ON FOURIER COEFFICIENTS OF NON-HOLOMORPHIC POINCARÉ SERIES

Eiji YOSHIDA

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Introduction

Let H be the complex upper half plane and Γ be a Fuchsian group of the first kind having at least a cusp ∞ . For an arbitrary nonzero integer m , $z = x + iy \in H$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, let us denote the non-holomorphic Poincaré series by $P_m(z, s, \Gamma)$. It is well known that the non-holomorphic Poincaré series $P_m(z, s, \Gamma)$ or its n th Fourier coefficient $a_m(y, s, n, \Gamma)$ plays an important role in the spectral theory of automorphic functions. As a matter of fact, since $P_m(z, s, \Gamma)$ turns out to be an element of the Hilbert space $L^2(\Gamma \backslash H)$, which consists of functions invariant under the transformations from Γ and square integrable on the quotient $\Gamma \backslash H$, the information concerning analytic continuation of $P_m(z, s, \Gamma)$ or $a_m(y, s, n, \Gamma)$ to a region $\operatorname{Re}(s) > 1/2$ is closely connected with not only the problem whether or not eigenvalues of the Laplace operator (so-called exceptional eigenvalues) are in existence but also the arithmetical one because, as indicated by Selberg, the Fourier coefficient $a_m(y, s, n, \Gamma)$ involves the Kloosterman zeta function as main term. But, in spite of such importance, representation for $a_m(y, s, n, \Gamma)$ seems to be complicated (cf. (1.11)).

On the other hand, in his article [11], Niebur introduced a certain automorphic function $F_m(z, s, \Gamma)$ and calculated its n th Fourier coefficient $b_m(y, s, n, \Gamma)$ explicitly (Theorem A). Then by virtue of the fact that F_m is an eigenfunction of the Laplace operator, the Fourier coefficient b_m is presented in a satisfactory form.

The difference between P_m and F_m surely exists, in fact the former belongs to $L^2(\Gamma \backslash H)$ and the latter does not, nevertheless as far as the complex variable s is concerned, both $P_m(z, s, \Gamma)$ and $F_m(z, s, \Gamma)$, and hence both $a_m(y, s, n, \Gamma)$ and $b_m(y, s, n, \Gamma)$ have quite similar analytic properties to each other.

In the present paper, it will be verified that $P_m(z, s, \Gamma)$ is expressible in

terms of a linear combination of $F_m(z, s + l, \Gamma)$, $l \geq 0$ (Proposition 1) and conversely $F_m(z, s, \Gamma)$ as that of $P_m(z, s + l, \Gamma)$, $l \geq 0$ (Proposition 2). And moreover as a consequence of Proposition 1, together with Theorem A, we will obtain the new expression for $a_m(y, s, n, \Gamma)$ (Theorem B). This result seems to be an analogue of the formula for Fourier coefficients of the classical holomorphic Poincaré series of weight k established by Petersson [12].

The assertion of Proposition 1 has already been stated in Hejhal [6: P. 257] without proof, which was acquainted the author by the referee. The author expresses his gratitude to the referee for valuable comments.

§1. Preliminaries and statement of results

Let $H = \{z = x + iy \in \mathbb{C} \mid \text{Im } z = y > 0\}$ be the complex upper half plane having the Poincaré metric

$$(1.1) \quad ds^2 = y^{-2}(dx^2 + dy^2).$$

Then, the group $G = PSL(2, \mathbb{R})$ acts on H as linear fractional transformation:

$$\gamma z = \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

The metric (1.1) gives rise naturally to a G -invariant measure whose explicit form is

$$(1.2) \quad d\mu(z) = y^{-2} dx dy.$$

A group Γ of linear fractional transformations is called Fuchsian of the first kind if Γ is a discrete subgroup of G with finite volume $\text{vol}(\Gamma \backslash H)$. Throughout this paper, we will assume that Γ is a Fuchsian group of the first kind and has at least a cusp ∞ . Hence the fundamental domain of Γ denoted by $\mathcal{D}_\Gamma (= \Gamma \backslash H)$ is always noncompact. Let q be the smallest positive number such that $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in \Gamma$ and put $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & nq \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$. The Laplace operator associated with the metric (1.1) is

$$(1.3) \quad D = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Suppose that m is a nonzero integer. In his article [11], Niebur then introduced the following series:

$$(1.4) \quad F_m(z, s, \Gamma) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} e\left(\frac{m}{q}x(\gamma z)\right) y(\gamma z)^{1/2} I_{s-(1/2)}\left(2\pi \frac{|m|}{q}y(\gamma z)\right)$$

for $z \in H$, $s \in \mathbb{C}$, where $e(x) = \exp(2\pi i x)$, $x(\gamma z) = \operatorname{Re}(\gamma z)$, $y(\gamma z) = \operatorname{Im}(\gamma z)$ and I_ν is the modified Bessel function of the first kind defined, for example, by

$$(1.5) \quad I_\nu(y) = \pi^{-1/2} \Gamma\left(\nu + \frac{1}{2}\right)^{-1} (y/2)^\nu \int_{-1}^1 e^{-yt} (1-t^2)^{\nu-(1/2)} dt$$

for $\nu \in \mathbb{C}$, $\operatorname{Re}(\nu) > -1/2$ and $y > 0$ ([3: Vol. 2, P. 81]). As is shown in [11], the series on the right side of (1.4) converges absolutely for $\operatorname{Re}(s) > 1$. From this fact it follows that $F_m(\gamma z, s, \Gamma) = F_m(z, s, \Gamma)$ for all $\gamma \in \Gamma$ and $DF_m(z, s, \Gamma) = s(s-1) F_m(z, s, \Gamma)$.

Let $S(m, n, c, \Gamma)$ be the generalized Kloosterman sum, i.e.,

$$(1.6) \quad S(m, n, c, \Gamma) = \sum_{\substack{0 \leq a < qc \\ 0 \leq d < qc}} e\left(\frac{1}{qc}(ma + nd)\right), \quad \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma$$

for $c > 0$ (cf. [5, 14]). Utilizing this sum, Niebur also presented Fourier coefficients of $F_m(z, s, \Gamma)$ in explicit forms.

THEOREM A ([11: Theorem 1]). *Let $b_m(y, s, n, \Gamma)$ be the n th Fourier coefficient of $F_m(z, s, \Gamma)$ and assume that $\operatorname{Re}(s) > 1$. Then we have*

$$(1.7) \quad b_m(y, s, n, \Gamma) = \delta_{m,n} y^{1/2} I_{s-(1/2)}\left(2\pi \frac{|n|}{q}y\right) + B_m(s, n) y^{1/2} K_{s-(1/2)}\left(2\pi \frac{|n|}{q}y\right)$$

for $n \neq 0$, where $\delta_{m,n}$ is the Kronecker symbol and

$$B_m(s, n) = \frac{2}{q} \sum_{c>0} S(m, n, c, \Gamma) c^{-1} M_{2s-1}(4\pi(mn)^{1/2}(qc)^{-1}),$$

in which $M_{2s-1}((mn)^{1/2}y)$ denotes $J_{2s-1}(|mn|^{1/2}y)$ or $I_{2s-1}(|mn|^{1/2}y)$ according to $mn > 0$ or $mn < 0$. Furthermore for $n = 0$,

$$(1.8) \quad \begin{aligned} b_m(y, s, 0, \Gamma) &= \frac{2\pi^s}{q} \left(\frac{|m|}{q}\right)^{s-(1/2)} (2s-1)^{-1} \Gamma(s)^{-1} y^{1-s} \\ &\quad \times \sum_{c>0} S(m, 0, c, \Gamma) c^{-2s}. \end{aligned}$$

Remark. The above result is shown in [1], [4], [6] and [10] as well. In his notation, Niebur denotes by $b_m(y, s; n)$ the m th Fourier coefficient of $F_n(z, s, \Gamma)$, while in Theorem A, $b_m(y, s, n, \Gamma)$ implies the n th Fourier coefficient of $F_m(z, s, \Gamma)$. In addition, we will point out here that in his expression for $b_m(y, s; n)$, $K_{s-(1/2)}(2\pi|n|y)$ should be replaced by $K_{s-(1/2)}(2\pi|m|y)$ and $M_{2s-1}(4\pi(mn)^{1/2}c)$ by $M_{2s-1}(4\pi(mn)^{1/2}c^{-1})$.

Let m be a nonzero integer. We will now define the non-holomorphic Poincaré series as follows:

$$(1.9) \quad P_m(z, s, \Gamma) = \sum_{\gamma \in F_\infty \setminus \Gamma} \exp \left\{ \frac{2\pi}{q} (imx(\gamma z) - |m|y(\gamma z)) \right\} y(\gamma z)^s$$

for $z \in H, s \in \mathbb{C}$ (cf. [10, 13]). The above series is introduced in a slightly general form to make a correspondence between F_m and P_m natural. If m is a positive integer, then it is actually an usual non-holomorphic Poincaré series (cf. [9, 14]). Let $L^2(\mathcal{D}_\Gamma)$ be the Hilbert space of square integrable functions defined on $\mathcal{D}_\Gamma$, i.e.,

$$L^2(\mathcal{D}_\Gamma) = \left\{ f \mid f(\gamma z) = f(z) \text{ for } \gamma \in \Gamma, \int_{\mathcal{D}_\Gamma} |f(z)|^2 d\mu(z) < \infty \right\}.$$

The series on the right in (1.9) converges absolutely for $\operatorname{Re}(s) > 1$, and becomes an element of the space $L^2(\mathcal{D}_\Gamma)$, while $F_m(z, s, \Gamma)$ does not belong to it. Furthermore $P_m(z, s, \Gamma)$ satisfies the following recursion relation:

$$(1.10) \quad \{s(1-s) + D\} P_m(z, s, \Gamma) = -4\pi \frac{|m|}{q} s P_m(z, s+1, \Gamma).$$

Next, let $a_m(y, s, n, \Gamma)$ be the n th Fourier coefficient of $P_m(z, s, \Gamma)$, i.e.,

$$\begin{cases} P_m(z, s, \Gamma) = \sum_{n=-\infty}^{\infty} a_m(y, s, n, \Gamma) e\left(\frac{n}{q}x\right), \\ a_m(y, s, n, \Gamma) = \frac{1}{q} \int_0^q P_m(x + iy, s, \Gamma) e\left(-\frac{n}{q}x\right) dx. \end{cases}$$

Here, a well-known representation for $a_m(y, s, n, \Gamma)$ is

$$(1.11) \quad \begin{aligned} & \delta_{m,n} y^s e^{-2\pi|n|y/q} \\ & + \frac{1}{q} \sum_{c \geq 0} S(m, n, c, \Gamma) c^{-2s} \int_{-\infty}^{\infty} \frac{y^{1-s}}{(1+x^2)^s} e\left(\frac{|m|}{qc^2} \cdot \frac{\varepsilon x + i}{y(x^2 + 1)} + \frac{n}{q}xy\right) dx \end{aligned}$$

where ε equals 1 or -1 according to $m > 0$ or $m < 0$ (cf. [9, 14]).

On the other hand, Proposition 1 in section 2, which asserts that $P_m(z, s, \Gamma)$ is expressible in terms of $F_m(z, s + l, \Gamma)$ ($l \geq 0$), enables us to study $a_m(y, s, n, \Gamma)$ from a different point of view. In fact, combining Proposition 1 with Theorem A, we can obtain the new expression for $a_m(y, s, n, \Gamma)$. Our main result is the following

THEOREM B. *For an arbitrary nonzero integer m , let $P_m(z, s, \Gamma)$ be the non-holomorphic Poincaré series defined by (1.9), and let $a_m(y, s, n, \Gamma)$ be its n th Fourier coefficient. Then under the condition $\operatorname{Re}(s) > 1$, we have*

$$(1.12) \quad a_m(y, s, n, \Gamma) = \delta_{m,n} y^s e^{-2\pi|n|y/q} + \frac{2}{\Gamma(s)} \left(\frac{2\pi}{q} \right)^{s+1} |m|^{(1-s)/2} |n|^{(3s-1)/2} A_m(y, s, n)$$

for $n \neq 0$ and

$$(1.13) \quad a_m(y, s, 0, \Gamma) = 2^{3-2s} \frac{\pi}{q\Gamma(s)} \left(\frac{\pi|m|}{q} \right)^{(1-s)/2} A_m(y, s, 0)$$

for $n = 0$. Furthermore, $A_m(y, s, n) = y^s \sum_{c>0} S(m, n, c, \Gamma) c^{-(1+s)} \tilde{A}_m(y, s, n, c)$, where $S(m, n, c, \Gamma)$ denotes the Kloosterman sum as in (1.6) and $\tilde{A}_m(y, s, n, c)$ is equal to one of the following formulas according to the condition of mn :

$$\int_0^\infty \exp\{-\beta y(1+2t^2)\} t^{2s-1} (1+t^2)^{(s-1)/2} J_{s-1}(\alpha(1+t^2)^{1/2}) dt$$

for $mn > 0$,

$$\int_0^\infty \exp\{-\beta y(1+2t^2)\} t^s (1+t^2)^{s-1} J_{s-1}(\alpha t) dt$$

for $mn < 0$ and

$$\int_0^\infty \exp(-yt^2) t^{3s-2} J_{s-1} \left(\left(4\pi \frac{|m|}{q} \right)^{1/2} \frac{t}{c} \right) dt$$

for $n = 0$, in which $\alpha = 4\pi|mn|^{1/2}(qc)^{-1}$, $\beta = 2\pi|n|/q$ and J_{s-1} is the Bessel function.

The Bessel function J_ν is defined, for example, by

$$(1.14) \quad J_v(y) = \pi^{-1/2} \Gamma\left(v + \frac{1}{2}\right)^{-1} (y/2)^v \int_{-1}^1 e^{iyt} (1-t^2)^{v-(1/2)} dt$$

for $v \in \mathbf{C}$, $\operatorname{Re}(v) > -1/2$ and $y > 0$ ([3: Vol. 2, P. 81, (7)]).

Let $E(z, s, \Gamma)$ be the Eisenstein series defined by the equation:

$$E(z, s, \Gamma) = \sum_{\gamma \in F_\infty \setminus \Gamma} y(\gamma z)^s$$

for $z \in H$, $s \in \mathbf{C}$, and let c_0 be the smallest among all c -parts of γ such that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $c > 0$. Then it is derived from [8: Theorem 2.1.1] that $E(z, s, \Gamma)$ satisfies the following estimate for $\operatorname{Re}(s) = \sigma > 1$ and $y = \operatorname{Im} z > 0$:

$$(1.15) \quad |E(z, s, \Gamma)| \leq E(x + iy, \sigma, \Gamma) = O\left(\frac{q}{\sigma - 1} Y^{\sigma-1}\right),$$

where $Y = \max(y, c_0^{-1})$. This estimate plays an essential role in proving the absolute convergence of $P_m(z, s, \Gamma)$ and $F_m(z, s, \Gamma)$ in $\operatorname{Re}(s) > 1$.

Before proceeding further, we shall introduce Gegenbauer's polynomial $C_l^\nu(t)$. As shown in [3: Vol. 1, P. 176 or Vol. 2, P. 175], it is defined by the equation:

$$(1.16) \quad \begin{aligned} C_l^\nu(t) &= (-2)^{-l} \frac{\Gamma(v + (1/2)) \Gamma(2v + l)}{\Gamma(2v) \Gamma(v + (1/2) + l)!} (1 - t^2)^{(1/2)-v} \\ &\quad \times \frac{d^l}{dt^l} [(1 - t^2)^{v-(1/2)+l}] \end{aligned}$$

for any non-negative integer l and $v, t \in \mathbf{C}$. Especially for $t = 1$, one knows from [3: Vol. 1, P. 176, (7)] that

$$(1.17) \quad C_l^\nu(1) = (-1)^l C_l^\nu(-1) = \Gamma(2v + l) / [l! \Gamma(2v)].$$

This polynomial appears in the proof of Proposition 1 or Theorem B, in which it plays an important role.

§ 2. $P_m(z, s, \Gamma)$ and $F_m(z, s, \Gamma)$

The purpose in this section is to establish the facts that $P_m(z, s, \Gamma)$ can be expressed as a linear combination of $F_m(z, s + l, \Gamma)$, $l \geq 0$ and conversely $F_m(z, s, \Gamma)$ as that of $P_m(z, s + l, \Gamma)$, $l \geq 0$. These properties are obtained from elementary arguments.

PROPOSITION 1 (cf. [6: P. 257, (4.9)]). *Let $P_m(z, s, \Gamma)$ and $F_m(z, s + l, \Gamma)$, $l \geq 0$ be automorphic functions defined respectively by (1.9) and (1.4). Then we have the following equality:*

$$(2.1) \quad P_m(z, s, \Gamma) = \frac{2\pi^{1/2}}{\Gamma(s)} \left(4\pi \frac{|m|}{q}\right)^{(1/2)-s} \sum_{l=0}^{\infty} d(s, l) F_m(z, s + l, \Gamma)$$

for $\operatorname{Re}(s) > 1$, where $d(s, l) = (-1)^l (s - (1/2) + l) \Gamma(2s - 1 + l) / l!$.

PROOF. By use of the definition (1.4) of $F_m(z, s + l, \Gamma)$, the series on the right in (2.1) can be described as

$$\sum_l d(s, l) \sum_{\gamma} e\left(\frac{m}{q} x(\gamma z)\right) y(\gamma z)^{1/2} I_{s-(1/2)+l} \left(2\pi \frac{|m|}{q} y(\gamma z)\right)$$

for $\operatorname{Re}(s) > 1$. We first need to verify the absolute convergence of the above double series. In view of (1.5), under the restriction $\operatorname{Re}(v) > 1/2$, it is found that

$$(2.2) \quad |I_v(y)| \leq 2\pi^{-1/2} \left| \Gamma\left(v + \frac{1}{2}\right) \right|^{-1} (y/2)^{\operatorname{Re}(v)} e^y$$

for $y > 0$. Thus subject to the condition $\operatorname{Re}(s) = \sigma > 1$, we have

$$(2.3) \quad \begin{aligned} \sum_l |d(s, l)| \sum_{\gamma} \left| e\left(\frac{m}{q} x(\gamma z)\right) y(\gamma z)^{1/2} I_{s-(1/2)+l} \left(2\pi \frac{|m|}{q} y(\gamma z)\right) \right| \\ \leq 2\pi^{-1/2} \sum_l \left(\pi \frac{|m|}{q} \right)^{\sigma+l-(1/2)} |d(s, l) \Gamma(s+l)^{-1}| A(z, \sigma+l), \end{aligned}$$

where $A(z, \sigma + l) = \sum_{\gamma} y(\gamma z)^{\sigma+l} e^{2\pi|m|y(\gamma z)/q}$, $\gamma \in \Gamma_{\infty} \setminus \Gamma$. Then (1.15) gives

$$A(z, \sigma + l) \leq e^{2\pi|m|Y/q} E(z, \sigma + l) = O\left(\frac{qY^{\sigma+l-1}}{\sigma + l - 1} e^{2\pi|m|Y/q}\right).$$

As for $|d(s, l) \Gamma(s + l)^{-1}|$, since we can take positive numbers a_0 and b_0 such that $a_0 = \max\{|(s - (1/2) + l)/l|, l \geq 1\}$ and $b_0 = \max\{|(2s - 2 + l)/(s - 1 + l)|, l \geq 1\}$ for any fixed complex variable s , it is majorized by

$$\begin{cases} \left| \frac{\Gamma(2s-1)}{\Gamma(s)} \left(s - \frac{1}{2}\right) \right| & \text{for } l = 0, \\ \left| \frac{\Gamma(2s-1) b_0^l}{\Gamma(s)(l-1)!} a_0 \right| & \text{for } l \geq 1. \end{cases}$$

Hence the double series in (2.3) satisfies the following estimate:

$$(2.4) \quad O \left\{ \left| \frac{\Gamma(2s-1)}{\Gamma(s)} \right| \frac{q Y^{\sigma-1}}{\sigma-1} e^{2\pi|m|Y/q} \left(\left| s - \frac{1}{2} \right| + a_0 b_0 Y e^{\pi|m|b_0 Y/q} \right) \right\}.$$

This means that the series on the right in (2.1) converges absolutely for $\operatorname{Re}(s) > 1$, and thus it may be written, after changing the order of summation with respect to l and γ , as

$$(2.5) \quad \frac{2\pi^{1/2}}{\Gamma(s)} \left(4\pi \frac{|m|}{q} \right)^{(1/2)-s} \sum_{\gamma} e \left(\frac{m}{q} x(\gamma z) \right) y(\gamma z)^{1/2} \\ \times \sum_l d(s, l) I_{s-(1/2)+l} \left(2\pi \frac{|m|}{q} y(\gamma z) \right).$$

The formula (2.4) also shows by a slight modification that the series in question converges uniformly on any compact set which lies in the half plane $\operatorname{Re}(s) > 1$.

Next, it is known from [3: Vol. 2, P. 98, (1)] that

$$(2.6) \quad \sum_{i=0}^{\infty} (v+l) C_l^v(\beta) I_{v+l}(z) = (z/2)^v e^{z\beta} / \Gamma(v),$$

where $C_l^v(\beta)$ is Gegenbauer's polynomial as in (1.6). If we put $\beta = -1$, then it follows from (1.17) that $(s - (1/2) + l) C_l^{s-(1/2)}(-1) = d(s, l) / \Gamma(2s-1)$. Hence the double series in (2.5) can be described as

$$2^{2(1-s)} \pi^{1/2} \frac{\Gamma(2s-1)}{\Gamma(s - (1/2)) \Gamma(s)} \sum_{\gamma} e \left(\frac{m}{q} x(\gamma z) \right) e^{-2\pi|m|y(\gamma z)/q} y(\gamma z)^s.$$

By means of $\Gamma(2s-1) = 2^{2(s-1)} \pi^{-1/2} \Gamma(s - (1/2)) \Gamma(s)$, the last series completely coincides with (1.9).

As a reverse of Proposition 1, we obtain the following

PROPOSITION 2. *Let $F_m(z, s, \Gamma)$ and $P_m(z, s + \Gamma)$, $l \geq 0$ be automorphic functions defined respectively by (1.4) and (1.9). Then we have*

$$(2.7) \quad F_m(z, s, \Gamma) = 2^{1-2s} \Gamma \left(s + \frac{1}{2} \right)^{-1} \sum_{l=0}^{\infty} f_m(s, l) P_m(z, s + l, \Gamma)$$

for $\operatorname{Re}(s) > 1$, where $f_m(s, l) = (4\pi|m|/q)^{s-(1/2)+l} (s)_l / [l! (2s)_l]$ and $(v)_l = v(v+1) \cdots (v+l-1)$ for $l \geq 1$, $(v)_0 = 1$.

PROOF. It is easily seen that $|(s)_l/(2s)_l| \leq 1$ for any non-negative integer l and for any complex number s with $\operatorname{Re}(s) = \sigma > 0$. This, together with (1.15), yields

$$\begin{aligned} & \sum_l |f_m(s, l)| \left| \sum_\gamma \exp \left\{ \frac{2\pi}{q} (imx(\gamma z) - |m|y(\gamma z)) \right\} y(\gamma z)^s \right| \\ & \leq \sum_l \left(4\pi \frac{|m|}{q} \right)^{\sigma - (1/2) + l} \frac{1}{l!} E(z, \sigma + l) \\ & = O \left(\frac{Y^{\sigma-1}}{\sigma-1} e^{4\pi|m|Y/q} \right). \end{aligned}$$

From this fact, we obtain that the series in (2.7) converges absolutely and uniformly for $\operatorname{Re}(s) > 1$ and equals

$$(2.8) \quad 2^{1-2s} \Gamma \left(s + \frac{1}{2} \right)^{-1} \sum_\gamma \exp \left\{ \frac{2\pi}{q} (imx(\gamma z) - |m|y(\gamma z)) \right\} \sum_l f_m(s, l) y(\gamma z)^{s+l}.$$

On the other hand, it is well known that the Whittaker function $M_{k,v}(z)$ is defined by the equation:

$$M_{k,v}(z) = z^{v+(1/2)} e^{-z/2} \sum_{l=0}^{\infty} \frac{(v-k+(1/2))_l}{(2v+1)_l} \frac{z^l}{l!},$$

and especially for $k=0$, the equality $M_{0,v}(z) = 2^{2v} \Gamma(v+1) z^{1/2} I_v(z/2)$ holds. These give

$$\begin{aligned} & \exp \left(-\frac{2\pi}{q} |m|y(\gamma z) \right) \sum_{l=0}^{\infty} f_m(s, l) y(\gamma z)^{s+l} \\ & = \left(4\pi \frac{|m|}{q} \right)^{-1/2} M_{0, s-(1/2)} \left(4\pi \frac{|m|}{q} y(\gamma z) \right) \\ & = 2^{2s-1} \Gamma \left(s + \frac{1}{2} \right) y(\gamma z)^{1/2} I_{s-(1/2)} \left(2\pi \frac{|m|}{q} y(\gamma z) \right). \end{aligned}$$

Hence, as a conclusion, the series in (2.8) turns out to be equal to (1.4).

We shall note here that an eigenfunction of the Laplace operator D associated with the non-holomorphic Poincaré series $P_m(z, s, \Gamma)$ can be constructed by use of the recursion relation (1.10). More specifically, it follows from $s(1-s) + (s+l)(s+l-1) = l(2s+l-1)$ and (1.10) that

$$\begin{aligned} \{s(1-s) + D\} P_m(z, s+l, \Gamma) &= l(2s+l-1) P_m(z, s+l, \Gamma) \\ &\quad - 4\pi \frac{|m|}{q} (s+l) P_m(z, s+l+1, \Gamma) \end{aligned}$$

for any non-negative integer l . Thus we find

$$\begin{aligned} \{s(1-s) + D\} \sum_{l=0}^N \left(4\pi \frac{|m|}{q}\right)^l \frac{(s)_l}{l! (2s)_l} P_m(z, s+l, \Gamma) \\ = - \left(4\pi \frac{|m|}{q}\right)^{N+1} \frac{(s)_{N+1}}{N! (2s)_N} P_m(z, s+N+1, \Gamma) \end{aligned}$$

for any positive integer N . Then from the same idea as in the proof of Proposition 2, it is verified that the right hand side of the preceding formula tends to 0 when $N \rightarrow \infty$, and that the infinite series $Q_m(z, s, \Gamma)$ defined by the equation:

$$Q_m(z, s, \Gamma) = \sum_{l=0}^{\infty} \left(4\pi \frac{|m|}{q}\right)^l \frac{(s)_l}{l! (2s)_l} P_m(z, s+l, \Gamma)$$

converges absolutely for $\text{Re}(s) > 1$. Therefore the series $Q_m(z, s, \Gamma)$ becomes an eigenfunction of the Laplace operator D , namely we have

$$\{s(1-s) + D\} Q_m(z, s, \Gamma) = 0.$$

In this case, the series $Q_m(z, s, \Gamma)$, constructed so as to be an eigenfunction of D , is essentially identical with $F_m(z, s, \Gamma)$, and hence the above method provides an alternative approach to the assertion of Proposition 2.

§3. Proof of Theorem B

From Proposition 1 and (2.4), it follows that

$$(3.1) \quad a_m(y, s, n, \Gamma) = \frac{2\pi^{1/2}}{\Gamma(s)} \left(4\pi \frac{|m|}{q}\right)^{(1/2)-s} \sum_{l=0}^{\infty} d(s, l) b_m(y, s+l, n, \Gamma)$$

for $\text{Re}(s) > 1$, where $a_m(y, s, n, \Gamma)$ and $b_m(y, s+l, n, \Gamma)$ are n th Fourier coefficients of $P_m(z, s, \Gamma)$ and $F_m(z, s+l, \Gamma)$ respectively. In this section, we will denote $a_m(y, s, n, \Gamma)$ and the Kloosterman sum $S(m, n, c, \Gamma)$ respectively by $a_m(y, s, n)$ and $S(m, n, c)$ for the sake of simplicity.

In case of $n \neq 0$, it is derived from Theorem A and (3.1) that

$$(3.2) \quad a_m(y, s, n) = \frac{2\pi^{1/2}}{\Gamma(s)} \left(4\pi \frac{|m|}{q}\right)^{(1/2)-s} \left\{ \delta_{m,n} y^{1/2} \sum_{l=0}^{\infty} d(s, l) I_{s-(1/2)+l} \left(2\pi \frac{|n|}{q} y\right) \right. \\ \left. + y^{1/2} \sum_{l=0}^{\infty} d(s, l) B_m(s+l, n) K_{s-(1/2)+l} \left(2\pi \frac{|n|}{q} y\right) \right\}$$

for $\operatorname{Re}(s) > 1$, where $d(s, l) = \Gamma(2s-1)(s-(1/2)+l) C_l^{s-(1/2)}(-1)$ and $B_m(s+l, n) = 2q^{-1} \sum_{c>0} S(m, n, c) c^{-1} M_{2(s+l)-1}(4\pi(mn)^{1/2}(qc)^{-1})$. It readily follows from (2.6) with $\beta = -1$ that part of the first term on the right side of (3.2) can be written as

$$\begin{aligned} & \Gamma(2s-1) \sum_{l=0}^{\infty} (s-(1/2)+l) C_l^{s-(1/2)}(-1) I_{s-(1/2)+l} \left(2\pi \frac{|n|}{q} y\right) \\ &= \frac{\Gamma(2s-1)}{\Gamma(s-(1/2))} \left(\pi \frac{|n|}{q} y\right)^{s-(1/2)} e^{-2\pi|n|y/q}. \end{aligned}$$

Thus using $\Gamma(2s-1) = 2^{2(s-1)} \pi^{-1/2} \Gamma(s-(1/2)) \Gamma(s)$, whole of the term in question turns out to be equal to $\delta_{m,n} y^s e^{-2\pi|n|y/q}$.

We will investigate, from now on, the second term on the right hand side of (3.2). In the beginning, it is known that the modified Bessel function $K_\nu(y)$ has an integral representation in the form:

$$K_\nu(y) = \pi^{1/2} \Gamma\left(\nu + \frac{1}{2}\right)^{-1} (y/2)^\nu \int_1^\infty e^{-yt} (t^2 - 1)^{\nu-(1/2)} dt$$

for $\operatorname{Re}(\nu) > -1/2$, $y > 0$ ([3: Vol. 2, P. 18, (15)]), where the principal branch is taken for $(t^2 - 1)^{\nu-(1/2)}$. We shall note here that, whenever the integrand $(t^2 - 1)^{\nu-(1/2)}$ is regarded as a product $(-1)^{\nu-(1/2)}(1 - t^2)^{\nu-(1/2)}$, their branches are so chosen that $\arg(-1) + \arg(1 - t^2) = 0$. Thus after integrating by parts l -times, it is found that the function $K_{s-(1/2)+l}(2\pi|n|y/q)$ can be expressed as

$$\begin{aligned} & \pi^{1/2} \Gamma(s+l)^{-1} \left(\pi \frac{|n|}{q} y\right)^{s-(1/2)} (-2)^{-l} (-1)^{s-1} \\ & \times \int_1^\infty \exp\left(-2\pi \frac{|n|}{q} yt\right) \frac{d^l}{dt^l} [(1 - t^2)^{s-1+l}] dt \end{aligned}$$

for $\operatorname{Re}(s) > 0$, and moreover from (1.16) and $(-1)^{s-1}(1 - t^2)^{s-1} = (t^2 - 1)^{s-1}$, the above may be rewritten as

$$\pi^{1/2} \left(\pi \frac{|n|}{q} y \right)^{s-(1/2)} \frac{\Gamma(2s-1)!!}{\Gamma(s)\Gamma(2s-1+l)} \\ \times \int_1^\infty \exp\left(-2\pi \frac{|n|}{q} yt\right) (t^2-1)^{s-1} C_l^{s-(1/2)}(t) dt.$$

Substituting the last formula for $K_{s-(1/2)+l}(2\pi|n|y/q)$ and considering (1.17), we obtain that the second term on the right in (3.2) is equivalent to

$$(3.3) \quad 2^{2(1-s)} \pi \frac{\Gamma(2s-1)}{\Gamma^2(s)} \left| \frac{m}{n} \right|^{(1/2)-s} y^s \sum_{l=0}^\infty (-1)^l \left(s - \frac{1}{2} + l \right) \\ \times B_m(s+l, n) \int_1^\infty \exp\left(-2\pi \frac{|n|}{q} yt\right) (t^2-1)^{s-1} C_l^{s-(1/2)}(t) dt$$

for $\operatorname{Re}(s) > 1$.

Next, as shown in [3: Vol. 2, P. 177, (31)], we know the following integral representation:

$$C_l^v(t) = \pi^{-1/2} \frac{\Gamma(v+(1/2))\Gamma(2v+l)}{l!\Gamma(v)\Gamma(2v)} \int_0^\pi (t+(t^2-1)^{1/2}\cos\theta)^l \sin^{2v-1}\theta d\theta$$

for $\operatorname{Re}(v) > 0$. Then an easy calculation yields

$$|C_l^{s-(1/2)}(t)| \leq \pi^{1/2} \left| \frac{\Gamma(s)}{\Gamma(s-(1/2))} \right| (2a_0 t)^l$$

for any complex number s with $\operatorname{Re}(s) > 1$ and for any $t > 1$, where $a_0 = \max\{|2s-1+l|/l, l \geq 1\}$. Hence we have

$$\int_1^\infty \left| \exp\left(-2\pi \frac{|n|}{q} yt\right) (t^2-1)^{s-1} C_l^{s-(1/2)}(t) \right| dt \\ \leq \pi^{1/2} \left| \frac{\Gamma(s)}{\Gamma(s-(1/2))} \right| (2a_0)^l \int_0^\infty \exp\left(-2\pi \frac{|n|}{q} yt\right) t^{2\sigma-2+l} dt \\ = O((2a_0)^l (2\sigma-1)_l y^{1-2\sigma-l})$$

for any fixed s which satisfies $\operatorname{Re}(s) = \sigma > 1$. As for $B_m(s+l, n)$, the inequality $|S(m, n, c)| \leq qc$ is readily shown to hold from the definition (1.6). Thus considering (2.2) and (1.14), it follows that

$$\left| S(m, n, c) c^{-1} M_{2(s+l)-1} \left(4\pi \frac{(mn)^{1/2}}{qc} \right) \right|$$

$$\begin{aligned}
&\leq 2q\pi^{-1/2} \left| \Gamma\left(2s + 2l - \frac{1}{2}\right) \right|^{-1} \left(2\pi \frac{|mn|^{1/2}}{qc} \right)^{2(\sigma+l)-1} \\
&\quad \times \exp\left(4\varepsilon\pi \frac{|mn|^{1/2}}{qc} \right) \\
&= O\left(\frac{1}{|2s - (1/2)|_{2l}} c^{1-2(\sigma+l)} \exp\left(4\varepsilon\pi \frac{|mn|^{1/2}}{qc_0} \right) \right)
\end{aligned}$$

for $\operatorname{Re}(s) = \sigma > 1$, where ε denotes 0 or 1 according as $mn > 0$ or $mn < 0$ and c_0 is as in (1.15). From these facts, we obtain

$$\begin{aligned}
(3.4) \quad &\sum_l \left| (-1)^l \left(s - \frac{1}{2} + l \right) \right| \left| \sum_c \left| S(m, n, c) c^{-1} M_{2(s+l)-1} \left(4\pi \frac{(mn)^{1/2}}{qc} \right) \right| \right. \\
&\quad \times \left. \int_1^\infty \left| \exp\left(-2\pi \frac{|n|}{q} yt \right) (t^2 - 1)^{s-1} C_l^{s-(1/2)}(t) \right| dt \right| \\
&= O \left\{ y^{1-2\sigma} \sum_c c^{1-2\sigma} \left(\left| s - \frac{1}{2} \right| + 2a_0 y^{-1} c^{-2} \exp\left(\frac{2a_0}{yc^2} \right) \right) \exp\left(4\varepsilon\pi \frac{|mn|^{1/2}}{qc_0} \right) \right\} \\
&= O \left\{ y^{1-2\sigma} \left(\frac{1}{\sigma - 1} + 2a_0 y^{-1} \exp\left(\frac{2a_0}{yc_0^2} \right) \right) \right\}
\end{aligned}$$

for $\operatorname{Re}(s) > 1$. This implies that the series in (3.3) is absolutely convergent for any s with $\operatorname{Re}(s) > 1$ and for any $y > 0$, and hence it is possible to change the order of summation with respect to l and c .

As a final step, we now consider the series:

$$(3.5) \quad \frac{\Gamma(2s-1)}{\Gamma(s)} \sum_{l=0}^{\infty} (-1)^l \left(s - \frac{1}{2} + l \right) M_{2(s+l)-1} \left(4\pi \frac{(mn)^{1/2}}{qc} \right) C_l^{s-(1/2)}(t).$$

By similar evaluation as in the above, we can derive the facts that the series in (3.5) converges absolutely for any s for which $\operatorname{Re}(s) > 1$ and for any $t > 1$, and moreover converges uniformly on any compact set in $\operatorname{Re}(s) > 1$ if $t > 1$. From the latter, we see that the series in question determines a regular function of s in the half-plane $\operatorname{Re}(s) > 1$, and the former, together with (3.4) again, shows that the order of summation over l and integration over t may be reversed for each fixed c . Next it follows from [3: Vol. 2, P. 176, (20)] that

$$\begin{aligned}
(3.6) \quad C_l^{s-(1/2)}(t) &= \frac{\Gamma(2s-1+l)}{l! \Gamma(2s-1)} F\left(-l, 2s-1+l; s; \frac{1-t}{2}\right) \\
&= (-1)^l \frac{\Gamma(2s-1+l)}{l! \Gamma(2s-1)} F\left(-l, 2s-1+l; s; \frac{1+t}{2}\right)
\end{aligned}$$

where $F = {}_2F_1$ is the hypergeometric function. By use of this equation, if $mn > 0$, the series in (3.5) is transformed into

$$\frac{1}{2} \sum_l \frac{\Gamma(2s-1+l)}{l! \Gamma(s)} F\left(-l, 2s-1+l; s; \frac{1+t}{2}\right) (2s-1+2l) J_{2s-1+2l}(\alpha)$$

where $\alpha = 4\pi|mn|^{1/2}(qc)^{-1}$. Then, as stated in [15: 5. 21, P. 140, (3)], one knows the following formula:

$$\begin{aligned}
(3.7) \quad (kz/2)^{-\nu} J_\nu(kz) \\
= (z/2)^{-\mu} \sum_{l=0}^{\infty} \frac{\Gamma(\mu+l)}{l! \Gamma(\nu+1)} F(-l, \mu+l; \nu+1; k^2) (\mu+2l) J_{\mu+2l}(z),
\end{aligned}$$

where μ, ν and $\nu - \mu$ are not negative integers. From this formula, the series in (3.5) supposed that $mn > 0$ can be written as

$$(3.8) \quad \frac{1}{2} \left(\frac{\alpha}{2}\right)^s \left(\frac{1+t}{2}\right)^{(1-s)/2} J_{s-1} \left(\left(\frac{1+t}{2}\right)^{1/2} \alpha \right)$$

in $\text{Re}(s) > 1$ except for $s = 2, 3, 4, \dots$. In the case of $mn < 0$, again from (3.6), the series in (3.5) is identified with

$$\frac{1}{2} \sum_l (-1)^l \frac{\Gamma(2s-1+l)}{l! \Gamma(s)} F\left(-l, 2s-1+l; s; \frac{1-t}{2}\right) (2s-1+2l) I_{2s-1+2l}(\alpha)$$

where $\alpha = 4\pi|mn|^{1/2}(qc)^{-1}$. Then since $\arg(\alpha) = 0$, we have

$$\begin{aligned}
I_{2s-1+2l}(\alpha) &= \exp\left(-\frac{2s-1+2l}{2}\pi i\right) J_{2s-1+2l}(e^{\pi i/2}\alpha) \\
&= (-1)^l \exp\left(\frac{1-2s}{2}\pi i\right) J_{2s-1+2l}(e^{\pi i/2}\alpha).
\end{aligned}$$

Thus by putting $k = e^{-\pi i/2}((t-1)/2)^{1/2}$ with $t > 1$, $\arg((t-1)/2) = 0$ and $z = e^{\pi i/2}\alpha$ in (3.7), the above series can be reduced to

$$(3.9) \quad \frac{1}{2} \left(\frac{\alpha}{2}\right)^s \left(\frac{t-1}{2}\right)^{(1-s)/2} J_{s-1} \left(\left(\frac{t-1}{2}\right)^{1/2} \alpha \right)$$

in $\text{Re}(s) > 1$ except for $s = 2, 3, 4, \dots$. However we know that the series in (3.5) is holomorphic for $\text{Re}(s) > 1$ in both cases $mn > 0$ and $mn < 0$, so for all s in this region, it turns out to be equal to (3.8) or (3.9) corresponding to the condition of mn .

From these arguments, we obtain, as a consequence, that the series in (3.3), and hence the second term on the right hand side of (3.2), can be described as follows:

$$\frac{2^{s+2}}{\Gamma(s)} \left(\frac{\pi}{q} \right)^{s+1} |m|^{(1-s)/2} |n|^{(3s-1)/2} A_m(y, s, n)$$

where $A_m(y, s, n) = y^s \sum_{c>0} S(m, n, c) c^{-(1+s)} \tilde{A}_m(y, s, n, c)$ and furthermore $\tilde{A}_m(y, s, n, c)$ equals

$$2^{-2s} \int_1^\infty \exp(-\beta y t) (t^2 - 1)^{s-1} \left(\frac{t + \varepsilon}{2} \right)^{(1-s)/2} J_{s-1} \left(\left(\frac{t + \varepsilon}{2} \right)^{1/2} \alpha \right) dt$$

in which ε is 1 or -1 in accordance with $mn > 0$ or $mn < 0$ and $\alpha = 4\pi|mn|^{1/2}(qc)^{-1}$, $\beta = 2\pi|n|/q$. After making a change of variable t to $1 + t'^2 = (1 + t)/2$ for $mn > 0$ and $t'^2 = (t - 1)/2$ for $mn < 0$, we arrive at the assertion (1.12) of Theorem B.

In case of $n = 0$, it is derived from Theorem A and (3.1) that

$$(3.10) \quad a_m(y, s, 0) = 2^{2(1-s)} \frac{\pi}{q\Gamma(s)} \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(2s-1+l)}{l! \Gamma(s+l)} \left(\pi \frac{|m|}{q} \right)^l \\ \times y^{1-(s+l)} \sum_{c>0} S(m, 0, c) c^{-2(s+l)}$$

for $\text{Re}(s) > 1$. Because $|S(m, 0, c)| \leq qc$, the following estimate holds:

$$(3.11) \quad \sum_l \left| \frac{(-1)^l \Gamma(2s-1+l)}{l! \Gamma(s+l)} \left(\pi \frac{|m|}{q} \right)^l y^{1-(s+l)} \right| \sum_c |S(m, 0, c) c^{-2(s+l)}| \\ = O \left(y^{1-\sigma} \sum_c c^{1-2\sigma} \exp \left(\frac{\pi a_0 |m|}{qyc_0^2} \right) \right) \\ = O \left(y^{1-\sigma} \frac{1}{\sigma-1} \exp \left(\frac{\pi a_0 |m|}{qyc_0^2} \right) \right)$$

for any fixed s with $\text{Re}(s) = \sigma > 1$, where $a_0 = \max \{|2s-2+l|/|s-1+l|, l \geq 1\}$. Therefore in (3.10), we can change the order of summation with respect to l and c . Next, we know the following integral representation:

$$y^{1-2s-l} \Gamma(2s-1+l) = \int_0^\infty e^{-yt} t^{2s-2+l} dt$$

for $\operatorname{Re}(s) > 1/2$ and $l \geq 0$. Then if we consider the series:

$$(3.12) \quad \sum_{l=0}^{\infty} \frac{(-1)^l t^l}{l! \Gamma(s+l)} \left(\pi \frac{|m|}{q} \right)^l c^{-2l},$$

it converges absolutely for any $t > 0$. Hence evaluation in (3.11) also shows that the order of summation over l in (3.12) and integration over t may be reversed. Moreover the series in (3.12) is equal to

$$\left(\pi \frac{|m|}{qc^2} t \right)^{(1-s)/2} J_{s-1} \left(2 \left(\pi \frac{|m|}{q} t \right)^{1/2} c^{-1} \right).$$

From these facts, the series in (3.10) coincides with

$$2^{3-2s} \frac{\pi}{q \Gamma(s)} \left(\pi \frac{|m|}{q} \right)^{(1-s)/2} A_m(y, s, 0)$$

where $A_m(y, s, 0) = y^s \sum_{c>0} S(m, 0, c) c^{-(1+s)} \tilde{A}_m(y, s, 0, c)$ and $\tilde{A}_m(y, s, 0, c)$ denotes

$$\frac{1}{2} \int_0^\infty \exp(-yt) t^{3(s-1)/2} J_{s-1} \left(2 \left(\pi \frac{|m|}{q} t \right)^{1/2} c^{-1} \right) dt.$$

By making the change of variable t to $t' = t^{1/2}$, we obtain (1.13) in Theorem B.

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Kagoshima National College of Technology
1460-1 Shinko, Hayato-cho, Airagun
Kagoshima 899-51
Japan