Complete hypersurface minimally immersed in a unit sphere

By

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1. Introduction

It is known that no closed minimal surfaces exist in an Euclidean space E_3 . S. B. Myers [2] generalized the result to higher dimensions and proved that there exist no closed minimal hypersurfaces in an open hemisphere. In this paper, we consider the following problem.

Let M be an m -dimensional Riemannian manifold which is minimally immersed in a unit sphere $S^{m+1}(1)$ of dimension $m+1$

$$I: M \rightarrow S^{m+1}(1)$$

Let $S_+^{m+1}(1) = \{u \in R^{m+2}; u = (u^1, \dots, u^{m+2}), |u| = 1, u^{m+2} \geq 0\}$ denote the closed hemisphere of $S^{m+1}(1)$. We shall study a minimally immersed hypersurface when $I(M) \subset S_+^{m+1}(1)$.

2. Preliminary

In this section, we shall make some preparations.

Let M be an m -dimensional Riemannian manifold. Then we shall use Δ to denote the Laplace Beltrami operator of M .

LEMMA 1 ([3], p. 38). Let M be a compact and connected Riemannian manifold without boundary. Then the integral

$$\int_M \Delta f \, dv = 0$$

holds for any function f on M , where Δ is the Laplace Beltrami operator.

LEMMA 2 ([1]. p. 176). Let M^* and M be connected Riemannian manifolds of the same dimension. Let $\Phi: M^* \rightarrow M$ be an isometric immersion. If M^* is complete, then M^* is a covering space of M with projection Φ and M is also complete.

LEMMA 3 ([4]. p. 660). If f is a subharmonic function defined on a complete Riemannian manifold M with $\int_M |\nabla f| dv < \infty$, then f is harmonic. Here ∇f is by definition the gradient field for f .

LEMMA 4 ([5]. p. 207). Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Then for any C^2 -function f with $\sup f < \infty$, there are points $\{q_k\}$ in M such that $\lim_{k \rightarrow \infty} f(q_k) = \sup f$ and $\lim_{k \rightarrow \infty} \Delta f(q_k) \leq 0$.

LEMMA 5 ([6]. p. 216). Let M be a complete Riemannian manifold with Ricci curvature bounded from below by a constant $-L$. Let f be a C^3 -function on M bounded from below such that, for some positive constants C_j ,

$$|\Delta f| \leq C_1 |\nabla f| + C_2 (f - \inf f)$$

$$|\nabla(\Delta f)| \leq C_3 \sqrt{\sum f_{ij}^2} + C_4 |\nabla f| + C_5 (f - \inf f).$$

Then we can find a constant C depending only on m , L and the C_j such that

$$|\nabla f| \leq C (f - \inf f).$$

The starting point of our investigation is the Myers' theorem which is stated as follows:

THEOREM (S. B. Myers [2]). Let M be a compact and connected Riemannian manifold without boundary which is minimally immersed in a unit sphere $S^{m+1}(1)$, $I: M \rightarrow S^{m+1}(1)$. If $I(M) \subset S_+^{m+1}(1)$, then $I(M)$ is totally geodesic.

PROOF. Because the immersion I is minimal, we have the component u^{m+2} of coordinate function $u = (u^1, \dots, u^{m+2})$ satisfying $\Delta u^{m+2} = -mu^{m+2}$, $u^{m+2} \geq 0$.

$$(3.1) \quad \int_M \Delta u^{m+2} dv = 0,$$

that is $\int_M u^{m+2} dv = 0$. We claim $u^{m+2} \equiv 0$. Hence,

$$I(M) \subset S^m(1) = \{u \in \mathbf{R}^{m+2}; |u| = 1, u^{m+2} = 0\}.$$

Because M is a compact and connected Riemannian manifold, we have from Lemma 2 $I(M) = S^m(1)$. This completes the proof of Theorem.

3. Main results

THEOREM 1. Let M be a complete and connected Riemannian manifold which is minimally immersed in a unit sphere $S^{m+1}(1)$, $I: M \rightarrow S^{m+1}(1)$. If $I(M) \subset S_+^{m+1}(1)$ and the component u^{m+2} of coordinate function $u = (u^1, \dots, u^{m+2})$ satisfies

$$\int_M |\nabla u^{m+2}| dv < +\infty,$$

then $I(M)$ is totally geodesic.

PROOF. Because the immersion I is minimal, we have

$$(3.2) \quad \Delta u^{m+2} = -m u^{m+2}, \quad u^{m+2} \geq 0.$$

Hence, $\Delta u^{m+2} \leq 0$, $-u^{m+2}$ is a subharmonic function. From

$$|\nabla u^{m+2}| = |\nabla(-u^{m+2})|,$$

Lemma 3 implies that $-u^{m+2}$ is harmonic. From (3.2), $u^{m+2} = 0$, that is, $I(M) \subset S^m(1)$. Because M is complete and connected, Lemma 2 implies $I(M) = S^m(1)$. Hence, $I(M)$ is totally geodesic.

THEOREM 2. Let M be a complete and connected Riemannian manifold which is minimally immersed in a unit sphere $S^{m+1}(1)$, $I: M \rightarrow S^{m+1}(1)$. If $I(M) \subset S_+^{m+1}(1)$ and the volume $\text{Vol}(M)$ of M is finite, then $I(M)$ is totally geodesic.

PROOF. Because the immersion I is minimal, we have

$$\Delta u^i = -m u^i, \quad i = 1, 2, \dots, m+2.$$

$$\sum_{i=1}^{m+2} (u^i)^2 = 1,$$

$$\Delta \sum_{i=1}^{m+2} (u^i)^2 = 0,$$

$$\sum_{i=1}^{m+2} \Delta(u^i)^2 = 0,$$

$$\Delta(u^i)^2 = 2|\nabla u^i|^2 + 2u^i \Delta u^i = 2|\nabla u^i|^2 - 2m(u^i)^2,$$

Hence

$$\sum_{i=1}^{m+2} |\nabla u^i|^2 = m,$$

$$|\nabla u^{m+2}| \leq \sqrt{m},$$

Hence
$$\int_M |\nabla u^{m+2}| dv \leq \sqrt{m} \text{Vol}(M) < \infty.$$

According to Theorem 1, we know that $I(M)$ is totally geodesic.

THEOREM 3. Let M be a complete and connected Riemannian manifold with Ricci curvature bounded from below by a constant $-L$ which is minimally immersed in a unit sphere $S^{m+1}(1)$, $I: M \rightarrow S^{m+1}(1)$. If $I(M) \subset S_+^{m+1}(1)$ and $\int_M u^{m+2} dv < \infty$, then $I(M)$ is totally geodesic.

PROOF. Because the immersion I is minimal, we have (3.2). For $-u^{m+2}$, according to Lemma 4, we can find a sequence $\{q_k\} \subset M$ such that

$$\lim_{k \rightarrow \infty} (-u^{m+2}(q_k)) = \sup(-u^{m+2})$$

$$\limsup_{k \rightarrow \infty} \Delta(-u^{m+2}(q_k)) \leq 0$$

that is,

$$(3.3) \quad \lim_{k \rightarrow \infty} (u^{m+2}(q_k)) = \inf u^{m+2}$$

$$(3.4) \quad \liminf_{k \rightarrow \infty} \Delta u^{m+2}(q_k) \geq 0$$

(3.2) and (3.4) imply $-\inf u^{m+2} = 0$. Hence, $\inf u^{m+2} = 0$. Because

$$|\Delta u^{m+2}| = m|u^{m+2}|$$

$$|\nabla(\Delta u^{m+2})| = m|\nabla u^{m+2}| \text{ and } \inf u^{m+2} = 0,$$

Lemma 5 implies

$$|\nabla u^{m+2}| \leq C u^{m+2}.$$

Hence,

$$\int_M |\nabla u^{m+2}| dv \leq \int_M C u^{m+2} dv < +\infty.$$

According to Theorem 1, we know that $I(M)$ is totally geodesic.

REMARK 1. If the volume $\text{Vol}(M)$ of M is infinite, then $I(M)$ is not totally geodesic.

In fact, if $I(M)$ is totally geodesic, then we can prove M is isometric globally to $S^m(1)$. Hence, $\text{Vol}(M) = \text{Vol}[S^m(1)] < \infty$. This is a contradiction.

REMARK 2. If the Ricci curvature of M is not bounded from below, then $I(M)$ is not totally geodesic.

In fact, if $I(M)$ is totally geodesic, then M is of constant curvature. Hence, the Ricci curvature of M is bounded. This is a contradiction.

Let $\sigma: [0, r(x)] \rightarrow M$ be any unit speed geodesic joining a point x to p . Let $\sigma'(t)$ be the tangent vector of σ and $\text{Ric}(\sigma'(t))$ be the Ricci curvature of M in the direction $\sigma'(t)$. Defined

$$K_\sigma(x) = \min_{0 \leq k \leq r(x)} \{ (m-1)/(r(x)-k) - [1/(r(x)-k)^2] \int_k^{r(x)} (t-k)^2 \text{Ric}(\sigma'(t)) dt \}.$$

If x is not on the cut locus of p , then we can take σ to be the unique shortest geodesic from p to x and define $K(x) = K_\sigma(x)$, where p is a fixed point in M . Otherwise we define $K(x) = \min_\sigma K_\sigma(x)$, where σ ranges over all the minimal geodesic from p to x . Here $r(x)$ is the distance function from p .

PROPOSITION 1. Let f be a C^2 -function defined on a complete Riemannian manifold M . Suppose

$$\liminf_{r(x) \rightarrow \infty} [f(x) - f(p)]/r(x) > -\infty.$$

Then either f attains its infimum at some point with $r(x) \leq 1$ or, for all $0 < \varepsilon < 1$, we can find a sequence $\{q_k\}$ in M such that $r(q_k) > 1$ and

$$\begin{aligned} \lim_{k \rightarrow \infty} f(q_k) &= \inf f, \\ \Delta f(q_k) &\geq [K(q_k) + (1 + \varepsilon)/r(q_k)] 2d_k/(\varepsilon - 1), \end{aligned}$$

with $d_k = [f(q_k) - f(p)]/r(q_k)$.

PROOF. For $-f$,

$$\limsup_{r(x) \rightarrow \infty} \{[-f(x)] - [-f(p)]\}/r(x) = -\liminf_{r(x) \rightarrow \infty} [f(x) - f(p)]/r(x) < +\infty.$$

According to Theorem 1 of [5], we know that the statement of Proposition 1 is true.

proposition 1 actually due to Yau.

THEOREM 4. Let M be a complete and connected Riemannian manifold which is minimally immersed in a unit sphere $S^{m+1}(1)$, $I: M \rightarrow S^{m+1}(1)$. If $I(M) \subset S_+^{m+1}(1)$ and $K(x) + 1/r(x) \geq 0$, then $I(M)$ is totally geodesic.

PROOF. Because the immersion is minimal, we have

$$\Delta u^{m+2} = -mu^{m+2}, \quad u^{m+2} \geq 0,$$

$$\Delta u^{m+2} \leq 0.$$

Hence, u^{m+2} is superharmonic.

1) If u^{m+2} attains its minimum in M , according to Hopf's theorem, we know $u^{m+2} = \text{constant}$. Hence, $u^{m+2} = 0$, $I(M) \subset S^m(1)$, that is, $I(M)$ is totally geodesic.

2) If u^{m+2} does not attain its minimum in M , we shall prove that it is not true. In fact, Because u^{m+2} is not constant, we can find a point p in M such that

$$(3.5) \quad u^{m+2}(p) > \inf u^{m+2} + \delta,$$

where $\delta > 0$ is a real number. From $|u^{m+2}| \leq 1$, we have

$$\lim_{r(x) \rightarrow \infty} [u^{m+2}(x) - u^{m+2}(p)]/r(x) = 0.$$

According to Proposition 1, for all $1 > \varepsilon > 0$, we can find a sequence $\{q_k\}$ in M such that $r(q_k) > 1$,

$$\lim_{k \rightarrow \infty} u^{m+2}(q_k) = \inf u^{m+2},$$

$$\Delta u^{m+2}(q_k) \geq [K(q_k) + (1 + \varepsilon)/r(q_k)]2d_k/(\varepsilon - 1)$$

with $d_k = [u^{m+2}(q_k) - u^{m+2}(p)]/r(q_k)$.

From (3.5), we can choose a subsequence $\{p_k\}$ of $\{q_k\}$ such that $u^{m+2}(p_k) - u^{m+2}(p) < 0$, that is, $d_k < 0$. According to $K(x) + 1/r(x) \geq 0$, we have

$$\Delta u^{m+2}(p_k) \geq 0.$$

Because $u^{m+2}(x)$ does not attain its minimum in M , we get that $u^{m+2}(x) > 0$ and $\Delta u^{m+2}(x) < 0$. This is a contradiction. Hence, $I(M)$ is totally geodesic.

REMARK 3. If the Ricci curvature of M satisfies, for all minimal geodesic $\sigma: [0, r(x)] \rightarrow M$, $\text{Ric}[\sigma'(t)] \leq 3m/r(x)$. Then $K(x) + 1/r(x) \geq 0$ holds.

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