of $\pi_{12}(S^5)$ with $E^\infty \sigma''' = 120\sigma$, where σ is a generator of $\pi_7^S(S^0) = \mathbf{Z}/240 [T]$. Then we have the following decomposition:

$$\begin{array}{ccc} S^{13} \cup_2 e^{14} & \xrightarrow{A} & S^5 \cup_2 e^6 \\ \uparrow & & \downarrow \\ S^{13} & \xrightarrow{\Sigma \sigma'''} & S^6 \end{array}$$

In this paper we shall show the existence of the following commutative diagram, namely, the existence of the best de-suspension of the Adams map:

$$\begin{array}{ccc} S^{12} \cup_2 e^{13} & \xrightarrow{A} & S^4 \cup_2 e^5 \\ \uparrow & & \downarrow \\ S^{12} & \xrightarrow{\sigma'''} & S^5. \end{array}$$

Although the former diagram can be obtained by the fact, $\{2\epsilon_5, \sigma''', 2\epsilon_{12}\} = 0 [T]$, the latter one can not obtained in this way. Using the mapping A , we obtain the following expressions $[T]$:

$$\mu_3 = \bar{\eta}_3 A i, \quad \rho^{1V} = p A^2 i.$$

Moreover, it would be possible to represent unstable elements related with $\text{Im} J$ in some more uniform context.

* Died October 30, 1984. Maruyama and Mukai edited this unpublished manuscript left by the late Dr. Oka.

§ 1. Homotopy Groups of $S^4 \cup e^8$

We will fix some generators of $\pi_i(S^n)$ (cf. [T]). $\pi_6(S^3) = \mathbb{Z}/12$ and it is generated by the Blakers-Massey element ω . There is a relation $\eta_3^3 = 6\omega$, where $\eta_2: S^3 \rightarrow S^2$ is the Hopf map, $\eta_n = \Sigma^{n-2}\eta_2$, $\eta_n^2 = \eta_n\eta_{n+1}$ and $\eta_n^3 = \eta_n\eta_{n+1}\eta_{n+2}$. Write $\alpha_1 = 4\omega$, an element of order 3. $\pi_7(S^4) = \mathbb{Z}\langle v_4 \rangle \oplus \mathbb{Z}/12\langle \Sigma\omega \rangle$, where $v_4: S^7 \rightarrow S^4$ is the Hopf map. Then $[\iota_4, \iota_4] = 2v_4 - \Sigma\omega$. Write $v_n = \Sigma^{n-4}v_4$ for $n \geq 4$. $\pi_{n+3}(S^n) = \mathbb{Z}/24\langle v_n \rangle$ for $n \geq 5$. There are relations $2v_n = \Sigma^{n-3}\omega$, $8v_n = \Sigma^{n-3}\alpha_1$ and $12v_n = \eta_n^3$ for $n \geq 5$. $\pi_9(S^3) = \mathbb{Z}/3\langle \alpha_1 \Sigma^3\alpha_1 \rangle$, $\pi_{10}(S^4) = \mathbb{Z}/24\langle v_4^2 \rangle \oplus \mathbb{Z}/3\langle (\Sigma\alpha_1)(\Sigma^4\alpha_1) \rangle$, where $v_n^2 = v_nv_{n+3}$ for $n \geq 4$.

Let $Y = S^4 \cup_{\alpha} e^8$, $\alpha \in \pi_7(S^4)$. Then we have the following

LEMMA 1.. $\pi_i(Y, S^4) = 0$ for $i < 8$. $\pi_8(Y, S^4) = \mathbb{Z}\langle u \rangle$. $\pi_9(Y, S^4) = \mathbb{Z}/2\langle u\eta_8 \rangle$. $\pi_{10}(Y, S^4) = \mathbb{Z}/2\langle u\eta_8^2 \rangle$. $\pi_{11}(Y, S^4) = \mathbb{Z}\langle [u, \iota_4] \rangle \oplus \mathbb{Z}/24\langle uv_8 \rangle$. $\pi_{12}(Y, S^4) = \mathbb{Z}/2\langle [u, \eta_4] \rangle$. $\pi_{13}(Y, S^4) = \mathbb{Z}/2\langle [u, \eta_4^2] \rangle$. Here $[\ , \]: \pi_i(Y, S^4) \times \pi_j(S^4) \rightarrow \pi_{i+j-1}(Y, S^4)$ is the relative Whitehead product.

PROOF. For $i \leq 8$, $\pi_i = \pi_i(Y, S^4)$ is obtained by the Hurewicz theorem. For $i = 9, 10$, it is obtained from the Blackers-Massey theorem. For $i = 11, 12, 13$, we apply James' exact sequence [J, Theorem 2.1]

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{i+1}(S^8) & \xrightarrow{H_\alpha} & \pi_{i-7}(S^4) \xrightarrow{[u, \cdot]} & \pi_i(Y, S^4) & \longrightarrow & \pi_i(Y/S^4) \xrightarrow{H_\alpha} & \pi_{i-8}(S^4), \\ & & & & & & & \parallel & \\ & & & & & & & \pi_i(S^8) & \end{array}$$

(the left end of the exact sequence is $\pi_{14}(Y, S^4)$)

where H_α is the composite

$$H_\alpha: \pi_i(S^8) \xrightarrow{H} \pi_i(S^{15}) \xleftarrow[\cong]{\Sigma^8} \pi_{i-8}(S^7) \xrightarrow{\alpha^*} \pi_{i-8}(S^4).$$

$H_\alpha = 0$ in our case, and we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi^4(S^4) \xrightarrow{[u, \cdot]} & \pi_{11}(Y, S^4) & \longrightarrow & \pi_{11}(S^8) & \longrightarrow & 0 \\ 0 & \longrightarrow & \pi_5(S^4) \xrightarrow{[u, \cdot]} & \pi_{12}(Y, S^4) & \longrightarrow & \pi_{12}(S^8) & \longrightarrow & 0 \\ & & & & & \parallel & & \\ 0 & \longrightarrow & \pi_6(S^4) \xrightarrow{[u, \cdot]} & \pi_{13}(Y, S^4) & \longrightarrow & \pi_{13}(S^8) & \longrightarrow & 0. \\ & & & & & \parallel & & \\ & & & & & 0 & & \end{array}$$

The first sequence is split, because $uv_8 \in \pi_{11}(Y, S^4)$ goes to a generator of $\pi_{11}(S^8)$. This completes the proof.

The element $[[\iota_4, \iota_4], \iota_4]$ is non-trivial and of order 3 [L, Th. 1.2.2]. It is a suspension element by $[T', \text{Lemma 4}]$. Therefore

$$[[\iota_4, \iota_4], \iota_4] = \pm(\Sigma\alpha_1)(\Sigma^4\alpha_1) \in \pi_{10}(S^4).$$

We now compute $\pi_i(Y)$, $Y = S^4 \cup_{\alpha} e^8$, $\alpha \in \pi_7(S^4)$, in cases ① $\alpha = [\iota_4, \iota_4]$, ② $\alpha = 2[\iota_4, \iota_4] + a\eta_4^3$ ($a \in \mathbb{Z}/2$). Write $Y = Y_1$ for ①, $Y = Y_2$ for ②. Let $j: Y \rightarrow (Y, S^4)$ be the inclusion. Then, v is an element such that $j_*v = 3[u, \iota_4]$. Let $\eta' = \eta_7^3$ or $6v_7$ according as ① or ②. Then, it will be shown that $\alpha\eta' = 0: S^{10} \rightarrow S^7 \rightarrow S^4$. This null-homotopy defines a coextension of η' ; $\tilde{\eta}': S^{11} \rightarrow S^4 \cup_{\alpha} e^8 = Y_i$ with the property $p\tilde{\eta}' = \Sigma\eta': S^{11} \rightarrow Y_i \xrightarrow{p} S^8$, where $p: Y_i \rightarrow S^8$ pinches S^4 to the base point. ϕ is a generator of $\pi_{11}(S^4) = \mathbb{Z}/15$, ϵ_4 is a generator of $\pi_{12}(S^4) = \mathbb{Z}/2$ (cf. [T]). $i: S^4 \rightarrow Y_i$ denotes the inclusion. Then we have the following

LEMMA 2.

$$\begin{aligned}\pi_{11}(Y_1) &= \mathbb{Z}\langle v \rangle \oplus \mathbb{Z}/2\langle \tilde{\eta}_7^3 \rangle \oplus \mathbb{Z}/15\langle i_*\phi \rangle, \\ \pi_{11}(Y_2) &= \mathbb{Z}\langle v \rangle \oplus \mathbb{Z}/4\langle \tilde{6}v_7 \rangle \oplus \mathbb{Z}/15\langle i_*\phi \rangle, \\ \pi_{12}(Y_k) &= \mathbb{Z}/2\langle v\eta_{11} \rangle \oplus \mathbb{Z}/2\langle i_*\epsilon_4 \rangle \quad \text{for } k=1, 2.\end{aligned}$$

PROOF. We compute the boundary

$$\partial: \pi_k(Y_i, S^4) \longrightarrow \pi_{k-1}(S^4) \quad \text{for } k=11, 12, 13$$

of the long exact sequence for pair (Y, S^4) . For $k=11$,

$$\begin{array}{ccc}\pi_{11}(Y_i, S^4) & \longrightarrow & \pi_{10}(S^4) \\ \parallel & & \parallel \\ \mathbb{Z} \oplus \mathbb{Z}/24 & & \mathbb{Z}/24 \oplus \mathbb{Z}/3 \\ [u, \iota_4], uv_8 & & v_4^2, (\Sigma\alpha_1)(\Sigma^4\alpha_1)\end{array}$$

Notice that $\partial(u) = \alpha$, $\partial[\gamma, \delta] = [\partial(\gamma), \delta]$, $\partial(u\Sigma\gamma) = (\partial u)\gamma = \alpha\gamma$, for

$$\gamma \in \pi_*(Y, S^4), \delta \in \pi_*(S^4).$$

Then

$$\partial[u, \iota_4] = [\alpha, \iota_4] = \begin{cases} [[\iota_4, \iota_4], \iota_4] = \pm(\Sigma\alpha_1)(\Sigma^4\alpha_1) & \textcircled{1}, \\ 2[[\iota_4, \iota_4] + a[\eta_4^3, \iota_4]] = \mp(\Sigma\alpha_1)(\Sigma^4\alpha_1) + a[\eta_4^3, \iota_4] & \textcircled{2}. \end{cases}$$

By [B, Cor. 7.4], $[\eta_4^3, \iota_4] = [\iota_4, \iota_4]\eta_4^3$ since η_4^3 is a suspension. Therefore $[\eta_4^3, \iota_4] = (2v_4 - \Sigma\omega)\eta_4^3 = (\Sigma\omega)\eta_4^3 = \Sigma(\omega\eta_6^3)$, but $\omega\eta_6^3 \in \pi_9(S^3) = \mathbf{Z}/3$ and $2(\omega\eta_6^3) = 0$, so $\omega\eta_6^3 = 0$. Therefore $[\eta_4^3, \iota_4] = 0$, and

$$\partial[u, \iota_4] = \begin{cases} \pm(\Sigma\alpha_1)(\Sigma^4\alpha_1) & \textcircled{1}, \\ \mp(\Sigma\alpha_1)(\Sigma^4\alpha_1) & \textcircled{2}. \end{cases}$$

Next,

$$\partial(uv_8) = \partial(u)v_7 = \alpha v_7 = \begin{cases} [\iota_4, \iota_4]v_7 & \textcircled{1}, \\ 2[\iota_4, \iota_4]v_7 + a\eta_4^3v_7 & \textcircled{2}. \end{cases}$$

We have $[\iota_4, \iota_4]v_7 = 2v_4^2 - (\Sigma\omega)v_7$ and $(\Sigma\omega)v_7 = \Sigma(\omega v_6)$ is a 3-primary element since $\pi_9(S^3) = \mathbf{Z}/3$. Then $-\omega v_6 = 32\omega v_6 = (4\omega)(8v_6) = \alpha_1(\Sigma^3\alpha_1)$. $\eta_4^3v_7 = \eta_4^2\eta_6v_7$ and $\eta_6v_7 \in \pi_{10}(S^6) = 0$ [T], so $\eta_4^3v_7 = 0$. Therefore

$$\partial(uv_8) = \begin{cases} 2v_4^2 + (\Sigma\alpha_1)(\Sigma^4\alpha_1) & \textcircled{1}, \\ 4v_4^2 - (\Sigma\alpha_1)(\Sigma^4\alpha_1) & \textcircled{2}. \end{cases}$$

Hence

Kernel of $\partial: \pi_{11}(Y_i, S^4) \rightarrow \pi_{10}(S^4)$

$$= \begin{cases} \begin{matrix} \mathbf{Z} & \oplus & \mathbf{Z}/2 \\ 3[u, \iota_4] & & 12uv_8 = u\eta_8^3 \end{matrix} & \textcircled{1}, \\ \begin{matrix} \mathbf{Z} & \oplus & \mathbf{Z}/4 \\ 3[u, \iota_4] & & 6uv_8 \end{matrix} & \textcircled{2}. \end{cases}$$

In cases $\textcircled{1}$ and $\textcircled{2}$, $\alpha\eta_7^3 = 0$ and $\alpha(6v_7) = 0$ respectively. Therefore we may define coextensions

$$\widetilde{\eta_7^3} \in \pi_{11}(Y_1) \quad \text{and} \quad \widetilde{6v_7} \in \pi_{11}(Y_2)$$

which go to $\eta_8^3 = \Sigma\eta_7^3$ and $6v_8 = \Sigma(6v_7)$ in $\pi_{11}(S^8)$ respectively. By the commutative diagram

$$\begin{array}{ccc}
\pi_{11}(Y_i) & \xrightarrow{i_*} & \pi_{11}(Y_i S^4) \\
& \searrow p_* & \downarrow p_* \\
& & \pi_{11}(S^8)
\end{array}$$

and $p_*(u\eta_8^3) = \eta_8^3$, $p_*(6uv_8) = 6v_8$ (the right p_* satisfies $p_*u = \epsilon_8$),

$$j_*\widetilde{\eta_7^3} = u\eta_8^3 \quad (1),$$

$$j_*\widetilde{6v_7} = 6uv_8 \quad (2).$$

The next boundary

$$\partial: \pi_{12}(Y, S^4) \longrightarrow \pi_{11}(S^4)$$

is trivial, since $\pi_{12}(Y, S^4) = \mathbf{Z}/2$ and $\pi_{11}(S^4) = \mathbf{Z}/15$. Since $\pi_{11}(S^4)$ is an odd torsion group and $j_*\pi_{11}(Y)$ is a 2-torsion plus free, the group extension at $\pi_{11}(Y)$ is trivial. Next we compute

$$\begin{array}{ccc}
\partial: \pi_{13}(X, S^4) & \longrightarrow & \pi_{12}(S^4). \\
\parallel & & \parallel \\
\mathbf{Z}/2\langle [u, \eta_4^2] \rangle & & \mathbf{Z}/2\langle \epsilon_4 \rangle
\end{array}$$

$\partial[u, \eta_4^2] = [\alpha, \eta_4^2]$, which is $[\alpha, \eta_4]\eta_{11}$ by [B, Cor. 7.4]. Since $[\alpha, \eta_4]$ is a 2-torsion element and $[\alpha, \eta_4] \in \pi_{11}(S^4) = \mathbf{Z}/15$, $[\alpha, \eta_4] = 0$. Therefore $\partial = 0$. Let $v \in \pi_{11}(Y_i)$ be an element with $j_*v = 3[u, \epsilon_4]$, as before. Then $j_*(v\eta_{11}) = 3[u, \epsilon_4]\eta_{11} = [u, \epsilon_4]\eta_{11} = [u, \eta_4]$ and it is a generator of $\pi_{12}(Y, S^4)$. Since $2(v\eta_{11}) = 0$, the group extension at $\pi_{12}(Y)$ is trivial. This completes the proof.

§ 2. Cohomology of the fibration $F \rightarrow S^4 \cup_2 e^5 \rightarrow S^5$

Let $M^n = S^{n-1} \cup_2 e^n$ $p: M^n \rightarrow S^n$ the map pinching S^{n-1} to the base point. Let F be the fibre of $p: M^5 \rightarrow S^5$. Consider the Serre spectral sequence

$$E_2^{s,t} = H^s(S^5; H^t(F)) \implies H^*(M^5) = \begin{cases} \mathbf{Z} & * = 0, \\ \mathbf{Z}/2 & * = 5, \\ 0 & * \neq 0, 5. \end{cases}$$

There is $v_1 \in H^4(F) = \mathbf{Z}$ with $d(v_1) = 2s \in H^5(S^5)$, since $H^5(M^5) = \mathbf{Z}/2$. Then

there exists $v_n \in H^{4n}(F) = \mathbb{Z}$ with $d(v_n) = s \otimes v_{n-1}$, $n \geq 2$, because the spectral sequence collapses in total degrees > 5 . We then have

$$H^i(F) = \begin{cases} \mathbb{Z}\langle v_n \rangle & \text{for } i = 4n, \\ 0 & \text{for } i \neq 4n. \end{cases}$$

By the induction, we shall prove $v_1^n = 2^{n-1}n!v_n$. If $v_1^n = 2^{n-1}n!v_n$,

$$\begin{aligned} d(v_1^{n+1}) &= (n+1)d(v_1)v_1^n = 2(n+1)s \otimes v_1^n \\ &= 2(n+1)2^{n-1}n!(s \otimes v_n) = 2^n(n+1)!s \otimes v_n \\ &= 2^n(n+1)!d(v_{n+1}). \end{aligned}$$

Since d is an isomorphism, we have the assertion. Therefore $v_1^m v_1^n = 2^{m+n-2}m!n!v_m v_n$. On the other hand $v_1^{m+n} = 2^{m+n-1}(m+n)!v_{m+n}$. Hence $v_m v_n = 2\binom{m+n}{m}v_{m+n}$. Thus we have the following

LEMMA 3.

$$H^i(F) = \begin{cases} \mathbb{Z}\langle v_n \rangle & \text{for } i = 4n, \\ 0 & \text{for } i \neq 4n. \end{cases}$$

$$v_1^n = 2^{n-1}n!v_n, \quad v_m v_n = 2\binom{m+n}{m}v_{m+n}.$$

Extend the fibration to the left to get a fibration

$$\Omega S^5 \xrightarrow{d} F \longrightarrow M^5.$$

It is well known (Serre) that

$$H^i(\Omega S^5) = \begin{cases} \mathbb{Z}\langle e_n \rangle & \text{for } i = 4n, \\ 0 & \text{for } i \neq 4n. \end{cases}$$

LEMMA 4. $d^*(v_n) = 2e_n$.

PROOF. $d^*(v_1) = 2e_1$ is obvious. The ring structure of $H^*(\Omega S^5)$ is given by $e_m e_n = \binom{m+n}{m}e_{m+n}$. d^* is a ring homomorphism and both of $H^*(F)$, $H^*(\Omega S^5)$ have no torsion. Hence we have the result.

REMARK. For $S^{n-1} \cup_m e^n$ with m odd, we may obtain the same result. In this case,

$$H^i(F) = \begin{cases} \mathbb{Z}\langle v_k \rangle & \text{for } i = k(n-1), \\ 0 & \text{for } i \not\equiv 0 \pmod{n-1}. \end{cases}$$

$$v_k v_l = m \binom{k+l}{k} v_{k+l}, \quad d^*(v_k) = m e_k.$$

§3. The map d in low skeletons

We may regard ΩS^5 and F as CW complexes, which consist of one cell in each $4n$ dimension. Let Z_1, Z_2 be the 15-skeleton of ΩS^5 and F . Then

$$Z_1 = Y_1 \cup_{\beta} e^{12}, \quad Y_1 = S^4 \cup_{\alpha} e^8.$$

$$Z_2 = Y_2 \cup_{\beta} e^{12}, \quad Y_2 = S^4 \cup_{\alpha} e^8.$$

We may assume d is cellular; the restrictions

$$d: Z_1 \longrightarrow Z_2, \quad d: Y_1 \longrightarrow Y_2$$

are also denoted by the same letter. They are of degree 2 in dimensions 4, 8, 12.

Y_1 is the James construction $J_2 S^4$, which is obtained from $S^4 \times S^4$ by identifying $(x, *)$ with $(*, x)$ for each $x \in S^4$. Therefore the attaching map α in Y_1 is the Whitehead product $[\iota_4, \iota_4]$. The existence of a map $Y_1 \rightarrow Y_2$ of degree (2, 2) leads to a homotopy commutative square

$$\begin{array}{ccccc} S^7 & \xrightarrow{\alpha=[\iota_4, \iota_4]} & S^4 & \longrightarrow & Y_1 \\ 2\iota_7 \downarrow & & \downarrow 2\iota_4 & & \downarrow d \\ S^7 & \xrightarrow{\alpha'} & S^4 & \longrightarrow & Y_2. \end{array}$$

By Lemma 3, α' has the Hopf invariant 4. Therefore we may put $\alpha' = 4v_4 + x\Sigma\omega$ for some $x \in \mathbb{Z}/12$. By the commutativity,

$$\begin{aligned} 8v_4 + 2x\Sigma\omega &= 2\alpha' = (2\iota_4)[\iota_4, \iota_4] = [2\iota_4, 2\iota_4] \\ &= 4[\iota_4, \iota_4] = 4(2v_4 - \Sigma\omega) = 8v_4 - 4\Sigma\omega. \end{aligned}$$

Then $2x \equiv -4 \pmod{12}$, $x \equiv -2 + 6a \pmod{12}$, $a = 0$ or 1 . Therefore $\alpha' = 4v_4 - 2\Sigma\omega + 6a\Sigma\omega = 2[\iota_4, \iota_4] + a\eta_4^3$, $a \in \mathbb{Z}/2$. Y_1, Y_2 are exactly same as in §1.

Now, $\beta \in \pi_{11}(Y_1)$ and $\Sigma\beta = 0$ because $\Sigma\Omega S^5 = \Sigma J_{\infty} S^4 = \bigvee_{i \geq 1} S^{4i+1}$ (James). $\widetilde{\eta}_7^3 \in \pi_{11}(Y_1)$ is non-trivial after suspension, because $p_* \widetilde{\eta}_7^3 = \eta_8^3 \neq 0$. Also we

have $i_*\phi \in \pi_{11}(Y_1)$. Therefore β is a multiple of $v \in \pi_{11}(Y_1)$. In $H^*(Z_1)$, $e_1e_2=3e_3$ ($|e_i|=4i$). This means $j_*\beta=3[u, \iota_4] \bmod uv_8$, by $[J']$. Therefore $\beta=v$ in $\pi_{11}(Y_1)$.

Similarly, $v_1v_2=6v_3$ in $H^*(Z_2)$ implies $j_*\beta'=6[u, \iota_4] \bmod uv_8$. Hence $\beta'=2v \bmod 6v_7$, $i_*\phi$ in $\pi_{11}(Y_2)$.

LEMMA 5. For $\widetilde{\eta_7^3} \in \pi_{11}(Y_1)$, $d_*\widetilde{\eta_7^3}=0$ in $\pi_{11}(Y_2)$.

$$d_*=0: \pi_{12}(Y_1) \longrightarrow \pi_{12}(Y_2).$$

PROOF. Since $\widetilde{\eta_7^3}$ is of order 2, $d_*\widetilde{\eta_7^3}=2x6v_7=x\widetilde{\eta_7^3}$, $x \in \mathbf{Z}/2$, because $26v_7=\widetilde{\eta_7^3}$ is the only element of order 2 in $\pi_{11}(Y_2)$. The map $p_*: \pi_{11}(Y_2) \rightarrow \pi_{11}(S^8)$ is monic on the factor $\mathbf{Z}/4$ generated by $6v_7$. Since $pd=(2\iota_8)p$, i.e.,

$$\begin{array}{ccc} Y_1 & \xrightarrow{q} & S^8 \\ \downarrow d & & \downarrow 2\iota_8 \\ Y_2 & \longrightarrow & S^8 \end{array}$$

commutes, $p_*d_*\widetilde{\eta_7^3}=(2\iota_8)p\widetilde{\eta_7^3}=(2\iota_8)\Sigma\eta_7^3=0$. Therefore $x p_*\widetilde{\eta_7^3}=0$ and $x=0$.

The group $\pi_{12}(Y_1)$ is generated by $v\eta_{11}$, $i_*\varepsilon_4$. By the commutativity

$$\begin{array}{ccc} S^{11} & \xrightarrow{v=\beta} & Y_1 \\ 2\iota_{11} \downarrow & & \downarrow d \\ S^{11} & \xrightarrow{\beta'} & Y_2, \end{array}$$

$d_*v\eta_{11}=d\beta\eta_{11}=\beta'(2\iota_{11})\eta_{11}=0$. By the commutativity

$$\begin{array}{ccc} S^4 & \xrightarrow{i} & Y_1 \\ \downarrow 2\iota_4 & & \downarrow d \\ S^4 & \xrightarrow{i} & Y_2, \end{array}$$

$d_*i_*\varepsilon_4=i_*(2\iota_4)\varepsilon_4=0$, since $\varepsilon_4=\Sigma\varepsilon_3$ and $(2\iota_4)\varepsilon_4=2\varepsilon_4=0$.

LEMMA 6. (i) Let $S^{11} \xrightarrow{i'} M^{12} \xrightarrow{p'} S^{12}$ be the cofibration. There is an element $\gamma \in [M^{12}, Y_1]$ such that $\gamma i'=\widetilde{\eta_7^3}$ and $d_*\gamma=0$. (ii) $[M^{12}, Y_1]=\mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$ generated by γ , $v\eta_{11}p'$, $i_*\varepsilon_4p'$. (iii) $d_*=0: [M^{12}, Y_1] \rightarrow [M^{12}, Y_2]$.

PROOF. The part (iii) is immediate from parts (i), (ii) and Lemma 5. We first prove (i). Since $2\widetilde{\eta_7^3}=0$, there is γ with $\gamma i'=\widetilde{\eta_7^3}$. By Lemma 5, we may put

$d_*\gamma = xv\eta_{11}p' + yi_*\varepsilon_4p'$, $x, y \in \mathbb{Z}/2$, that is the Toda bracket $\{d, \widetilde{\eta}_7^3, 2\iota_{11}\}$ contains $xv\eta_{11} + yi_*\varepsilon_4$.

Consider the Toda bracket $\{[\iota_4, \iota_4], \eta_7^3, 2\iota_{10}\}$, which is contained in $\pi_{11}(S^4) = \mathbb{Z}/15$. Since the bracket has the indeterminacy $\pi_{11}(S^4) \circ (2\iota_{10}) = \pi_{11}(S^4)$, it contains 0. Therefore, there is a complex $A = S^5 \cup e^{11} \cup e^{12}$ together with a map $f: A \rightarrow S^4$ such that $A^{(11)} = S^7 \cup_{\eta_7^3} e^{11}$, $A/S^7 = S^{11} \cup_{2\iota_{11}} e^{12}$, $f|_{S^7} = [\iota_4, \iota_4]$. Notice that the first two conditions do not determine the homotopy type of A and the assertion holds only for a specific complex A . More precisely, let σ and τ be null-homotopies of $[\iota_4, \iota_4]\eta_7^3 = 0$ and of $\eta_7^3(2\iota_{10}) = 0$, and let $B = S^7 \cup_{\eta_7^3} e^{11}$. Then σ defines an extension $g: B \rightarrow S^4$ of $[\iota_4, \iota_4]$, and τ defines a coextension $g': S^{11} \rightarrow B$ of $2\iota_{10}$. There is a choice of σ and τ such that $gg' \simeq *$. Let $A = B \cup_{g'} e^{12}$. Then, for such a choice, there is an extension $f: A \rightarrow S^4$ of g . The homotopy σ defines a coextension $\widetilde{\eta}_7^3: S^{11} \rightarrow S^4 \cup_{[\iota_4, \iota_4]} e^8 = Y_1$, which is just the same $\widetilde{\eta}_7^3$ as in Lemma 2 (with a specific homotopy). Then σ gives a commutative square

$$\textcircled{1} \quad \begin{array}{ccc} B & \xrightarrow{g} & S^4 \\ \downarrow \pi & & \downarrow i \\ S^{11} & \xrightarrow{\widetilde{\eta}_7^3} & Y_1, \end{array}$$

where π collapses S^7 to the base point. Since $if|_{S^7}: S^7 \rightarrow A \rightarrow S^4 \rightarrow Y_1$ is $ig|_{S^7}: S^7 \rightarrow B \rightarrow S^4 \rightarrow Y_1$, which is $i[\iota_4, \iota_4] = 0$, there is a commutative square

$$\textcircled{2} \quad \begin{array}{ccc} A & \xrightarrow{f} & S^4 \\ \downarrow \pi' & & \downarrow i \\ S^{11} \cup_{2\iota_{11}} e^{12} = M^{12} & \xrightarrow{\gamma} & Y_1, \end{array}$$

where π' collapses S^7 to the base point. Hence a square

$$\begin{array}{ccc} B & \hookrightarrow & A \\ \downarrow \pi & & \downarrow \pi' \\ S^{11} & \xrightarrow{i} & M^{12} \end{array}$$

commutes. Now the restriction of $\textcircled{1}$ to B gives $\textcircled{2}$, because both of them are obtained from the same homotopy σ (the other homotopy τ gives an extension of the diagram $\textcircled{1}$ to $\textcircled{2}$). Therefore $\gamma i' = \eta_7^3$, and our γ may be obtained in this way.

Next, the Toda bracket $\{\eta_7^3, 2\epsilon_{10}, \eta_{10}\}$ contains 0, because $\{\eta_7^3, 2\epsilon_{10}, \eta_{10}\} \supset \eta_7^2\{\eta_9, 2\epsilon_{10}, \eta_{10}\} \in \eta_7^2(2v_9)=0 [T]$. The indeterminacy is $\eta_7^3\pi_{12}(S^{10}) + \pi_{11}(S^7)\eta_{11}=0$. Therefore $\{\eta_7^3, 2\epsilon_{10}, \eta_{10}\}$ consists of a unique element 0. Then there is a map $h: S^{13} \rightarrow A$ with $\eta_{12} = \Sigma^2\eta_{10} = p'\pi'h: S^{13} \rightarrow A \xrightarrow{\pi'} M^{12} \xrightarrow{p'} S^{12}$. The triviality of the indeterminacy enables us to construct h for any choice of A . Consider the commutative diagram

$$\begin{array}{ccccccc}
 S^{13} & \xrightarrow{h} & A & \xrightarrow{f} & S^4 & \xrightarrow{2\epsilon_4} & S^4 \\
 & \searrow \eta_{12} & \downarrow \pi' & & \downarrow i & & \downarrow i \\
 & & M^{12} & \xrightarrow{\gamma} & Y_1 & \xrightarrow{d} & Y_2 \\
 & & \downarrow p' & \nearrow xv\eta_{11} + yi\epsilon_4 & & & \\
 & & S^{12} & & & &
 \end{array}$$

The composite fh is in $\pi_{13}(S^4)$, which is $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by $v_4^3, \mu_4, \eta_4\epsilon_5 [T]$. Since μ_4 and $\eta_4\epsilon_5$ are suspension elements, $(2\epsilon_4)\mu_4 = 2\mu_4 = 0$ and $(2\epsilon_4)\eta_4\epsilon_5 = 2\eta_4\epsilon_5 = 0$. By Hilton [H, Theorem 6.9],

$$(2\epsilon_4)v_4 = 2v_4 + [\epsilon_4, \epsilon_4]H(v_4) = 4v_4 - \Sigma\omega.$$

Then $(2\epsilon_4)v_4^3 = 0$. Hence $(2\epsilon_4)fh = 0$. Therefore,

$$xv\eta_{11}^2 + yi\epsilon_4\eta_{12} = (xv\eta_{11} + yi\epsilon_4)\eta_{12} = d\gamma\pi'h = i(2\epsilon_4)fh = 0.$$

This is a relation in $\pi_{13}(Y_2)$. This goes to

$$3x[u, \eta_{11}^2] = 0 \quad \text{in } \pi_{12}(Y_2, S^4).$$

By Lemma 1, $x=0$. To show $y=0$, it is enough to show $i\epsilon_4\eta_{12} \neq 0$ in $\pi_{13}(Y_2)$. We prove it in the stable range, i.e., $\Sigma^\infty i\epsilon_4\eta_{12} \neq 0$ in $\pi_{13}^s(Y_2)$. The attaching map of Y_2 is $2[\epsilon_4, \epsilon_4] + a\eta_4^2$, $a \in \mathbb{Z}/2$, hence stably $a\eta_4^3$. Since $(\Sigma^\infty \eta_4^3)\pi_6^s(S^0) = 0$, $\Sigma^\infty \epsilon_4\eta_{12}$ can not be divisible by $\Sigma^\infty \eta_4^3$, hence $\Sigma^4 i\epsilon_4\eta_{12} \neq 0$.

We next prove (ii). Consider the exact sequence

$$\begin{array}{ccccc}
 \pi_{12}Y_1 & \xrightarrow{\times 2} & \pi_{12}Y_1 & \xrightarrow{p'^*} & [M^{12}, Y_1] \xrightarrow{i'^*} \pi_{11}Y_1 \xrightarrow{\times 2} \pi_{11}Y_1 \\
 \parallel & & & & \parallel \\
 \mathbb{Z}/2 \oplus \mathbb{Z}/2 & & \omega & & \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/15 \\
 & & \gamma \downarrow & & \downarrow \widetilde{\eta}_7^3
 \end{array}$$

Then, it is enough to show $2\gamma=0$. The group $[M^{12}, M^{12}]$ is in the stable range, hence $2id_{M^{12}} = i'\eta_{11}p'$ in $[M^{12}, M^{12}]$ by $[B']$. Therefore

$$2\gamma = \gamma i' \eta_{11} p' = \widetilde{\eta_7^3} \eta_{11} p' \in i_* \{[\epsilon_4, \epsilon_4], \eta_7^3, \eta_{10}\} p' \subset i_* \pi_{12}(S^4) p'.$$

Hence we have $2\gamma = x i_* \epsilon_4 p'$ with $x \epsilon_4 \in \{[\epsilon_4, \epsilon_4], \eta_7^3, \eta_{10}\}$, $x \in \mathbb{Z}/2$. Stabilize it to get $x \Sigma^\infty \epsilon_4 \equiv 0 \pmod{(\Sigma^\infty \eta_{11}) \pi_7^3(S^0)}$. Then $x=0$, and $2\gamma=0$. This completes the proof.

REMARK. From the result on $\pi_{13}(S^4)$ and $\pi_{13}(Y, S^4)$ in Lemma 1, the above discussion on stabilizing leads to the following result:

$$\begin{array}{ccccccc} \pi_{13}(Y) = \mathbb{Z}/2 & \oplus & \mathbb{Z}/2 & \oplus & \mathbb{Z}/2 & \oplus & \mathbb{Z}/2 & (Y=Y_1 \text{ or } Y_2). \\ \parallel & & \parallel & & \parallel & & \parallel & \\ \langle v \eta_{11}^2 \rangle, & & \langle i_* v_4^3 \rangle, & & \langle i_* \mu_4 \rangle, & & \langle i_* \eta_4 \epsilon_5 \rangle \end{array}$$

§ 4. Existence of the Adams map

LEMMA 7. $\pi_{11}(Z_1) = \mathbb{Z}/2 \oplus \mathbb{Z}/15$,

$$k_* \widetilde{\eta_7^3}, \quad k_* i_* \phi$$

$$\pi_{12}(Z_1) = \mathbb{Z}/2,$$

$$k_* i_* \epsilon_4$$

$$\pi_{13}(Z_1) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

$$k_* i_* v_4^3, \quad k_* i_* \mu_4, \quad k_* i_* \eta_4 \epsilon_5$$

where $Z_1 = J_3 S^4$ and $k: Y_1 = J_2 S^4 \rightarrow Z_1$ is the inclusion.

PROOF. By the Blakers-Massey theorem, for $i \leq 13$,

$$\pi_{i+1}(S^{12}) \xrightarrow{\partial} \pi_i(Y_1) \xrightarrow{k_*} \pi_i(Z_1) \longrightarrow \pi_i(S^{12}).$$

is exact. ∂ is given by $\partial(\Sigma\psi) = \beta\psi$, $\beta = v$. Then the result is immediate.

THEOREM. There is a map $A: M^{13} \rightarrow M^5$ with a commutative square

$$\begin{array}{ccc} M^{13} & \xrightarrow{A} & M^5 \\ \uparrow i' & & \downarrow p' \\ S^{12} & \xrightarrow{\sigma''} & S^5. \end{array}$$

PROOF. Consider the diagram

$$\begin{array}{ccccc}
[M^{13}, M^5] & \xrightarrow{p'_*} & [M^{13}, S^5] & \xrightarrow{\partial} & [M^{12}, F] \\
& & \wr & & \parallel \\
& & [M^{12}, \Omega S^5] & \xrightarrow{d_*} & [M^{12}, F] \\
& \cong \uparrow & & & \cong \uparrow \\
[M^{12}, Z_1] & \xrightarrow{d_*} & [M^{12}, Z_2] & & \\
& \uparrow \scriptstyle{epi} & \uparrow \scriptstyle{k_*} & & \uparrow \scriptstyle{k_*} \\
[M^{12}, Y_1] & \xrightarrow{d_*} & [M^{12}, Y_2] & &
\end{array}$$

Since $\pi_{12}(S^5) = \mathbf{Z}/2 \oplus \mathbf{Z}/15$ and $\pi_{13}(S^5) = \mathbf{Z}/2$,

$$\begin{array}{ccc}
\sigma'' & \Sigma\phi & \varepsilon_5
\end{array}$$

$$[M^{13}, S^5] = \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

with generators $\gamma', \varepsilon_5 p'$, where $\gamma' i' = \sigma''$. From Lemma 2,

$$[M^{12}, Y_1] = \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

with generators $\gamma'', v\eta_{11}p', i_*\varepsilon_4 p'$, where $\gamma'' i' = \widetilde{\eta_7^3}$. From Lemma 7,

$$[M^{12}, Z_1] = \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

with generators $k_*\gamma''$ and $k_*i_*\varepsilon_4 p'$. Through the isomorphism

$$\pi_{12}(S^5) \cong \pi_{11}(\Omega S^5) \cong \pi_{11}(Z_1),$$

σ''' corresponds with $k_*\widetilde{\eta_7^3}$ (this fact is equivalent to $H(\sigma''') = \eta_3^3$), hence we have a commutative diagram

$$\begin{array}{ccc}
\Omega S^5 & \xrightarrow{H} & \Omega S^9 \\
\uparrow & & \downarrow \\
Z_1 & \longrightarrow & S^8 \\
\uparrow & \nearrow & \\
Y_1 & &
\end{array},$$

where the mappings are canonical. Therefore γ' corresponds with $k_*\gamma''$ (with changes of γ', γ'' if necessary). Also $\varepsilon_5 p'$ corresponds with $k_*i_*\varepsilon_4 p'$. By Lemma 6 (iii), $\partial=0$: $[M^{13}, S^5] \rightarrow [M^{12}, F]$. Hence, there is an element $A \in [M^{13}, M^5]$ with $p'_*A = \gamma'$. This completes the proof.

REMARK. (1) The argument is applied to construct A with a factorization

$$\begin{array}{ccc} S^{14} \cup_8 e^{15} & \xrightarrow{A} & S^6 \cup_8 e^7 \\ \uparrow i & & \downarrow p \\ S^{14} & \xrightarrow{\sigma'} & S^7. \end{array}$$

- (2) The theorem answers to the question posed by Walker [W, p. 486].
 (3) By more computations,

$$[M^{13}, M^5] = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

$$\begin{array}{cccccc} A & i\bar{\varepsilon}_4 & \tilde{\varepsilon}_4 p & i\nu_4^3 p & \alpha \end{array}$$

Here $\bar{\varepsilon}_4: M^{13} \rightarrow S^4$ is an extension of ε_4 , $\tilde{\varepsilon}_4: S^{13} \rightarrow M^5$ is a coextension of ε_4 and α is the image of $v\bar{\eta}_{11} \in [M^{13}, Y_2]$, i.e.,

$$M^{13} \xrightarrow{\bar{\eta}_{11}} S^{11} \xrightarrow{v} Y_2 \hookrightarrow F \longrightarrow M^5.$$

The stable group was computed by Mukai [M]:

$$\{M^8, M^0\} = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

$$\begin{array}{ccccc} A & i\bar{\varepsilon} & \tilde{\varepsilon} p & i\sigma\bar{\eta} & \bar{\eta}\sigma p \end{array}$$

The last two summands of $[M^{13}, M^5]$ vanish in the stable range, the first three summands of $[M^{13}, M^5]$ survive in the stable range and the last two summands of the stable group are not in $\Sigma^\infty[M^{13}, M^5]$.

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