FAMILIES OF COMPLEX ANALYTIC CURVES AND MEROMORPHIC MAPPINGS OF POINCARÉ OF SEVERAL COMPLEX VARIABLES.

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Introduction.

We have flown a trial balloon into the field of meromorphic mappings of Poincaré of two complex variables from the new point of view in the previous paper [2]. Here, we generalize the conclusions of that paper to the case of several complex variables. For this purpose, Section 2 is devoted to the generalization of the reduction theorem of rational ruled surfaces to the case of normal complex spaces of arbitrary dimension—Theorem 1. The proof is a natural application of Nishino's idea (cf. [3]). In Section 1, we prepare two lemmas. Lemma 1 also gives us an elementary proof of the fact that the Riemann domain of the two-valued analytic function $\sqrt{xy}$ over $\mathbf{C}^2$ has the singular point over the origin in any neighborhood of which the Cousin II problem is not universally solvable.

In Section 3, we give a general theorem on a local holomorphic mapping—Theorem 2. Let $(f): \mathbf{C}^n \rightarrow \Pi^n \mathbf{P}$ be a meromorphic mapping of Poincaré derived by a rational transformation (R) of $\Pi^n \mathbf{P}$ where $\Pi^n \mathbf{P}$ is the n -times direct product of Riemann sphere $\mathbf{P}^1$ —the n -dimensional Osgood space. Denote by l_z the complex line passing the origin in $\mathbf{C}^n$ with the direction ratio $z = [z_1: z_2: \cdots: z_n]$. The proper image L_z of l_z under (f) is called an *invariant image curve* of (f) . We say that L_z is *algebraic* if L_z is contained in an algebraic curve in $\Pi^n \mathbf{P}$ in the set-theoretic sense. By the abovementioned Theorem 2, we know the following: if the invariant image curves L_z are algebraic for sufficiently many values of z , L_z are algebraic for all values of z and there exists an algebraic family $\{F_z\}$ of algebraic curves in $\Pi^n \mathbf{P}$ such that L_z is contained in F_z for each $z \in \mathbf{P}^{n-1}$. In this case, the family $\{F_z\}$ is one-folded—Theorem 3 in Section 4. Since L_z is the image of the Gaussian plane

$\mathbf{C}^1$, the universal covering of the normalization $\hat{L}_z$ of L_z is not hyperbolic. So $\hat{L}_z$ must be elliptic or rational. Suppose that all invariant image curves L_z of (f) are algebraic and all $\hat{L}_z$ are rational. Then we can reduce (f) essentially to a simple one each invariant image curve of which is contained in a complex line passing the origin—Theorem 4 in Section 4.

§ 1. Two lemmas.

1. We have employed the terminology of $H.$ Grauert [1] in this paper. Consider a finite-sheeted ramified covering $\mathcal{D}$ with no relative boundary over an n -dimensional complex manifold M , the projection of which we denote by $\tilde{\omega}: \mathcal{D} \rightarrow M$. A complex function f in an open subset δ of $\mathcal{D}$ is said to be holomorphic in δ if f is continuous in δ and is holomorphic in $\delta - \tilde{\omega}^{-1}(K)$ where K is the critical locus of $\tilde{\omega}$. Denote by $\mathcal{O}_{\mathcal{D}}$ the sheaf of germs of holomorphic functions on $\mathcal{D}$. As is well-known, the ringed space $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ is isomorphic to an n -dimensional normal complex space. When we mention a finite-sheeted ramified covering with no relative boundary over a complex manifold, we are always considering this ringed space. Conversely, the normal complex space can be defined from this direction all over again (cf. K. Oka [4], K. Stein [6]).

Let Σ be a purely n -dimensional normal complex space ($n \geq 2$) and let S be a purely $(n-1)$ -dimensional analytic set in Σ —a hypersurface in Σ . We say that S is a *hypersurface of order one* in Σ if an arbitrary point of S has a neighborhood U such that there exists a holomorphic function f on U which takes the value zero only on $S \cap U$ and whose order of zero at each point of $(S - \text{sing}(\Sigma) - \text{sing}(S)) \cap U$ is one.

EXAMPLE 1. Let $\mathcal{D}$ be the Riemann domain of the two-valued analytic function $\sqrt{xy}$ on the polydisc $V = \{(x, y) \in \mathbf{C}^2 \mid |x| < 1, |y| < 1\}$, the projection of which we denote by $\tilde{\omega}: \mathcal{D} \rightarrow V$. Denote by l a complex line $\{(x, y) \in V \mid y = cx\}$ for a non-zero constant c . As is easily seen, the analytic set $\tilde{\omega}^{-1}(l)$ has two irreducible components, one of which we denote by S . Then S is not a hypersurface of order one in $\mathcal{D}$.

We can prove this fact easily by the following lemma.

LEMMA 1. Let S be a hypersurface of order one in a purely n -dimensional normal complex space Σ . Suppose that there is a holomorphic mapping g of Σ

onto an $(n - 1)$ -dimensional complex manifold M and that the restriction $g|_S: S \rightarrow M$ of g to S is a homeomorphism. Then (i) $S \cap \text{sing}(\Sigma) = \emptyset$ and (ii) S is a smooth analytic set in Σ .

PROOF. An arbitrary point P_0 of S has a neighborhood U in Σ such that the open ringed subspace $(U, \mathcal{O}_U)$ of $(\Sigma, \mathcal{O}_\Sigma)$ is isomorphic to a normal complex model space, that is, an analytic set in some domain of a complex Euclidean space equipped with a sheaf of rings of germs of holomorphic functions on it. Let f be a holomorphic function in U taking the value zero only on $S \cap U$ such that the order of zero of f is one at each point of $(S - \text{sing}(\Sigma)) \cap U$. Such a function f exists if U is sufficiently small. We may suppose that $g(U)$ is contained in a coordinate neighborhood of $g(p_0)$ in M , so that the mapping g is given by $n - 1$ holomorphic functions $g_1, g_2, \dots, g_{n-1}$.

Consider the analytic polyhedron $\Delta = \{p \in U \mid |f(p)| < \varepsilon, |g_j(p) - g_j(p_0)| < \varepsilon; j = 1, 2, \dots, n - 1\}$. Since the set $\{p \in U \mid f(p) = 0, g_j(p) = g_j(p_0); j = 1, 2, \dots, n - 1\}$ consists of only one point, for a sufficiently small positive number ε , Δ is relatively compact in U . And the mapping $\varphi: \Delta \rightarrow \Gamma$ defined by the equations $w_0 = f(p)$, $w_j = g_j(p)$ for $p \in \Delta$ is a proper finite open holomorphic mapping where Γ is the polydisc $\{(w_0, w_1, \dots, w_{n-1}) \in \mathbb{C}^n \mid |w_j| < \varepsilon; j = 0, 1, \dots, n - 1\}$. Hence φ defines a finite-sheeted (ramified) covering $\mathcal{D}$ with no relative boundary over Γ which is isomorphic to the ringed space $(\Delta, \mathcal{O}_\Delta)$. Let us prove that this covering is one-sheeted, which establishes our lemma.

Let p_1 be a point of $\Delta \cap (S - \text{sing}(\Sigma) - \text{sing}(S))$. Since the order of zero of f at p_1 is one, there exists a neighborhood U_1 of p_1 and $n - 1$ holomorphic functions $\zeta_1, \zeta_2, \dots, \zeta_{n-1}$ in U_1 such that the set $(f, \zeta_1, \zeta_2, \dots, \zeta_{n-1})$ of n functions defines a system of local coordinates at U_1 . The jacobian matrix $J(\varphi)$ of the mapping φ on this coordinate system is

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ * & \frac{\partial g_1}{\partial \zeta_1} & \frac{\partial g_1}{\partial \zeta_2} & \dots & \frac{\partial g_1}{\partial \zeta_{n-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ * & \frac{\partial g_{n-1}}{\partial \zeta_1} & \frac{\partial g_{n-1}}{\partial \zeta_2} & \dots & \frac{\partial g_{n-1}}{\partial \zeta_{n-1}} \end{pmatrix}.$$

On the other hand, since $g|_S: S \rightarrow M$ is a homeomorphism,

$$\det \frac{\partial(g_1, g_2, \dots, g_{n-1})}{\partial(\zeta_1, \zeta_2, \dots, \zeta_{n-1})} \neq 0$$

at p_1 . Hence $\det J(\varphi) \neq 0$ at p_1 . Since $\varphi^{-1}(\varphi(p_1)) = \{p_1\}$, $\mathcal{D}$ is one-sheeted over Γ . (Q.E.D.)

2. Let us prepare another lemma.

LEMMA 2. (Stein factrization) *Let Σ be a connected normal complex space and M be a connected complex manifold such that $\dim \Sigma = \dim M$. Suppose that Σ has a proper holomorphic mapping $\varphi: \Sigma \rightarrow M$ into M and that there exists a point p of Σ which is isolated in the set $\varphi^{-1}(\varphi(p))$. Then (i) there exists a finite-sheeted (ramified) covering $\mathcal{D}$ with no relative boundary over M having the canonical projection $\tilde{\omega}: \mathcal{D} \rightarrow M$ and the proper holomorphic mapping $\sigma: \Sigma \rightarrow \mathcal{D}$ onto $\mathcal{D}$ which satisfies $\varphi = \tilde{\omega} \circ \sigma$ and (ii) the space $\mathcal{D}$ has an analytic set J^* with codimension at least two in $\mathcal{D}$ such that each point $p \in \sigma^{-1}(J^*)$ is not isolated in $\sigma^{-1}(\sigma(p))$ and such that the restriction $\sigma|_{\Sigma - \sigma^{-1}(J)}$ of σ to $\Sigma - \sigma^{-1}(J)$ is a biholomorphic mapping onto $\mathcal{D} - J$ where $J = J^* \cup \sigma(\text{sing}(\Sigma))$.*

PROOF. Let T be the set of critical points of the restriction of φ to $\Sigma - \text{sing}(\Sigma)$. Since the set T is an analytic set in $\Sigma - \text{sing}(\Sigma)$ with codimension one, the closure $\bar{T}$ of T in Σ is an analytic set in Σ . By Remmert's proper mapping theorem, $\varphi(\bar{T})$ is an analytic set in M . Set $K = \varphi(\bar{T} \cup \text{sing}(\Sigma))$. The restriction $\varphi|_{\Sigma - \varphi^{-1}(K)}$ of φ to $\Sigma - \varphi^{-1}(K)$ is a proper finite mapping onto $M - K$, which defines a finite-sheeted covering space $\mathcal{D}_0$ over the domain $M - K$ in M . Adding the boundary points of $\mathcal{D}_0$ over K as usual, we get a finite-sheeted (ramified) covering $\mathcal{D}$ over M which is the completion of $\mathcal{D}_0$ over M . Denote by $\tilde{\omega}: \mathcal{D} \rightarrow M$ the canonical projection of $\mathcal{D}$ onto M . By Riemann's continuation theorem, there exists a canonical proper holomorphic mapping σ such that $\varphi = \tilde{\omega} \circ \sigma$. Denote by T_1 the set of the critical points of the restriction $\sigma|_{\Sigma - \text{sing}(\Sigma)}$ of σ to $\Sigma - \text{sing}(\Sigma)$. (A point of $\sigma^{-1}(\text{sing}(\mathcal{D})) - \text{sing}(\Sigma)$ is considered to be a critical point of $\sigma|_{\Sigma - \text{sing}(\Sigma)}$.) Since $T_1 \subset T$, the closure $\bar{T}_1$ of T_1 in Σ is an analytic set in Σ . We call a point p of $\bar{T}_1$ a critical point of σ of the first kind if p is isolated in $\sigma^{-1}(\sigma(p))$. Suppose that T_1 has a critical point of σ of the first kind. Then there exists a neighborhood V of p and a neighborhood W of $\sigma(p)$ such that $\sigma(V) = W$ and the restriction $\sigma|_V$ of σ to V is a proper finite mapping onto W (cf. The proof of Lemm 1). But the restriction $\sigma|_{V_0}$ of σ to $V_0 = V - \varphi^{-1}(K)$ is a biholomorphic mapping, so that p is not a critical point of $\sigma|_{\Sigma - \text{sing}(\Sigma)}$ (Riemann's continuation theorem), a contradiction. Hence $\bar{T}_1$ has no critical point of σ of the first kind. By the

theorem on functional relations, we know that $\dim \sigma(\bar{T}_1) \leq \dim \mathcal{D} - 2$. The restriction $\sigma|_{\sum - \varphi^{-1}(J)}$ of σ to $\sum - \varphi^{-1}(J)$ is a biholomorphic mapping onto $\mathcal{D} - J$ where we denote by J the set $\sigma(\bar{T}_1) \cup \sigma(\text{sing}(\sum))$. (Q.E.D.)

§2. Ruled variety with basis $\mathbf{P}^{n-1}$.

3. Let us recall an algebraic manifold obtained by one blowing-up at a point of $\mathbf{P}^n$ ($n \geq 2$). Let $z = [z_1 : z_2 : \dots : z_n]$ be a homogeneous coordinate of $\mathbf{P}^{n-1}$ and let $x = (x_1, x_2, \dots, x_n)$ be an inhomogeneous coordinate of $\mathbf{P}^n$. Denote by F_1^n a compact submanifold of $\mathbf{P}^{n-1} \times \mathbf{P}^n$ defined by the equations $z_j x_k = z_k x_j$ ($1 \leq j \leq n, 1 \leq k \leq n$). Denote by V_j the open subset of F_1^n defined by $z_j \neq 0$ ($1 \leq j \leq n$). Then $\{V_j\}$ is an open covering of F_1^n . Set $E = \{(z, x) \in \mathbf{P}^{n-1} \times \mathbf{P}^n | x_1 = x_2 = \dots = x_n = 0\}$, which is smooth hypersurface of order one in F_1^n defined by the system $\{(V_j, (x_j|F_1^n))\}$. Each open set V_j is biholomorphically equivalent to $\mathbf{C}^{n-1} \times \mathbf{P}^1$. Let γ be the canonical projection of $\mathbf{P}^{n-1} \times \mathbf{P}^n$ onto $\mathbf{P}^{n-1}$, that is, $\gamma(z, x) = z$. Set $\tilde{\gamma} = (\gamma|F_1^n)$. The triple $(F_1^n, \tilde{\gamma}, \mathbf{P}^{n-1})$ defines a $\mathbf{P}^1$ -bundle on $\mathbf{P}^{n-1}$.

We prove the following theorem in this section.

THEOREM 1. *Let S be a hypersurface in an n -dimensional compact normal complex space $\sum$. Suppose that $\sum$ has a holomorphic mapping $g: \sum \rightarrow \mathbf{P}^{n-1}$ onto $\mathbf{P}^{n-1}$ whose restriction $g|_S: S \rightarrow \mathbf{P}^{n-1}$ to S is a homeomorphism. Assume that the analytic set $g^{-1}(z)$ is a connected complex analytic curve for each $z \in \mathbf{P}^{n-1}$. Denote by C_z the irreducible component of $g^{-1}(z)$ which passes S . Suppose that the normalization $\hat{C}_z$ of each C_z is biholomorphically equivalent to the Riemann sphere $\mathbf{P}^1$ and there exist n meromorphic functions $X_1, X_2, \dots, X_n$ on $\sum$ which is holomorphic in a neighborhood V of S and which satisfies the following two conditions for a homogeneous coordinate $[z_1 : z_2 : \dots : z_n]$ of $\mathbf{P}^{n-1}$: (i) Set $\sum_j = g^{-1}(U_j)$ where $U_j = \{z \in \mathbf{P}^{n-1} | z_j \neq 0\}$. The function $X_j|_{V \cap \sum_j}$ takes the value zero only on $S \cap \sum_j$ whose order on $S \cap \sum_j$ is one. (ii) $(X_j/X_k)|_{S \cap \sum_k} = g^*(z_j/z_k)|_{S \cap \sum_k}$.*

Then there exists a bimeromorphic holomorphic mapping Ψ of $\sum$ onto F_1^n which maps a neighborhood of S onto a neighborhood of E biholomorphically and satisfies $g = \tilde{\gamma} \circ \Psi$ and such that the mapping $\Psi|_{C_z}: C_z \rightarrow \tilde{\omega}^{-1}(z)$ is a biholomorphic mapping of C_z onto $\tilde{\omega}^{-1}(z)$.

REMARK. Suppose that $\sum$ is an n -dimensional projective algebraic ruled variety with basis $\mathbf{P}^{n-1}$. If a section S of $\sum$ satisfies $[S]|S = [-H]$ where H is

a hyperplane in $S \cong \mathbf{P}^{n-1}$, then there exists n meromorphic functions $X_1, X_2, \dots, X_n$ on Σ satisfying the conditions in Theorem 1.

4. Under the assumption of Theorem 1, consider the meromorphic mapping $\psi_i: \Sigma \rightarrow \mathbf{P}^{n-1} \times \mathbf{P}^1$ defined by the equations $z = g(p)$, $X = X_i(p)$ ($p \in \Sigma$) for fixed i where X is an inhomogeneous coordinate of $\mathbf{P}^1$. Denote by $\tau_1: \Sigma \times (\mathbf{P}^{n-1} \times \mathbf{P}^1) \rightarrow \Sigma$ the canonical projection of the product space $\Sigma \times (\mathbf{P}^{n-1} \times \mathbf{P}^1)$ onto Σ and denote by τ_2 the canonical projection of $\Sigma \times (\mathbf{P}^{n-1} \times \mathbf{P}^1)$ onto $\mathbf{P}^{n-1} \times \mathbf{P}^1$. Let $\mathfrak{G}$ be the graph of ψ_i in the direct product space $\Sigma \times (\mathbf{P}^{n-1} \times \mathbf{P}^1)$, that is, $\mathfrak{G} = \{(p, (z, X)) \in (\Sigma - I) \times (\mathbf{P}^{n-1} \times \mathbf{P}^1) | z = g(p), X = X_i(p)\}$ where I is the set of indeterminate points of X_i . The closure $\bar{\mathfrak{G}}$ of $\mathfrak{G}$ in $\Sigma \times (\mathbf{P}^{n-1} \times \mathbf{P}^1)$ is a purely n -dimensional irreducible analytic set in $\Sigma \times (\mathbf{P}^{n-1} \times \mathbf{P}^1)$. Denote by $v_1: \tilde{\Sigma} \rightarrow \bar{\mathfrak{G}}$ the normalization of the reduced complex space $\bar{\mathfrak{G}}$. The mapping $\varphi = \tau_2 \circ v_1$ satisfies the condition in Lemma 2. Hence φ admits a factorization $\varphi = \tilde{\omega} \circ \sigma$ where $\tilde{\omega}: \mathcal{D} \rightarrow \mathbf{P}^{n-1} \times \mathbf{P}^1$ is the projection of a finite-sheeted (ramified) covering $\mathcal{D}$ with no relative boundary over $\mathbf{P}^{n-1} \times \mathbf{P}^1$ and $\sigma: \tilde{\Sigma} \rightarrow \mathcal{D}$ is a proper holomorphic mapping of $\tilde{\Sigma}$ onto $\mathcal{D}$. And the normal complex space $\mathcal{D}$ has an analytic set J with codimension at least two such that the restriction $\sigma|_{\tilde{\Sigma} - \sigma^{-1}(J)}$ of σ to $\tilde{\Sigma} - \sigma^{-1}(J)$ is a biholomorphic mapping onto $\mathcal{D} - J$ and such that $\sigma^{-1}(J)$ contains $\text{sing}(\tilde{\Sigma})$. Set $v = \tau_1 \circ v_1$. The restriction $v|_{\tilde{\Sigma} - v^{-1}(I)}$ of v to $\tilde{\Sigma} - v^{-1}(I)$ is a biholomorphic mapping onto $\Sigma - I$. Hence the composition $\mu = \sigma \circ v^{-1}$ is a bimeromorphic mapping of $\tilde{\Sigma}$ to $\mathcal{D}$ whose restriction $\mu|_{\Sigma - H}$ to $\Sigma - H$ is a biholomorphic mapping of $\Sigma - H$ onto $\mathcal{D} - \sigma(v^{-1}(I)) - J$ where $H = (I \cup v(\sigma^{-1}(J)))$.

By lemma 1, we know that $S \cap \text{sing}(\Sigma) = \emptyset$. Since the restriction $X_i|_{C_z}$ of X_i to C_z is non-constant for $z \in U_i$, the set $S \cap \Sigma_i$ is contained in $\Sigma_i - H$. Set $\mathcal{D}_i = \tilde{\omega}^{-1}(U_i \times \mathbf{P}^1)$. Since $\psi_i(p) = \tilde{\omega} \circ \mu(p)$ for $p \in \Sigma - H$ and since X_i takes the value zero of order one on $S \cap \Sigma_i$, $\mu(S \cap \Sigma_i)$ has a neighborhood W contained in $\mathcal{D}_i - (\sigma(v^{-1}(I))) - J$ such that the restriction $\tilde{\omega}|_W: W \rightarrow \tilde{\omega}(W)$ of $\tilde{\omega}$ to W is a one-sheeted covering over an open neighborhood $\tilde{\omega}(W)$ of $U_i \times \{0\}$.

5. Denote by $\mathcal{D}_\alpha$ the complex analytic curve in $\mathcal{D}_i$ defined by $z = \alpha$, that is, $\mathcal{D}_\alpha = \tilde{\omega}^{-1}(\{\alpha\} \times \mathbf{P}^1)$. The open set $W_\alpha = \mathcal{D}_\alpha \cap W$ in $\mathcal{D}_\alpha$ is a one-sheeted part of $\mathcal{D}_\alpha$ over the open set $\tilde{\omega}(W_\alpha)$ in $\{\alpha\} \times \mathbf{P}^1$. Let c_1, c_2 be two complex numbers such that $(\alpha, c_j) \in \tilde{\omega}(W_\alpha)$; $j = 1, 2$. The point α has a neighborhood δ in U_i such that $\delta \times \{c_j\} \subset \tilde{\omega}(W)$; $j = 1, 2$. Set $\mathcal{D}_\delta = \tilde{\omega}^{-1}(\delta \times \mathbf{P}^1)$. Denote by R_z the proper image of C_z under the meromorphic mapping $\mu: \tilde{\Sigma} \rightarrow \mathcal{D}$. Then W_z

$\subset R_z \subset \mathcal{D}_z$. Let $\lambda_z: \hat{R}_z \rightarrow R_z$ be the normalization of the reduced complex space R_z . By the assumption, $\hat{R}_z$ is biholomorphically equivalent to $\mathbf{P}^1$. Set $p_z^0 = (\tilde{\omega}|W)^{-1}((z, 0))$ and $p_z^j = (\tilde{\omega}|W)^{-1}((z, c_j))$; $j = 1, 2$. Consider the rational function $\Phi_\delta^z(p)$ of degree one on $\hat{R}_z \cong \mathbf{P}^1$ for $z \in \delta$ which takes the value 0 on $\lambda_z^{-1}(p_z^0)$, the value 1 on $\lambda_z^{-1}(p_z^1)$ and the value ∞ on $\lambda_z^{-1}(p_z^2)$. Let q be the canonical projection of the product space $\mathbf{P}^{n-1} \times \mathbf{P}^1$ onto $\mathbf{P}^1$. Set $K = q(\tilde{\omega}(J)) \cup g(I)$, which is a nowhere dense analytic set in $\mathbf{P}^{n-1}$. Let T be the set of the critical points of the mapping g in $g^{-1}(\delta - K)$. By Remmert's proper mapping theorem and Sard's theorem, the image $g(T)$ of T under g is a nowhere dense analytic set in $\delta - K$. Let B be the branch locus of $\tilde{\omega}^{-1}((\delta - K) \times \mathbf{P}^1) \subset \mathcal{D}$ over $(\delta - K) \times \mathbf{P}^1$. If $z \in \delta - K - g(T)$, $\mathcal{D}_z$ is irreducible so that $R_z = \mathcal{D}_z$ and $\mathcal{D}_z$ intersects B at a finite number of points of $\mathcal{D}$. Set $\mathcal{D}_\delta^0 = \tilde{\omega}^{-1}((\delta - K - g(T)) \times \mathbf{P}^1)$ and define the function $\Phi_\delta(p)$ on $\mathcal{D}_\delta^0 - B$ by the equation $\Phi_\delta(p) = (\lambda_z^{-1})^* \Phi_\delta^z(p)$ where $z = q \circ \tilde{\omega}(p)$.

Denote by $p_1, p_2, \dots, p_r$ the points of $\mathcal{D}_{z'} \cap B$ for $z' \in \delta - K - g(T)$. Each point p_j has an open neighborhood Γ_j ($j = 1, 2, \dots, r$) in $\mathcal{D}$ such that (i) $(\tilde{\omega}| \Gamma_j)^{-1}(\tilde{\omega}(p_j))$ consists only of one point p_j and (ii) $\tilde{\omega}| \Gamma_j: \Gamma_j \rightarrow \tilde{\omega}(\Gamma_j)$ is a proper finite open mapping and such that $\Gamma_1, \Gamma_2, \dots, \Gamma_r$ are pairwise disjoint. The set $B_j = (\tilde{\omega}| \Gamma_j)(B \cap \Gamma_j)$ is an analytic hypersurface in $\tilde{\omega}(\Gamma_j)$. Let $d_1, d_2, \dots, d_r$ be the X -coordinates of the points $\tilde{\omega}(p_1), \tilde{\omega}(p_2), \dots, \tilde{\omega}(p_r)$, respectively. Set $\gamma_j = \{X \in \mathbf{P}^1 \mid |X - d_j| < \varepsilon\}$ for a positive number ε . (When $d_j = \infty$, we set $\gamma_j = \{X \in \mathbf{P}^1 \mid |1/X| < \varepsilon\}$.) Let ε be sufficiently small so that $(\{z'\} \times \bar{\gamma}_j) \subset \tilde{\omega}(\Gamma_j)$ and $(\{z'\} \times \partial\gamma_j) \cap B_j = \emptyset$. Then there exists a connected open neighborhood δ' of z' in $\delta - K - g(T)$ such that $(\delta' \times \bar{\gamma}_j) \subset \tilde{\omega}(\Gamma_j)$ and $(\delta' \times \partial\gamma_j) \cap B_j = \emptyset$. Set $\Gamma_j^0 = (\tilde{\omega}| \Gamma_j)^{-1}(\delta' \times \gamma_j)$ and set $\mathcal{D}_{\delta'} = \tilde{\omega}^{-1}(\delta' \times \mathbf{P}^1)$. Since $(\mathcal{D}_{\delta'} - \bigcup_j \Gamma_j^0) \cap B = \emptyset$, $\mathcal{D}_z - \bigcup_j \Gamma_j^0$ is canonically identified with $\mathcal{D}_{z'} - \bigcup_j \Gamma_j^0$ for $z \in \delta'$. Denote by $\iota_z: (\mathcal{D}_z - \bigcup_j \Gamma_j^0) \rightarrow (\mathcal{D}_{z'} - \bigcup_j \Gamma_j^0)$ this canonical identification mapping. The domain $\mathcal{D}_{\delta'} - \bigcup_j \Gamma_j^0$ is biholomorphically equivalent to the product space $\delta' \times (\mathcal{D}_{z'} - \bigcup_j \Gamma_j^0)$. The family of functions $\{\Phi_\delta(\iota_z^{-1}(p))\}_{z \in \delta'}$ is a normal family on $\mathcal{D}_{z'} - \bigcup_j \Gamma_j^0$. Since there is only one rational function of degree one on $\hat{R}_z$ taking the value 0 on $\lambda_z^{-1}(p_z^0)$, the value 1 on $\lambda_z^{-1}(p_z^1)$ and the value ∞ on $\lambda_z^{-1}(p_z^2)$ and since ε was arbitrary, the function Φ_δ must be continuous on $\mathcal{D}_\delta^0 - B$ (Since a non-constant uniform limit of univalent functions is univalent, the normal family must have the unique limit.).

6. Let u be another inhomogeneous coordinate of $\mathbf{P}^1$ such that

$$u = \frac{(c_1 - c_2) \cdot X}{c_1(X - c_2)}.$$

Denote by m the number of sheets of the covering $\mathcal{D}$ over $U_i \times \mathbf{P}^1$. When $z \in \delta - K - g(T)$, the inverse of the mapping $(\lambda_z^{-1})^* \Phi_\delta^z: (\mathcal{D}_z - B) \rightarrow \{z\} \times \mathbf{P}^1$ is represented by

$$\zeta(z, w) = \frac{w(w^{m-1} + a_2(z)w^{m-2} + \cdots + a_m(z))}{b_1(z)w^{m-1} + \cdots + b_{m-1}(z)w + b_m(z)}$$

as $u = \zeta(z, w)$ where $a_2(z), a_3(z), \dots, a_m(z), b_2(z), b_3(z), \dots, b_m(z)$ are continuous functions on $\delta - K - g(T)$ and $b_1(z) = 1 + a_2(z) + a_3(z) + \cdots + a_m(z) - b_2(z) - b_3(z) \cdots - b_m(z)$.

Each analytic set $B \cap \Gamma_j^0$ in Γ_j^0 in the previous paragraph is represented by an (many-valued) analytic function $\xi_j^0(z)$ on δ' as $u = \xi_j^0(z)$. We may suppose that $\xi_j^0(z) \neq \infty$ because c_2 was an arbitrary number such that $(\alpha, c_2) \in \tilde{\omega}(W_\alpha)$. Denote by κ the set of points on δ' at which the functions ξ_j^0 ramify or tend to ∞ . Let z_1 be a point of $\delta' - \kappa$ and let δ'' be a small ball with the center z_1 such that $\delta'' \subset \delta' - \kappa$. The functions ξ_j^0 ($j = 1, 2, \dots, r$) define one-valued functions $\xi_1(z), \xi_2(z), \dots, \xi_s(z)$ on δ'' ($r \leq s$). We denote by $\eta_j(z)$ the value of Φ_δ at the critical point represented by $\xi_j(z)$. Let e_j be the order of ramification of $\mathcal{D}$ at the branch point represented by $\xi_j(z)$. By Hurwitz's formula, $\sum_{j=1}^s e_j = 2m - 2$. The function $\zeta(z, w)$ must satisfy the following $2m - 2$ systems of two equations.

$$(1) \quad \begin{aligned} \frac{\partial^l}{\partial w^l} \zeta(z, \eta_j(z)) &= 0 \\ \zeta(z, \eta_j(z)) &= \xi_j(z) \\ (l = 1, 2, \dots, e_j; j = 1, 2, \dots, s). \end{aligned}$$

Let us substitute unknowns $X_2, X_3, \dots, X_m, Y_2, Y_3, \dots, Y_m$ for $a_2(z), a_3(z), \dots, a_m(z), b_2(z), b_3(z), \dots, b_m(z)$ in each system in the above and eliminate $\eta_j(w)$ from each system. We get $2m - 2$ equations

$$(2) \quad \begin{aligned} \mathfrak{P}_k(z; X_2, X_3, \dots, X_m, Y_2, Y_3, \dots, Y_m) &= 0 \\ (k = 1, 2, \dots, 2m - 2) \end{aligned}$$

where $\mathfrak{P}_k$ is a polynomial in $X_2, X_3, \dots, X_m, Y_2, Y_3, \dots, Y_m$ whose coefficients are

holomorphic functions of z in δ'' .

Consider (2) as the system of $2m - 2$ equations which has a continuous solution $X_2 = a_2(z), X_3 = a_3(z), \dots, X_m = a_m(z), Y_2 = b_2(z), Y_3 = b_3(z), \dots, Y_m = b_m(z)$. By (1) and Hurwitz's formula, the system of equations (2) has only one solution and hence the functions $a_2(z), a_3(z), \dots, a_m(z), b_2(z), b_3(z), \dots, b_m(z)$ are holomorphic on δ'' (cf. Nishino [3], p. 230). By Riemann's continuation theorem and Picard's theorem, Φ_δ can be analytically continued to a meromorphic function in $\mathcal{D}_\delta$, which we denote by the same letter Φ_δ . Since the points p_z^0, p_z^1, p_z^2 are not indeterminate points of Φ_δ , the restriction $\Phi_\delta|_{R_z}: R_z \rightarrow \mathbf{P}^1$ of Φ_δ to R_z induces a biholomorphic mapping of $\hat{R}_z$ onto $\mathbf{P}^1$. Set $\Psi_\delta = \mu^* \Phi_\delta$. By Riemann's continuation theorem, Ψ_δ is a holomorphic function on $\sum_\delta - \mu^{-1}(W)$ where $\sum_\delta = g^{-1}(\delta)$. The restriction $\Psi_\delta|_{C_z}$ of Ψ_δ to C_z is a biholomorphic mapping of C_z onto $\mathbf{P}^1$.

7. An arbitrary point α in $\mathbf{P}^{n-1}$ has a neighborhood δ contained in U_i for some i such that we can construct a meromorphic function Ψ_δ on $\sum_\delta$ in the previous paragraph. By Heine-Borel's theorem, $\mathbf{P}^{n-1}$ can be covered by a finite number of such neighborhoods $\delta_1, \delta_2, \dots, \delta_h$ which satisfies $\delta_j \subset U_{v(j)}$ for a function $v(j)$ of the set of index j into the set $\{1, 2, \dots, n\}$. Set $\Psi_j = \Psi_{\delta_j}$. Then

$$1/\Psi_j(p) = \{\alpha_{jk}(g(p))/\Psi_k(p)\} + \beta_{jk}(g(p))$$

for $p \in g^{-1}(\delta_j \cap \delta_k)$ where $\alpha_{jk}(z)$ is a non-zero holomorphic function in $\delta_j \cap \delta_k$ and $\beta_{jk}(z)$ is a holomorphic function in $\delta_j \cap \delta_k$. We can prove easily that $\alpha_{ij}\beta_{jk} + \alpha_{ik}\beta_{ki} + \beta_{ij} = 0$. Since

$$\Psi_j(p) = \frac{\Psi_k(p)}{\alpha_{jk}(g(p)) + \beta_{jk}(g(p)) \Psi_k(p)},$$

$\Psi_j(p)/\Psi_k(p)|_{\delta_j^* \cap \delta_k^*} = 1/\alpha_{jk}(g(p))$ where $\delta_j^* = S \cap g^{-1}(\delta_j)$. The function $\chi_j = (X_{v(j)}/\Psi_j)|_{S \cap \delta_j^*}$ is a non-zero holomorphic function on $S \cap \delta_j^*$. Hence we may suppose that $(X_{v(j)}/\Psi_j)|_{S \cap \delta_j^*} \equiv 1$, if necessary, substituting $\hat{\chi}_j \cdot \Psi_j$ for Ψ_j where $\hat{\chi}_j = g^*\{(g|S)^{-1}\}^* \chi_j$. By the assumption, we get $1/\alpha_{jk}(g(p)) = g^*(z_{v(j)}/z_{v(k)})|_{\delta_j^* \cap \delta_k^*}$. We may suppose that each δ_j is holomorphically complete. As is well-known, there exists a holomorphic function β_j in each δ_j which satisfies $\alpha_{ij}(z)\beta_j(z) - \beta_i(z) = \beta_{ij}(z)$. Set

$$\Psi_j^* = \frac{\Psi_j}{1 + \beta_j \Psi_j} \quad (j = 1, 2, \dots, h).$$

We get $1/\Psi_j^*(p) = \alpha_{jk}(g(p))/\Psi_k^*(p)$ for $p \in g^{-1}(\delta_j \cap \delta_k)$. The mappings $g \times \Psi_j^*: g^{-1}(\delta_j) \rightarrow \delta_j \times \mathbf{P}^1$ defined by $z = g(p)$, $w = \Psi_j^*(p)$ construct the mapping $\Psi: \Sigma \rightarrow F_1^n$ in Theorem 1.

§3. A theorem on a local holomorphic mapping.

8. Let $f: V \rightarrow f(V)$ be a biholomorphic mapping of the polydisc $V = \{(x_1, x_2, \dots, x_n) \in \mathbf{C}^n \mid |x_j| < 1, 1 \leq j \leq n\}$ onto an open set $f(V)$ in $\mathbf{C}^n$ which is defined by n holomorphic functions $f_j(x_1, x_2, \dots, x_n)$ ($1 \leq j \leq n$) in V such that $f_j(0, 0, \dots, 0) = 0$. Denote by l_z the part of the complex line passing the origin with direction ratio $z = [z_1: z_2: \dots: z_n]$ in V , that is, $l_z = \{(x_1, x_2, \dots, x_n) \in V \mid x_j = z_j t \ (1 \leq j \leq n), t \in \mathbf{C}\}$. We say that the complex analytic curve $f(l_z)$ is *algebraic* if $f(l_z)$ is contained in an algebraic curve in $\mathbf{C}^n$ as a subset. Let us denote by $\pi_{jk}: \mathbf{C}^n \rightarrow \mathbf{C}^2$ the canonical projection of $\mathbf{C}^n$ onto the $X_j X_k$ -space $\mathbf{C}^2$, that is, $\pi_{jk}(X_1, X_2, \dots, X_n) = (X_j, X_k)$. If $\pi_{jk}(f(l_z))$ is algebraic for each j, k ($1 \leq j, k \leq n$), then $f(l_z)$ is algebraic. We call the minimum of the homogeneous degree of algebraic curves containing $\pi_{jk}(f(l_z))$ the *homogeneous degree* of the image curve $\pi_{jk}(f(l_z))$ of l_z under the mapping $\pi_{jk} \circ f$.

LEMMA 3. Set $A_{jk}(m) = \{z \in \mathbf{P}^{n-1} \mid \text{The image curve } \pi_{jk}(f(l_z)) \text{ of } l_z \text{ under the mapping } \pi_{jk} \circ f \text{ is an algebraic curve with homogeneous degree at most } m\}$ for each positive integer m . Then $A_{jk}(m)$ is an algebraic set in $\mathbf{P}^{n-1}$.

PROOF. Set $U_i = \{z \in \mathbf{P}^{n-1} \mid z_i \neq 0, |z_j/z_i| < 2 \ (1 \leq j \leq n)\}$. If $z_i \neq 0$, the functions $\varphi_j^z(t) = f_j(z_1 t/z_i, z_2 t/z_i, \dots, z_n t/z_i)$ is holomorphic in $D_z^i = \{t \in \mathbf{C} \mid (z_1 t/z_i, z_2 t/z_i, \dots, z_n t/z_i) \in V\}$. The set $\pi_{jk}(f(l_z))$ is an algebraic curve of degree at most m if and only if the Wronskian $W^z(t)$ of the set of the $_{m+2}C_2 - 1$ functions $\varphi_j^z, \varphi_k^z, (\varphi_j^z)^2, \varphi_j^z \varphi_k^z, (\varphi_k^z)^2, \dots, \varphi_j^z (\varphi_k^z)^{m-1}, (\varphi_k^z)^m$ is identically zero. Since $W^z(t)$ defines a holomorphic function in $D^i = \{(z, t) \in U_i \times \mathbf{C} \mid (z_1 t/z_i, z_2 t/z_i, \dots, z_n t/z_i) \in V\}$, the set $A_{jk}^i(m) = \{z \in U_i \mid \pi_{jk}(f(l_z)) \text{ is an algebraic curve with homogeneous degree at most } m\}$ is an analytic set in U_i . Because $\{U_i\}$ is an open covering of $\mathbf{P}^{n-1}$, the set $A_{jk}(m)$ is an algebraic set in $\mathbf{P}^{n-1}$ (Theorem of Chow). (Q.E.D.)

9. The n -dimensional Osgood space $\prod^n \mathbf{P}$ —the n -times direct product of $\mathbf{P}^1$ is the most intimate compactification of $\mathbf{C}^n$ for us.

THEOREM 2. Let f be a biholomorphic mapping of the unit polydisc V around

PROOF. We keep the notation in the proof of Lemma 3. We may suppose that the jacobian matrix $J(f)$ is the unit matrix at the origin. Set $A(m) = \bigcap_{j,k} A_{jk}(m)$. By Lemma 3, $A(m)$ is an algebraic set in $\mathbf{P}^{n-1}$. Since $A = \bigcup_m A(m)$ and $A(1) \subset A(2) \subset \dots \subset A(m) \subset \dots$, if A is not contained in any countable sum of nowhere dense algebraic set in $\mathbf{P}^{n-1}$, then $A(m_0) = \mathbf{P}^{n-1}$ for some m_0 . Let us fix the index i . For convenience's sake, denote by $\psi_1, \psi_2, \dots, \psi_N$ the functions $\varphi_i^z, \varphi_j^z, (\varphi_i^z)^2, \varphi_i^z \varphi_j^z, (\varphi_j^z)^2, \dots, \varphi_i^z (\varphi_j^z)^{m_0-1}, (\varphi_j^z)^{m_0}$ where $N =_{m_0+2} C_2 - 1$ ($j \neq i, 1 \leq j \leq n$). Denote by ${}^{ij}W_k(z, t)$ the wronskian $W(\psi_1, \psi_2, \dots, \psi_k)$ of the k functions $\psi_1, \psi_2, \dots, \psi_k$ of the complex variable t . Since $f_i(x_1, \dots, x_n) = x_i + \dots$ at the origin, ${}^{ij}W_1(z, t) = \varphi_i^z(t) \not\equiv 0$ as a function of the variable t . Since ${}^{ij}W_{m_0}(z, t) \equiv 0$ in D^i , there is a (unique) number k such that ${}^{ij}W_k(z, t) \not\equiv 0$ in D^i and ${}^{ij}W_{k+1}(z, t) \equiv 0$ in D^i . Consider the system of linear equations

$$\begin{aligned}
 & \alpha_1 \psi_1 + \alpha_2 \psi_2 + \cdots + \alpha_k \psi_k = \psi_{k+1} \\
 & \alpha_1 \psi_1^{(1)} + \alpha_2 \psi_2^{(1)} + \cdots + \alpha_k \psi_k^{(1)} = \psi_{k+1}^{(1)} \\
 & \dots\dots\dots \\
 & \alpha_1 \psi_1^{(k-1)} + \alpha_2 \psi_2^{(k-1)} + \cdots + \alpha_k \psi_k^{(k-1)} = \psi_{k+1}^{(k-1)}
 \end{aligned}
 \tag{3}$$

$$\sum_{1 \leq \mu + \nu \leq m_0} {}^{ij} \beta_{\mu\nu}(z) \cdot (\varphi_i^z)^\mu (\varphi_j^z)^\nu = 0$$

valid in D_i where ${}^{ij}\beta_{\mu\nu}(z)$ is a holomorphic function in U_i for each μ, ν such that the all N functions ${}^{ij}\beta_{\mu\nu}(z)$ ($1 \leq \mu + \nu \leq m_0$) do not vanish simultaneously for

any $z \in U_i$ (cf. [2]). Hence the coordinate (X_i, X_j) of the point on $\pi_{ij} \circ f(l_z)$ satisfies the equation

$$(i, j) \quad \sum_{1 \leq \mu + \nu \leq m_0} {}^{ij}\beta_{\mu\nu}(z) \cdot X_i^\mu \cdot X_j^\nu = 0$$

for each $z \in U_i$. Denote by F_z the algebraic curve in $\prod^n \mathbf{P}$ defined by the system E_i of $n-1$ equations $(i, j); j \neq i, 1 \leq j \leq n$. Since ${}^{ij}W_k(z, t) \not\equiv 0$ in D^i for each j (and k), almost all algebraic curves F_z are irreducible. Hence the system $\{(U_i, E_i)\}_{1 \leq i \leq n}$ defines an irreducible algebraic variety $\sum_0$ in $\mathbf{P}^{n-1} \times \prod^n \mathbf{P}$. The family $\{F_z\}$ is the algebraic family of algebraic curves in $\prod^n \mathbf{P}$ defined by $\sum_0$. We get $f(l_z) \subset F_z \cap \mathbf{C}^n$ for each $z \in \mathbf{P}^{n-1}$ where $\mathbf{C}^n$ denotes the finite space of $\prod^n \mathbf{P}$. (Q.E.D.)

§4. Meromorphic mappings of Poincaré.

10. Let a be a complex number with the absolute value greater than 1. Let $R_j(X_1, X_2, \dots, X_n)$ ($j = 1, 2, \dots, n$) be n rational functions in variables $X_1, X_2, \dots, X_n$ which are holomorphic at the origin and have the expansions $R_j = aX_j + \dots$ ($j = 1, 2, \dots, n$) at the origin. In the above, $\dots$ represents the sum of terms with order greater than one. Under this condition, as H. Poincaré [5] showed in 1890, there are the unique set of n meromorphic functions $f_j(x_1, x_2, \dots, x_n)$ ($j = 1, 2, \dots, n$) which are holomorphic at the origin having the expansions $f_j = x_j + \dots$ at the origin and satisfy the system of functional equations $(f)(ax_1, ax_2, \dots, ax_n) = (R) \circ (f)(x_1, x_2, \dots, x_n)$. Here, (R) is the rational transformation of the Osgood space $\prod^n \mathbf{P}$ defined by $(R_1, R_2, \dots, R_n)$ and (f) is the meromorphic mapping of $\mathbf{C}^n$ to $\prod^n \mathbf{P}$ defined by $(f_1, f_2, \dots, f_n)$. We call (f) the *meromorphic mapping of Poincaré* derived by (R) .

Let l_z be the complex line passing the origin with the direction ratio $z = [z_1 : z_2 : \dots : z_n]$ in $\mathbf{C}^n$. The proper image L_z of l_z under the meromorphic mapping (f) of Poincaré is called an *invariant image curve* of (f) . An invariant image curve L_z of (f) is a complex analytic curve in the wide sense which is invariant under (R) . Since L_z is the image of the Gaussian plane $\mathbf{C}^1$ under the holomorphic mapping $f|_{l_z}$, the normalization $\hat{L}_z$ of L_z is biholomorphically equivalent to the Gaussian plane $\mathbf{C}^1$, the punctured Gaussian plane $\mathbf{C}^*$, the Riemann sphere $\mathbf{P}^1$ or a compact Riemann surface of genus one. In the case where $\hat{L}_z$ is a compact Riemann surface of genus one, we say that L_z is of *elliptic type*. In the other cases, we say that L_z is of *rational type*.

EXAMPLE 2. $(R): X'_j = 3X_j - 4X_j^3; j = 1, 2, \dots, n.$
 $(f): X_j = \sin x_j; j = 1, 2, \dots, n.$

In this case, $f(l_z)$ is algebraic if and only if z is a rational point of $\mathbf{P}^{n-1}$. If z is a rational point of $\mathbf{P}^{n-1}$, then the mapping $f|_{l_z}$ is infinite-sheeted. If z is an irrational point of $\mathbf{P}^{n-1}$, then the mapping $f|_{l_z}$ is one-sheeted. Since L_z is not compact, all L_z are of rational type.

In the previous paper [2], we have given an interesting example in which general invariant image curves L_z are of elliptic type.

LEMMA 4. Let C be a complex analytic curve defined in a neighborhood U of the origin O in $\mathbf{C}^n$ —the finite space of $\prod^n \mathbf{P}$. Suppose that $O \in C$ and C is irreducible at O and that C is invariant under the transformation (R) . Then C coincides with a curve L_z in some neighborhood of O for some $z \in \mathbf{P}^{n-1}$.

The proof is just the same as that of Lemma 1 in the paper [2]. So it may be omitted.

In the following example, all invariant image curves of (f) are of rational type.

EXAMPLE 3. Let $R^0(X_1, X_2, \dots, X_n)$ be a rational function such that $R^0(0, 0, \dots, 0) = a$. Then each complex line passing the origin is invariant under the rational transformation

$$(R): X'_j = X_j R^0(X_1, X_2, \dots, X_n); j = 1, 2, \dots, n.$$

By Lemma 4, an invariant image curve L_z of the meromorphic mapping of Poincaré derived by (R) is contained in the complex line passing the origin with direction ratio z .

Conversely, if L_z is contained in a complex line for each $z \in \mathbf{P}^{n-1}$, (R) must be of this type. Theorem 4 in this section asserts that, if all L_z are algebraic and of rational type, (f) can be essentially reduced to a meromorphic mapping of Poincaré of the type of this example.

11. By Theorem 2 in Section 3, if L_z are algebraic for sufficiently many values of z , then L_z are algebraic for all values of $z \in \mathbf{P}^{n-1}$. So let us suppose that L_z is algebraic for each $z \in \mathbf{P}^{n-1}$ in the followings. Also by Theorem 2, there exists an algebraic family $\{F_z\}$ of algebraic curves in $\prod^n \mathbf{P}$ such that $L_z \subset F_z$ for each $z \in \mathbf{P}^{n-1}$. Let U be a polydisc around the origin of $\mathbf{C}^n$ such that the restriction $(f)|_U$ of (f) to U is a biholomorphic mapping onto a

neighborhood W of the origin O of $\prod^n \mathbf{P}$. Suppose that the homogeneous degree of each algebraic curve $\pi_{ij}(L_z \cap W)$ is at most m_0 where π_{ij} is the canonical projection of $\mathbf{C}^n$ onto $\mathbf{C}_{X_i X_j}^2$. If $F_z \neq F_{z'}$, then F_z intersects $F_{z'}$ at $(2m_0^2)^{n-1}$ points at most. A point p of $L_z \cap L_{z'}$ is called a *point of intersection* of the family $\{L_z\}$ if $F_z \neq F_{z'}$. Since L_z is invariant under the transformation (R), we get the following easily (cf. The proof of Lemma 2 in the paper [2]).

LEMMA 5. *The set of points of intersection of the family $\{L_z\}$ is contained in a nowhere dense algebraic set in $\prod^n \mathbf{P}$.*

Let $\sum_0$ be the algebraic variety in $\mathbf{P}^{n-1} \times \prod^n \mathbf{P}$ which defines $\{F_z\}$. Let $v: \sum \rightarrow \sum_0$ be the normalization of $\sum_0$. The mapping $\varphi = \tau \circ v$ is a proper holomorphic mapping of $\sum$ onto $\prod^n \mathbf{P}$ where τ is the canonical projection of the product space $\mathbf{P}^{n-1} \times \prod^n \mathbf{P}$ onto $\prod^n \mathbf{P}$. By Lemma 2 in Section 1, there exists a finite-sheeted (ramified) covering $\mathcal{D}$ with no relative boundary over $\prod^n \mathbf{P}$, the projection of which we denote by $\tilde{\omega}: \mathcal{D} \rightarrow \prod^n \mathbf{P}$, such that a proper holomorphic mapping $\sigma: \sum \rightarrow \mathcal{D}$ of $\sum$ onto $\mathcal{D}$ satisfies $\varphi = \tilde{\omega} \circ \sigma$. And there exists an analytic set J in $\mathcal{D}$ with codimension at least two such that the restriction $\sigma_0 = \sigma|_{\sum - \sigma^{-1}(J)}$ of σ to $\sum - \sigma^{-1}(J)$ is a biholomorphic mapping onto $\mathcal{D} - J$. Set $g = \mu \circ v$ where μ is the canonical projection of $\mathbf{P}^{n-1} \times \prod^n \mathbf{P}$ onto $\mathbf{P}^{n-1}$. Consider the meromorphic function $h_{jk} = (\sigma_0^{-1})^* g^*(z_j/z_k)$ on $\mathcal{D} - J$. By Levi's extension theorem, h_{jk} can be analytically continued to a meromorphic function on $\mathcal{D}$. The composition $\xi_{jk} = h_{jk} \circ (\tilde{\omega}^{-1})$ is, in general, a many-valued function on $\mathcal{D}$. The Riemann domain $\mathcal{D}$ over $\prod^n \mathbf{P}$ is the *Durchschnitt* of the functions ξ_{jk} ; $1 \leq k < j \leq n$. In the following example, the functions ξ_{j1} ($2 \leq j \leq n$) are two-valued.

EXAMPLE 4.

$$(R): X'_1 = (X_1 + 1)^3 - 1, \quad X'_j = \frac{X_j}{X_1} (X_1 + 1) \{ (X_1 + 1)^3 - 1 \}$$

$$; j = 2, 3, \dots, n.$$

$$(f): X_1 = \exp(x_1) - 1, \quad X_j = \frac{x_j}{x_1} \cdot \exp(x_1/2) (\exp(x_1) - 1)$$

$$; j = 2, 3, \dots, n.$$

In this case, $\xi_{j1} = X_j/X_1 \sqrt{X_1 + 1}$; $j = 2, 3, \dots, n$.

12. Set $\tilde{F}_z = \sigma(g^{-1}(z))$. The analytic set $\tilde{F}_z$ is the closure of the set

$\{p \in \mathcal{D} - J \mid z_j/z_k = h_{jk}(p); 1 \leq k < j \leq n\}$ in $\mathcal{D}$. Denote by $\tilde{L}_z$ the irreducible component of $\tilde{F}_z$ over the closure of L_z in $\prod^n \mathbf{P}$ with respect to the projection $\tilde{\omega}$. Denote by C_z the proper image of $\tilde{L}_z$ under the mapping σ^{-1} .

Recall the polydisc U around the origin such that the restriction $(f)|U$ of (f) to U is a biholomorphic mapping onto the open neighborhood $W = (f)(U)$ of the origin O of $\prod^n \mathbf{P}$. Denote by ξ_{jk}^1 the branch of ξ_{jk} in W which takes the value z_j/z_k on $(f)(l_z \cap U)$ where $z = [z_1 : z_2 : \dots : z_n]$. Denote by $\tilde{W}$ the part of $\mathcal{D}$ which corresponds to W by the set of function elements ξ_{jk}^1 ($1 \leq k < j \leq n$). The restriction $\tilde{\omega}_0 = (\tilde{\omega}| \tilde{W})$ of $\tilde{\omega}$ to $\tilde{W}$ is a one-to-one holomorphic mapping onto W . Set $\tilde{O} = \tilde{\omega}_0^{-1}(O)$ and set $S = \sigma^{-1}(\tilde{O})$. Since the open neighborhood $V = \sigma^{-1}(\tilde{W})$ of S is biholomorphically equivalent to the complex manifold made from U by one blowing-up at the origin, the analytic set S is a smooth hypersurface in $\sum$ such that the restriction $g|S$ of g to S is a homeomorphism onto $\mathbf{P}^{n-1}$. Set $U_j = \{z \in \mathbf{P}^{n-1} \mid z_j \neq 0\}$ and set $\sum_j = g^{-1}(U_j)$. The holomorphic function $\varphi^* X_j|V \cap \sum_j$ on $V \cap \sum_j$ takes the value zero only on $S \cap \sum_j$, the order of which on $S \cap \sum_j$ is one. Since the inverse $((f)|U)^{-1}$ of $(f)|U$ is represented by the equations $x_j = X_j + \dots$ ($j = 1, 2, \dots, n$), we get

$$z_j/z_k = (x_j/x_k)|l_z \cap U = [(X_j + \dots)/(X_k + \dots)]|(f)(l_z \cap U).$$

Hence we get $\lim_{P \rightarrow S, P \in C_z} \varphi^* X_j / \varphi^* X_k = z_j/z_k$. It means that $(\varphi^* X_j / \varphi^* X_k)|S \cap \sum_k = g^*(z_j/z_k)|S \cap \sum_k$.

13. In spite of many valuedness of ξ_{jk} , we can prove that the family $\{F_z\}$ is one-folded. Denote by I the sum of the set of indetermination points of the meromorphic functions h_{jk} ($1 \leq k < j \leq n$) on $\mathcal{D}$. Set $K = \tilde{\omega}(I \cup J)$. Suppose that F_z and $F_{z'}$ have a common point p in $\prod^n \mathbf{P} - K$ and that $z \neq z'$. Denote by P_1 a point of $\tilde{F}_z$ over p with respect to the projection $\tilde{\omega}$ and denote by P_2 a point of $\tilde{F}_{z'}$ over p . The points P_1, P_2 have pairwise disjoint open neighborhoods V_1, V_2 such that (i) $\tilde{\omega}|V_j: V_j \rightarrow W^*$ ($j = 1, 2$) is a proper finite mapping onto the open neighborhood $W^* = \tilde{\omega}(V_1) = \tilde{\omega}(V_2)$ of p in $\prod^n \mathbf{P} - K$ and (ii) each $V_j \cap \tilde{\omega}^{-1}(p)$ consists of only one point P_j . The composition $\xi_{jk}^0 = (h_{jk}|V_2) \circ (\tilde{\omega}|V_2)^{-1}$ is a (many-valued) meromorphic function on W^* . By Lemma 5, we can prove that $\xi_{jk}^0 \circ (\tilde{\omega}|V_1)$ is constant on $\tilde{F}_z \cap V_1$ (cf. p. 62 of the paper [2]). Hence we get the following theorem.

THEOREM 3. *There exists an analytic set K with codimension at least two in*

$\prod^n \mathbf{P}$ such that, if F_z and $F_{z'}$ have a common point in $\prod^n \mathbf{P} - K$, then $F_z = F_{z'}$ as a subset of $\prod^n \mathbf{P}$.

14. Let H be a hyperplane passing an ordinary point p_0 of $L_{z_0} - K$ which intersects L_{z_0} transversally. The point p_0 has an open neighborhood W_0 such that F_{z_0} intersects $H \cap W_0$ at only one point p_0 . Denote by $n(z, H)$ the number of intersection points of F_z and $H \cap W'_0$ where W'_0 is a neighborhood of p_0 contained in W_0 which is relatively compact in W_0 . We call the number $\rho = \limsup_{z \rightarrow z_0} n(z, H)$ the *order* of L_{z_0} .

Denote by $\xi_{j1}^1, \xi_{j1}^2, \dots, \xi_{j1}^b$ the branches of ξ_{j1} ($j = 2, 3, \dots, n$) in W . Let us consider the fundamental symmetric expressions $s_1 = \xi_{j1}^1 + \xi_{j1}^2 + \dots + \xi_{j1}^b$, $s_2 = \xi_{j1}^1 \xi_{j1}^2 + \dots + \xi_{j1}^{b-1} \xi_{j1}^b$, $s_b = \xi_{j1}^1 \xi_{j1}^2 \dots \xi_{j1}^b$. These functions are constant on each L_z and is analytically continued to a rational function on $\prod^n \mathbf{P}$. At least one function H_{j1} among the functions $s_1, s_2, \dots, s_b$ is non-constant. Denote by $\{G_z\}$ the algebraic family of algebraic curves in $\prod^n \mathbf{P}$ defined by the equations $z_j/z_1 = H_{j1}(X_1, X_2, \dots, X_n); j = 2, 3, \dots, n$. Each F_z is contained in an element $G_{z'}$ of the family $\{G_z\}$ for some z' . Hence, by Sard's theorem, there exists a nowhere dense algebraic set N in $\mathbf{P}^{n-1}$ such that, if $z \in \mathbf{P}^{n-1} - N$, then L_z is of order one. Hence the restriction $\tilde{\omega}|_{\tilde{L}_z}: \tilde{L}_z \rightarrow \tilde{L}_z$ of $\tilde{\omega}$ to $\tilde{L}_z$ is an one-sheeted covering of the closure $\tilde{L}_z$ of L_z if $z \in \mathbf{P}^{n-1} - N$. Therefore, the normalization $\hat{C}_z$ of C_z is $\mathbf{P}^1$ or a compact Riemann surface of genus one for each $z \in \mathbf{P}^{n-1}$. If L_z are of rational type for all values of $z \in \mathbf{P}^{n-1}$, then $\hat{C}_z$ is biholomorphically equivalent to $\mathbf{P}^1$ for all values of $z \in \mathbf{P}^{n-1}$.

15. Suppose that L_z are of rational type for all values of $z \in \mathbf{P}^{n-1}$. Then $\sum$ satisfies the conditions of Theorem 1. Hence there exists a bimeromorphic holomorphic mapping $\Psi: \sum \rightarrow F_1^n$ of $\sum$ onto F_1^n satisfying the statements of Theorem 1. Denote by $\tilde{\sigma}: F_1^n \rightarrow \mathbf{P}^n$ the holomorphic mapping which constructs the hypersurface E in F_1^n . The composition $\tilde{\Psi} = \tilde{\sigma} \circ \Psi \circ (\sigma^{-1})$ is a bimeromorphic mapping of $\mathcal{D}$ to $\mathbf{P}^n$ which maps biholomorphically a neighborhood $\tilde{W}_1$ of $\tilde{O}$ such that $\tilde{W}_1 \subset \tilde{W}$ onto a neighborhood $\hat{W}_1$ of the origin $\tilde{\sigma}(E)$ of $\mathbf{P}^n$. Hence the mapping $\tilde{\omega} \circ (\tilde{\Psi}^{-1})$ is a rational mapping of $\mathbf{P}^n$ to $\prod^n \mathbf{P}$ which maps $\hat{W}_1$ onto $W_1 = \tilde{\omega}(\tilde{W}_1)$ biholomorphically. There exists an inhomogeneous coordinate $(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$ of $\mathbf{P}^n$ with the origin $\tilde{\sigma}(E)$ such that the proper image of $\hat{X}_j$ -axis under $\tilde{\omega} \circ (\tilde{\Psi}^{-1})$ is the X_j -axis of $\prod^n \mathbf{P}$ for each j and such that the proper image of the complex line $\hat{X}_1 = \hat{X}_2 = \dots = \hat{X}_n$ in $\mathbf{P}^n$ is the complex line $X_1 = X_2 = \dots = X_n$ in $\prod^n \mathbf{P}$.

The composition $(\hat{f}) = \tilde{\Psi} \circ (\tilde{\omega}_0|_{\tilde{W}_1})^{-1} \circ (f)$ defines a biholomorphic mapping of $(f)^{-1}(W_1)$ onto $\tilde{W}_1$ which has the developments $\hat{X}_j = \alpha x_j + \dots$ ($j = 1, 2, \dots, n$) at the origin of $\mathbf{C}^n$ for some complex number α . Denote by $\lambda: \mathbf{P}^n \rightarrow \prod^n \mathbf{P}$ the birational mapping defined by the equations $\tilde{X}_j = \alpha^{-1} \hat{X}_j$ ($j = 1, 2, \dots, n$) where $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n)$ is a global coordinate of $\prod^n \mathbf{P}$. The composition $(\tilde{f}) = \lambda \circ (\hat{f})$ defines a biholomorphic mapping of $(f)^{-1}(W_1)$ onto $\lambda(\tilde{W}_1)$ which has the developments $\tilde{X}_j = x_j + \dots$ ($j = 1, 2, \dots, n$) at the origin of $\mathbf{C}^n$. (Of course, we may suppose that $\tilde{W}_1$ has no point at infinity.)

Consider the rational transformation (R') of $\mathbf{P}^{n-1} \times \prod^n \mathbf{P}$ defined by the equations $z' = z, X'_j = R_j(X_1, X_2, \dots, X_n)$ ($j = 1, 2, \dots, n$). The composition $v^{-1} \circ (R'|_{\sum_0}) \circ v$ defines a one-valued meromorphic transformation of $\sum$ because the mapping $v: \sum \rightarrow \sum_0$ is one-sheeted, which defines a meromorphic transformation $(\tilde{R})$ of $\mathcal{D}$. Hence $(\tilde{R}) = \tilde{\Psi} \circ (\tilde{R}) \circ (\tilde{\Psi}^{-1})$ is a rational transformation of $\mathbf{P}^n$, under which each complex line passing the origin $\tilde{\sigma}(E)$ is invariant. The local mapping $(\tilde{f})$ satisfies the mapping equation

$$(\tilde{f})(ax_1, ax_2, \dots, ax_n) = (\tilde{R}) \circ (\tilde{f})(x_1, x_2, \dots, x_n)$$

where $(\tilde{R}) = \lambda \circ (\tilde{R}) \circ (\lambda^{-1})$. Hence $(\tilde{f})$ is a restriction of the meromorphic mapping of Poincaré derived by $(\tilde{R})$ which we denote by the same letter $(\tilde{f})$. The rational mapping $(T) = \tilde{\omega} \circ (\tilde{\Psi}^{-1}) \circ (\lambda^{-1})$ satisfies $(f) = (T) \circ (\tilde{f})$.

We say that the family $\{L_z\}$ is *one-sheeted* if the canonical projection $\tilde{\omega}: \mathcal{D} \rightarrow \prod^n \mathbf{P}$ of $\mathcal{D}$ onto $\prod^n \mathbf{P}$ is one-sheeted. Let us state the conclusion.

THEOREM 4. *Suppose that all the invariant image curves L_z of the meromorphic mapping (f) of Poincaré derived by the rational transformation (R) are algebraic and of rational type. Then there exists a rational mapping (T) of $\prod^n \mathbf{P}$ to $\prod^n \mathbf{P}$ which maps a neighborhood of the origin of $\prod^n \mathbf{P}$ onto a neighborhood of the origin of $\prod^n \mathbf{P}$ biholomorphically and which satisfies the following two conditions: (i) A branch $(\tilde{R})$ of $(T)^{-1} \circ (R) \circ (T)$ is a one-valued rational transformation of $\prod^n \mathbf{P}$ such that each complex line passing the origin of $\prod^n \mathbf{P}$ is invariant under $(\tilde{R})$. (ii) The meromorphic mapping $(\tilde{f})$ of Poincaré derived by $(\tilde{R})$ satisfies $(f) = (T) \circ (\tilde{f})$. When the family $\{L_z\}$ is one-sheeted, the rational mapping (T) is a birational transformation of $\prod^n \mathbf{P}$.*

Hence, under the condition of Theorem 4, (f) can be essentially reduced to a meromorphic mapping of Poincaré $(\tilde{f})$ derived by a rational transformation $(\tilde{R})$ of the type of Example 3. In the case of Example 4, we get the following.

$$\begin{aligned}
(T): X_1 &= (\tfrac{1}{2} \cdot \tilde{X}_1 + 1)^2 - 1, \quad X_j = (\tilde{X}_j / \tilde{X}_1) (\tfrac{1}{2} \cdot \tilde{X}_1 + 1) \{ (\tfrac{1}{2} \cdot \tilde{X}_1 + 1)^2 - 1 \} \\
&\quad ; j = 2, 3, \dots, n. \\
(\tilde{R}): \tilde{X}'_1 &= 2 \{ (\tfrac{1}{2} \cdot \tilde{X}_1 + 1)^3 - 1 \}, \quad \tilde{X}'_j = (2 \tilde{X}_j / \tilde{X}_1) \{ (\tfrac{1}{2} \cdot \tilde{X}_1 + 1)^3 - 1 \} \\
&\quad ; j = 2, 3, \dots, n. \\
(\tilde{J}): \tilde{X}_j &= (2x_j/x_1) \{ \exp(x_1/2) - 1 \}; j = 1, 2, \dots, n.
\end{aligned}$$

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