THE KÜHNEL TRIANGULATION OF THE COMPLEX PROJECTIVE PLANE FROM THE VIEW POINT OF COMPLEX CRYSTALLOGRAPHY

PART I

By

Bernard MORIN and Masaaki YOSHIDA

(Received Sept. 10, 1990)

TABLE OF CONTENTS

Introduction

Part I

Chapter I	The Kühnel complex viewed as a PL-manifold
§1	Kühnel's simplicial complex K
§2	The Grünbaum-Brückner link $\mathcal{M}$
§3	A topological insight into $ K $
§4	The direct proof that $ K \cong \mathbb{CP}^2$
Chapter II	The Kühnel complex viewed as an orbifold
§5	The G -invariant torus T in K Appendix (by A. Marin): Strangeness of Brückner's triangulation; The trefoil knot pleads guilty
§6	The six fold cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$
§7	Some lattices and crystallographic groups in $\mathbb{C}^2$
§8	The rectilinear triangulation $\tilde{K}$ of $\mathbb{C}^2$
§9	An explicit triangulation of $\mathbb{CP}^2$ by K (the crystallographic proof)
§10	The Dirichlet domain P with respect to $e_1 = k_{10}$
§11	The prismatic Marin fundamental domain MaD

References

Part II (by P. Arnoux and A. Marin)

Chapter III	The ontological proof forcing $\mathbb{CP}^2$ to admit the triangulation K . Looking for $(n+1)^2$ vertices triangulations of $\mathbb{CP}^n$
Chapter IV	Homology projective spaces and Kühnel's complexes
Chapter V	The lattice $L[f]$ decorated with 81 colors
Chapter VI	How to make paper models

Introduction

In this paper we give three new proofs of the fact that the geometric realization $|K|$ of the Kühnel triangulation K defined in §1 is homeomorphic to the complex projective plane $\mathbf{CP}^2$.

The original proof is due to W. Kühnel and T. F. Banchoff and can be found in [K-B]. It compares an explicit map from $|K| - |\dot{\Delta}|$ to the sphere S^2 with the Hopf map $\mathbf{CP}^2 - B \rightarrow \mathbf{CP}^1 \cong S^2$. (Here Δ is any 4-simplex in K and B is the complement of a closed tubular neighborhood of a complex line in $\mathbf{CP}^2$.) In fact, using the same idea, there also exists a yet unpublished but completely straightforward combinatorial argument (see end of §2, where we reproduce A. Marin's version of it). Moreover Freedman's classification of simply connected closed 4-manifolds tells us that there are only two topological manifolds having the homotopy type of $\mathbf{CP}^2$, one of which admits no PL-structure. Therefore it suffices to show that K has the homotopy type of $\mathbf{CP}^2$ and has a PL-structure (see §2 Power hammer proof).

The first of our three proofs (referred to in Part I Chapter I as the *direct proof*) was presented at Oberwolfach in 1981 and notes of this talk have been circulated by K. Jänich, whom we thank for having drawn the wonderful pictures they contain. This proof uses the fact that $\mathbf{CP}^2$ maps naturally onto the 2-simplex Δ_2 ; this map is compared with a simplicial mapping from a subdivision $\bar{K}$ of K to a subdivision $\bar{\Delta}_2$ of Δ_2 constructed in §3. The second proof (referred to in Part I Chapter II as the *crystallographic proof*) is by showing that the Kühnel triangulation K is in fact the quotient of a rectilinear triangulation $\tilde{K}$ of a $\mathbf{C}^2$ by the action of complex crystallographic group Γ (defined in §7) preserving $\tilde{K}$. An explicit mapping $Y: \mathbf{C}^2 \rightarrow \mathbf{CP}^2$ is then defined (see §9) which factorizes through $|K| \cong \mathbf{C}^2/\Gamma$. The last proof (referred to in Part II Chapter III as the *ontological proof*) is a very efficient attack on the problem that was developed by Alexis Marin after reading drafts of the present work. It is included in Part II of this paper entirely due to Marin. Other contributions by Marin will be mentioned as we encounter them, so that this paper should be considered as a dialogue between the authors and him, dialogue in which he has the last word (see epilogue in Part II Chapter V and §8 Remark (3)).

A question naturally arises: "Why three new proofs when at least two

satisfactory ones already exist ?” Answer: Many mathematicians asked for a spacial presentation of the phenomenon discovered by Kühnel. In order to respond to this demand, we thought that as many view points as possible should be given. Each proof (including the one in [K-B]) does indeed throw a different light from a different angle of vision, on a simple but tricky four dimensional fact. The trickiness comes from the constraint that human vision has been devised to grasp only three dimensional happenings. We have tried our best to make the four dimensional world—at least one corner of it—as palpable as possible to the reader.

In §1 (Part I Chapter I) we recall what is the Kühnel triangulation K (a simplicial complex of dimension 4 with 9 vertices, the stars of any two vertices being isomorphic to one another). In §2 we recall that the link $\mathcal{M}$ at a given vertex is homeomorphic to the 3-sphere which implies that $|K|$ is a topological 4-manifold. We show that K is PL , which allows Freedman’s Power hammer proof to work, and we cite Marin’s version of the straightforward combinatorial proof. §3 defines a piecewise linear projection $p: |K| \rightarrow \Delta_2$ and studies its topological properties. §4 introduces a map $\pi: \mathbf{CP}^2 \rightarrow \Delta_2$ and compares it with p thus showing $|K| \cong \mathbf{CP}^2$ and making the triangulation explicit. §5 (Part I Chapter II) shows that K contains a combinatorial torus T with remarkable properties. This fact enables one to connect Kühnel’s result with complex crystallography. §6 constructs a 6 fold branched cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$ which will turn out to be isomorphic to the link of any vertex of $\tilde{K}$. In order to explain the crystallographic proof, §7 introduces lattices and various crystallographic ingredients needed later, in particular, the vertices of $\tilde{K}$ and the groups Γ and Γ'' . The construction of $\tilde{K}$ is completed in §8. In §9, a Γ -invariant holomorphic mapping $Y: \mathbf{C}^2 \rightarrow \mathbf{CP}^2$ is constructed by using Weierstrass functions, producing a second explicit homeomorphism $|K| \xrightarrow{\sim} \mathbf{CP}^2$. It is also shown there how the real projective plane $\mathbf{RP}^2 \subset \mathbf{CP}^2$ fits inside our picture. As a dessert §10 constructs a highly symmetric Dirichlet domain for the action of Γ in $\mathbf{C}^2$ and uses it to exhibit a fundamental domain for $\tilde{K}$. §11 prepares for the construction of Part II Chapter III by embedding in $\tilde{K}$ the fundamental domain discovered by A. Marin. By considering a fundamental domain of $|\tilde{K}|$ which is isomorphic to the product of Δ_2 with a hexagon, and by using a priori requirements, A. Marin in Part II Chapter III forces the Kühnel triangulation to exist. This provides a way for the hurried mathematician, not interested in visualizing things, to avoid most of the effort

imposed on him in the present article. In Part II Chapter IV A. Marin and Pierre Arnoux pose and almost completely solve a combinatorial topology problem in which Kühnel's triangulation appears naturally as part of the solution as well as the triangulation of the Hamiltonian projective plane $\mathbf{HP}^2$ with fifteen vertices also due to Kühnel ([B-K]). Part II Chapter V studies the quotient of the lattice $L[f]$ of vertices of $\tilde{K}$ by a sublattice of index 81 which should provide the reader with a global understanding of the entire scene. The chapter concludes by identifying $\mathbf{C} \times \mathbf{C}$ with $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ which enables to interpret the lattice $L[f]$ as the tensor product over the integers of the hexagonal lattice by itself. Part II Chapter VI explains how to obtain 3-dimensional paper models of various constructions used in the course of the article, this time satisfying the unhurried reader willing to enjoy the scenery.

The pictures of K. Jänich, already cited, were of invaluable help in communicating our ideas, and are reproduced here with the logo (the Katakana signs pronounced as the English word "Yea"). We also include four drawings made for us by George K. Francis which we attribute to him as they occur.

We would like to acknowledge the importance to us of the contribution of Jänich, numerous discussions with Kühnel (who gave us much useful information and made many helpful suggestions) and the deep interest shown by A. Marin, in particular his stern checking, his rigorous criticisms and the solidity of his contribution. We are grateful to Alex Flegmann who tried his best to make our Franco-Japanese English look less clumsy. Without the facilities offered both by the IRMA (ULP Strasbourg, France) and by the MPI für Mathematik (Bonn, Germany), the authors could not thank each other for their mutual collaboration. A special mention should be made of Marie-Claire Grima, who showed great patience in tabulating a huge quantity of tedious diagrams. Some of these appear below while the remainder will, we hope, serve as the basis for a computerised audiovisual animation of the pictures we have in mind.

Chapter I The Kühnel complex viewed as a PL-manifold

§1 Kühnel's simplicial complex K

Let the vertices $1, 2, \dots, 9$ of the 8-simplex Δ_8 be arranged in an array as

follows:

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

In the course of this paper we like to keep in mind the idea that the three columns of this matrix are respectively blue, white and red, so that for instance 1 belongs to the blue family, 5 to the white family, and 9 to the red family.

Let G be the subgroup of the symmetric group $\mathfrak{S}_9$ (acting on Δ_8) generated by α , β and γ , where α cyclically permutes the columns and β (does the same to) the rows of the above matrix, while γ is β on the first column, β^{-1} on the second and the identity on the third:

$$\begin{aligned} \alpha \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} &= \begin{pmatrix} 4 & 7 & 1 \\ 5 & 8 & 2 \\ 6 & 9 & 3 \end{pmatrix}, & \beta \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} &= \begin{pmatrix} 2 & 5 & 8 \\ 3 & 6 & 9 \\ 1 & 4 & 7 \end{pmatrix}, \\ \gamma \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} &= \begin{pmatrix} 2 & 6 & 7 \\ 3 & 4 & 8 \\ 1 & 5 & 9 \end{pmatrix}. \end{aligned}$$

Thus we have

$$\alpha\beta = \beta\alpha, \quad \beta\gamma = \gamma\beta, \quad \alpha^{-1}\gamma\alpha\gamma^{-1} = \beta,$$

showing that the order of G is 27. Moreover G acts transitively on the vertices of Δ_8 . For later use, let us also define δ and ε :

$$\delta \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 8 \\ 1 & 4 & 7 \\ 3 & 6 & 9 \end{pmatrix}, \quad \varepsilon \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 7 \\ 5 & 2 & 8 \\ 6 & 3 & 9 \end{pmatrix}.$$

As an alternative possibility, let us indicate a sub-simplex of Δ_8 by marking the positions of its vertices in the 3×3 -matrix (Braille notation), for instance,

$$(1 \ 2 \ 4 \ 5 \ 9) = \begin{array}{|c|c|c|} \hline \bullet & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array}, \quad (1 \ 2 \ 4 \ 5 \ 6) = \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array}.$$

The *type* of a sub-simplex of Δ_8 is the collection of non-zero occupation numbers of its vertices in the columns, for instance, $(1 \ 2 \ 4 \ 5 \ 9)$ is of type 2-2-1 and $(1 \ 2 \ 4 \ 5 \ 6)$ is of type 3-2. The *marked vertex* of a 2-2-1 simplex is the

vertex which lives alone in its column. For example 9 is the marked vertex in the example above. The color of a 2-2-1 simplex is the color of its marked vertex; thus (1 2 4 5 9) is red. The color of a 3-2 simplex is the color of the missing column; thus (1 2 4 5 6) is red. Obviously the type is invariant under the five actions defined above, while α and ε permute the colors which are β , γ and δ -invariant.

Let K' and K'' be the sub-complexes of Δ_8 respectively generated by the G -orbits of (1 2 4 5 9) and (1 2 4 5 6) and let K be the G -invariant complex $K' \cup K''$. Notice that K contains the 2-skeleton of Δ_8 and that it has 36 4-simplices. Figure 1-1 shows the G -orbit of (1 2 4 5 9), which consists of 27 4-simplices of type 2-2-1. The segments indicate that the joined simplices share a common hyperface. Note that the figure actually contains 30 simplices, but of course the opposite corners of the hexagon must be identified in order to produce a toroidal arrangement. Note also that α acts on the diagram as an anticlockwise rotation of $2\pi/3$ about the center of the figure and that β and γ are respectively the four step eastwards and the two step southeastwards

The Kühnel complex

Figure 1-1

Figure 1-2

translations on the torus. The intersection of K' with the star of K at vertex 9 is represented on the diagram by the central northwest-southeast strip of width three. Figure 1-2 shows the G -orbit of $(1\ 2\ 4\ 5\ 6)$, which consists of 9 4-simplices of type 3-2.

§2 shows that the topological realization $|K|$ of K is a PL-manifold and recalls two proofs of the fact that $|K| \cong \mathbf{CP}^2$. §3 studies the topology of $|K'|$ and $|K''|$, and §4 presents the first of our new demonstrations showing that $|K| \cong \mathbf{CP}^2$.

Note. The present definitions of β and γ have been interchanged compared with those in [K-B]. It is essentially due to the fact that, as we will see in §12, our γ (that is to say Kühnel and Banchoff's β) is somehow less important than our β , although α and γ generate G .

§2 The Grünbaum-Brückner link $\mathcal{M}$

In this section we show that $|K|$ is a topological and also a PL-manifold. We study the stars of vertices of K (cf. [G-S]). Since G is transitive on these vertices, all stars are combinatorially isomorphic. The link in K of vertex 9 (i.e. the boundary of the star at this vertex) is the complex generated by the twenty 3-simplices in Figure 2-1. (We remark that (b) and (c) are obtained by respectively applying γ and γ^2 to (a), while the two simplices in boxes (d^+) and (d^-) are γ -invariant.)

The union of the topological realizations of the six simplices displayed in each of the boxes (a), (b) and (c) is a 3-ball. Now $(a) \cup (b) \cup (c)$ and $(d^+) \cup (d^-)$ are two 3-balls glued along their common boundary, showing that the link of vertex

Figure 2-1

9 in K is a 3-sphere (and hence that $|K|$ is a topological manifold).

The complex displayed in Figure 2-1 (denoted $\mathcal{M}$ in [G-S]) is one of the four ways described by M. Brückner (1909) and B. Grünbaum (1967) to triangulate the 3-sphere S^3 with 8 vertices, 28 edges, 40 triangles and 20 tetrahedra.

Description of $\mathcal{M}$. In order to have a good grasp on $\mathcal{M}$ one can describe it as follows.

(1) Notice first that the union of the three simplices appearing on the right of boxes (a), (b) and (c) can be non-rectilinearly arranged into a *quilted annulus* as shown in Figure 2-2.

(2) On the axis of symmetry of Figure 2-2 let us place vertex 7 and vertex 8 respectively below and above the plane of the picture. The simplices appearing on the left of boxes (a), (b) and (c) are obtained by joining 7 to 8 and then (78) respectively to (56), (45) and (64).

The quilted annulus formed by three 3-simplices (1635), (2514) and (3426)

Figure 2-2

(3) Notice now that the collection of simplices respectively listed on lines (e^+) and (e^-) can be interpreted as representing the visual cone when one looks at the quilted annulus from 8 and 7, respectively. From 8 you see the top part of the annulus namely the six triangles:

(316), (615), (125), (524), (234), (436).

From 7 you see the bottom part of the annulus namely the six triangles:

$$(135), (536), (214), (415), (326), (624).$$

(4) The union of the 18 tetrahedra already constructed is a ball whose complement in S^3 can be decomposed into the two tetrahedra in (d^+) and (d^-) which share the face (123) (a triangle having the point at infinity as its center).

The combinatorial manifold K . There are several reasons why the topological manifold K is in fact PL. Firstly because S^3 has a unique PL-structure (see [Moi]); secondly because the way we described the triangulation is obviously C^1 in the sense of J.H.C. Whitehead (see [Wh 2]); and lastly because, by adding the three midpoints $\hat{16}$, $\hat{25}$ and $\hat{34}$ of respective edges (16) , (25) and (34) as new vertices, one easily gets a subdivision of $\mathcal{M}$ with eleven vertices and twenty nine 3-simplices, which is isomorphic to a linear subdivision of $\partial\Delta^4$. For instance, the ball considered above consisting of six non-rectilinear tetrahedra now decomposes into the following nine rectilinear simplices:

$$\begin{array}{lll} (1 \ 2 \ 8 \ \hat{25}) & (2 \ 4 \ 8 \ \hat{25}) & (4 \ 5 \ 8 \ \hat{25}) \\ (5 \ 1 \ 8 \ \hat{25}) & (1 \ 4 \ 5 \ \hat{25}) & (1 \ 2 \ 4 \ \hat{25}) \\ (1 \ 2 \ 4 \ 7) & (1 \ 4 \ 5 \ 7) & (4 \ 5 \ 7 \ 8) \end{array}$$

Claim 0. $|K| \cong \mathbf{CP}^2$.

Power hammer proof (mentioned by W. Kühnel). Since K contains the 2-skeleton of Δ_8 , $|K|$ is simply connected. Its Euler characteristic is 3. Hence $|K|$ has the homotopy type of $\mathbf{CP}^2$ (c.f. [E-K]). Using the Freedman classification of simply connected 4-manifolds (c.f. [S]), one sees that the combinatorial manifold K is homeomorphic to $\mathbf{CP}^2$. ■

Nevertheless in §3-§12 we intend to present three new proofs of this fact. Before doing this, we conclude the present section by giving the straightforward combinatorial argument following the idea of Kühnel and Banchoff mentioned in the introduction and as explained by A. Marin.

The straightforward combinatorial proof of claim 0 (A. Marin). Let Δ be any one of the 4-simplices of K . The subcomplex Σ generated by the vertices outside Δ is a 2-sphere. Let $r: K \rightarrow [0, 1]$ be the simplicial mapping which sends Δ to 1 and Σ to 0. Consider the decomposition:

$$|K| = r^{-1}([0, 1/2]) \bigcup_{r^{-1}(1/2)} r^{-1}([1/2, 1]) = T \bigcup_{\partial} B.$$

The second term B of this decomposition is a regular neighbourhood of Δ in the 4-manifold $|K|$ (cf. [Wh 1] or [R-S, chap 3]) and hence it is a 4-ball. The first term T is a regular neighbourhood of the sphere Σ in $|K|$. Therefore by claims 1 and 2 below, T is PL-isomorphic to a disc bundle over Σ . Since the boundary of this bundle is the 3-sphere ∂B , the Euler class of the bundle is ± 1 . With a suitable orientation, T is PL-isomorphic to a tube around a complex line in $\mathbf{CP}^2$, i.e. the complement of a 4 ball in $\mathbf{CP}^2$. By the Alexander trick there is essentially one way to glue ∂T to ∂B , so $|K| = T \bigcup_{\partial} B$ is PL-isomorphic to $\mathbf{CP}^2$.

Claim 1. The sphere Σ is locally flat in the manifold $|K|$.

Proof. Notice that Σ is combinatorially the boundary of a 3-simplex not contained in K . In order to prove Claim 1, it suffices to check that the boundary of each 2-simplex in Σ is unknotted in the link in K of the remaining vertex of Σ . Now it suffices to look at Figure 2-2 to observe that any closed curve formed by three edges is unknotted in M . More generally one can prove that all closed curves formed by three edges of any triangulation of S^3 with eight vertices are unknotted. ■

Claim 2. A regular neighbourhood of a locally flat 2-sphere S in a PL 4-manifold V is PL-isomorphic to a 2-disc bundle over S .

Proof. A regular neighbourhood in the manifold pair (V, S) (cf. [R-S, Chap. 4]) of a point interior to a triangle of S is an unknotted ball pair (B_1, D_1) . The closed disc $D_2 = S - \mathring{D}_1$ is locally flat in the manifold $V - \mathring{B}_1$. Hence a regular neighbourhood of this disc in $(V - \mathring{B}_1, D_2)$ is also an unknotted ball pair (B_2, D_2) . The pair $(B_1 \cup B_2, D_1 \cup D_2)$ is then a regular neighbourhood of S in V . It is the union of two unknotted ball pairs which are glued along regular neighbourhoods of the circles bounding the 2-balls in the 3-spheres bounding the 4-balls (therefore along unknotted solid tori), the pair $(B_1 \cup B_2, D_1 \cup D_2)$ is a disc bundle. ■

§3 A topological insight into $|K|$

This section introduces a piecewise linear mapping $|K| \rightarrow \Delta_2$ in order to give in §4 an intuitive approach to Claim 0 of §2 which is completely different to the attack adopted in [K-B].

As suggested by Figure 1-2 the complex K'' decomposes into three complexes called Ka , Kb and Kc :

$$Ka = \partial(456) * (789),$$

$$Kb = \partial(789) * (123),$$

$$Kc = \partial(123) * (456),$$

where ∂ denotes the boundary of a simplex and $*$ the join operator between complexes. Thus $|K''|$ appears as a union of three 4-balls touching each other

along a great circle of their boundaries; the boundary of each ball being thought of as the join of the two circles of contact which it contains. If we let 10, 11 and 12 denote the barycenters of the simplices (789), (123) and (456) then we can interpret Ka , Kb and Kc as cones with vertices 10, 11 and 12 and bases

$$\partial Ka = \partial(456) * \partial(789),$$

$$\partial Kb = \partial(789) * \partial(123),$$

$$\partial Kc = \partial(123) * \partial(456).$$

Let us denote by K_{10} , K_{11} and K_{12} the subdivisions of Ka , Kb and Kc so defined and by $\bar{K}$ the complex $K' \cup K_{10} \cup K_{11} \cup K_{12}$ which is a subdivision of K . Continuing the suggestion in §1, we like to think of 10, 11 and 12 as respectively belonging to the blue, the white and the red family. Accordingly Ka , Kb and Kc as well as their subdivisions K_{10} , K_{11} and K_{12} will be thought of as combinatorial decompositions of a blue, a white and a red 4-ball, respectively.

Define:

$$\alpha \quad : \quad 10 \mapsto 11, \quad 11 \mapsto 12, \quad 12 \mapsto 10$$

$$\beta, \gamma, \delta : \quad 10 \mapsto 10, \quad 11 \mapsto 11, \quad 12 \mapsto 12$$

$$\varepsilon \quad : \quad 10 \mapsto 10, \quad 11 \mapsto 12, \quad 12 \mapsto 11$$

so that now $\alpha, \beta, \gamma, \delta, \varepsilon$ act on $\bar{K}$ and hence so too does the group G .

In order to understand the topology of the entire space $|K|$, we construct a combinatorial map

$$p: \bar{K} \rightarrow \overline{\Delta_2},$$

where $\overline{\Delta_2}$ is the subdivision of the 2-simplex (abc) using the midpoints a' , b' and c' of the sides and the central triangle

$$\Delta'_2 = a'b'c'$$

as shown in the following figure.

$$\Delta'_2 \subset \overline{\Delta_2}$$

Keep in mind that the vertices a and a' , sides bc and $b'c'$ as well as the corner triangle $(ac'b')$ must be thought of as being blue. Accordingly vertices b and c , vertices b' and c' as well as their opposite sides and the corresponding corner triangles must be considered as being white and red, respectively. Check that the map between vertices shown in the following table does indeed extend to a G -invariant combinatorial map $p: \bar{K} \rightarrow \overline{\Delta_2}$, where G acts in an obvious manner on $\overline{\Delta_2}$.

$$\begin{array}{cccccc}
 & 1 & 4 & 7 & 10 & 11 & 12 \\
 & 2 & 5 & 8 & & & \\
 p: & 3 & 6 & 9 & & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & a' & b' & c' & a & b & c
 \end{array}$$

Then

$$p^{-1}(ac'b') = K_{10},$$

$$p^{-1}(ba'c') = K_{11},$$

$$p^{-1}(cb'a') = K_{12},$$

$$p^{-1}(\Delta'_2) = K'.$$

Let us think of Δ'_2 as an equilateral triangle in the euclidian plane $\mathbf{E}^2$ with its barycenter at the origin. Furthermore let H be the regular hexagon with vertices a_1, c_2, b_1, a_2, c_1 and b_2 where

$$a_2 = -a_1 = 3a'/4,$$

$$b_2 = -b_1 = 3b'/4,$$

$$c_2 = -c_1 = 3c'/4,$$

as shown in the following picture.

Let $\bar{H}$ be the torus obtained by identifying opposite sides of H .

It is the quotient of $\mathbf{E}^2$ by the lattice L generated by $a_1 + b_2$ and $b_1 + c_2$. Let π_f be the canonical projection of $\bar{H} = \mathbf{E}^2/L$ on $\mathbf{E}^2/\{(f\mathbf{R})^\perp + L\} \cong S^1$ where $f = a', b', c'$ and $\perp$ stands for the orthogonal complement. Let U be the topological space obtained by factoring $\Delta'_2 \times \bar{H}$ by the following equivalence relation:

$(d, e) \sim (d', e')$ whenever $(d', e') = (d, e)$ or $d = d' = f$ and $\pi_f(e) = \pi_f(e')$ where $f = a', b'$ or c' .

Claim. The geometric realization $|K'|$ is homeomorphic to U .

Proof. Let S be a 2-2-1 simplex in K and v its marked vertex (see §1). We will say S of class f if $p(v) = f$. The column decomposition of S shows the marked vertex v and two 1-simplices of type 2-0-0 called S_1 and S_2 . Now S can be thought of as the triple join $S_1 * S_2 * \{v\}$ and $p: S \rightarrow \Delta'_2$ as the canonical projection of the triple join onto its base.

Let us define the *standard realization* of a 2-2-1 simplex of class f to be the following simplex S_f (where $f = a', b', c'$) in $\Delta'_2 \times \mathbf{R}^2$, which canonically projects onto Δ'_2 using the first projection:

$$S_{a'} = ((b', b_1), (b', b_2)) * ((c', c_1), (c', c_2)) * \{(a', 0)\},$$

$$S_{b'} = ((c', c_1), (c', c_2)) * ((a', a_1), (a', a_2)) * \{(b', 0)\},$$

$$S_{c'} = ((a', a_1), (a', a_2)) * ((b', b_1), (b', b_2)) * \{(c', 0)\}.$$

Notice that all 2-1-1 hyperfaces of the three standard realizations are linearly isometric by type preserving isomorphisms.

For any simplex S of class f , there are four type-preserving identifications $S \rightarrow S_f$ which also preserve projections to $\overline{\Delta_2}$. They can be deduced from each other by a linear isometry of S_f .

For all $d \in \Delta'_2$, let d_a, d_b and d_c be the projections of d onto the sides of Δ'_2 . Let

$$P_{a'} = d d_b d_{a'} d_c,$$

$$P_{b'} = d d_c d_{b'} d_a,$$

$$P_{c'} = d d_a d_{c'} d_b$$

be the three parallelograms shown in the following figure:

Notice that P_f degenerates into a segment when d is on a side of Δ'_2 containing f , and into a point when $d = f$. Notice also that the inverse image of d in $S_f (\subset \{d\} \times \mathbf{R}^2)$ projects to the second factor $\mathbf{R}^2$ as the parallelogram obtained by translating P_f so that it becomes centered at the origin.

Now one can readily realize K' by identifying to S_f its simplices of class f (using any type-preserving identification) and by gluing along 2-1-1 hypersurfaces the copies of the S_f 's thus obtained by linear isometries (commuting with projections to $\overline{\Delta}_2$) according to the pattern shown in Figure 1-1.

Now for all $d \in \Delta'_2$, $p^{-1}(d)$ appears to be a patchwork made of the 27 parallelograms (some of which may be degenerate) corresponding to the intersections of $p^{-1}(d)$ with the 27 2-2-1 simplices of K . For instance the inverse image of the origin is the torus obtained by identifying the opposite sides of the hexagon shown in Figure 3-1 where the parallelograms are now rhombi. Notice that this hexagon is homothetic to H and can therefore be canonically identified with it. When d moves away from the center, the hexagonal pattern gets continuously distorted according to, for instance, the dispositions in Figure 3-2 and 3-3. But after the suitable identifications are done, the patchwork remains topologically a torus. The universal cover of the patchwork torus $p^{-1}(d)$ is obtained by letting the lattice

$$\mathbf{Z}(d_c - d_a) + \mathbf{Z}(d_b - d_a)$$

act on the union of the three parallelograms $P_{a'}$, $P_{b'}$ and $P_{c'}$. One can also use the triangular fundamental domain obtained by considering the union of the three half parallelograms cut along the long diagonals as shown below and tessellating under the action of the group generated by rotations of π centered at the midpoints of the sides of the triangle.

When d arrives to the interior of one of the sides of Δ'_2 18 out of the 27 parallelograms degenerate into segments but $p^{-1}(d)$ still remains a topological torus (Figure 3-2). When d finally reaches a vertex f then 9 of the parallelograms become points while the 18 others are segments. So the torus degenerates to a circle according to projection π_f (Figure 3-3). This completes the proof of the claim. ■

In Figure 3-1 each rhombus shows the marked vertex of the 2-2-1 simplex it represents. In order to list all the vertices of the represented simplex, consider its marked vertex as well as those of the four adjacent rhombi. For example, the union of the two halves of the hexagon marked with a cross is the intersection with the torus of the simplex (1 2 6 7 8). The picture also shows the blue white red coloring of each of the 2-2-1 represented simplexes using the Heraldic convention for colors, namely for blue, for white and for red.

Strasbourg 14-07-81

Figure 3-1

イエ

Figure 3-2

Figure 3-3

イエ

§4 The direct proof that $|K| \cong \mathbf{CP}^2$ [Jä]

In this section, we introduce a map $\pi: \mathbf{CP}^2 \rightarrow \mathcal{A}_2$ and show that the decomposition of $\mathbf{CP}^2$ induced by the inverse image under π of the subdivision $\overline{\mathcal{A}_2}$ is isomorphic to the decomposition

$$|K| = |K_a| \cup |K_b| \cup |K_c| \cup |K'|.$$

This shows that $|K| \cong \mathbf{CP}^2$.

Let us now think of $\mathcal{A}_2$ as being the standard 2-simplex in $\mathbf{R}^3$ with coordinates t_0, t_1, t_2 , so that the points previously introduced now become

$$\begin{aligned} a = e_0 &= (1, 0, 0), \quad b = e_1 = (0, 1, 0), \quad c = e_2 = (0, 0, 1), \\ a' &= (0, 1/2, 1/2), \quad b' = (1/2, 0, 1/2), \quad c' = (1/2, 1/2, 0). \end{aligned}$$

Let us identify the standard 5-sphere S^5 in $\mathbf{C}^3$ (with coordinates y_0, y_1, y_2) with the join $S^1 * S^1 * S^1$ which embeds into $\mathbf{C}^3$ as the set of points of the form

$$y = (\sqrt{t_0} \exp 2\pi i s_0 / t_0, \sqrt{t_1} \exp 2\pi i s_1 / t_1, \sqrt{t_2} \exp 2\pi i s_2 / t_2),$$

where $t = (t_0, t_1, t_2) \in \mathcal{A}_2$ and $s = (s_0, s_1, s_2) \in \mathbf{R}^3$. Notice that the formula makes sense even if some t_j 's equal zero. Now the canonical mapping $S^1 * S^1 * S^1 \rightarrow \mathcal{A}_2$ identifies with the map $S^5 \rightarrow \mathcal{A}_2$ given by

$$(y_0, y_1, y_2) \mapsto (|y_0|^2, |y_1|^2, |y_2|^2).$$

Furthermore, considered as action on $S^1 * S^1 * S^1$ the Hopf action of S^1 on S^5 becomes the diagonal action on the join defined by (θ, y) to

$$(\sqrt{t_0} \exp i(\theta + 2\pi s_0 / t_0), \sqrt{t_1} \exp i(\theta + 2\pi s_1 / t_1), \sqrt{t_2} \exp i(\theta + 2\pi s_2 / t_2)),$$

where $\theta \in \mathbf{R}$ and y is as above. The quotient $\mathbf{CP}^2$ of S^5 under the Hopf action now identifies with the quotient of $S^1 * S^1 * S^1$ under the diagonal action. Let $\pi: \mathbf{CP}^2 \rightarrow \mathcal{A}_2$ be the quotient under the diagonal action of the canonical mapping of the join so that we now have the commutative diagram:

$$\begin{array}{ccc} S^5 = S^1 * S^1 * S^1 & \longrightarrow & \mathbf{CP}^2 \\ & \searrow \quad \swarrow \pi & \\ & \mathcal{A}_2 & \end{array}$$

Notice that $S^1 * S^1 * S^1$ is the quotient of $\Delta_2 \times \mathbf{R}^3$ under the equivalence relation defined by $(t, s) \sim (t', s')$ if

$$t = t' \text{ and } s - s' \in L_t := t_0 \mathbf{Z} \times t_1 \mathbf{Z} \times t_2 \mathbf{Z},$$

where it is assumed that $0\mathbf{Z} = \mathbf{R}$. Therefore $\mathbf{CP}^2$ is seen to be the quotient of $\Delta_2 \times \mathbf{R}^3$ under the equivalence relation defined by $(t, s) \sim (t', s')$ if

$$t = t' \text{ and } s - s' \in L_t + D_t,$$

where D_t is the line in $\mathbf{R}^3$ generated by $t = (t_0, t_1, t_2) = \Delta_2$.

In order to study $\pi^{-1}(t)$ for $t \in \Delta_2$, it suffices to look at the image of L_t in the quotient of the euclidean space $\mathbf{R}^3$ by the greatest vector space V_t contained in the subgroup $L_t + D_t$. We note that

$$V_t = \begin{cases} D_t & t \in \text{the interior of } \Delta_2, \\ D_t + \mathbf{R}e_i & t \in \text{the interior of the side opposite to } e_i, \\ \mathbf{R}^3 & t \text{ is a vertex of } \Delta_2. \end{cases}$$

For each t in the interior of Δ_2 , the Dirichlet domain of the image of L_t is given by the following figure, where α_0, α_1 and α_3 are the images of $(t_0, 0, 0)$, $(0, t_1, 0)$, $(0, 0, t_2)$. α_0, α_1 and α_2 generate the image lattice, one has $\alpha_0 + \alpha_1$

$+\alpha_2 = 0$ and all angles $\alpha_j \alpha_k (j \neq k)$ are in the interval $] \pi/2, \pi[$. When t approaches some interior point of the side opposite to the vertex e_i , the angle $\alpha_j \alpha_k (i \neq j \neq k \neq i)$ tends to π and α_i tends to 0. In the limit, α_j and α_k are on the same line but the dimension of V_t has increased and one has to project everything orthogonally onto the line $\alpha_j \alpha_k$. Now $\mathbf{R}^3/V_t$ is isomorphic to $\mathbf{R}$ in which the image of L_t has to be a discrete additive subgroup tending to $\mathbf{R}$ itself when t remains on the side and tends to a vertex. Notice that when t is at the center of A_2 , the Dirichlet fundamental domain of $\pi^{-1}(t)$ is a regular hexagon, that L_t is the cubical lattice and that D_t is the diagonal of $\mathbf{R}^3$.

The previous remarks can be summarized as follows: $\pi^{-1}(t)$ is a torus which degenerates into a circle when t tends to a side and to a point when t tends to a vertex. There are three ways (analogous to those studied in §3) in which the torus can degenerate to the circle, according to which side t approaches. It therefore should now be obvious that

$$\pi^{-1}((ab'c') \cup (ca'b') \cup (bc'a'))$$

is the union of three smooth 4-balls touching each other in $\mathbf{CP}^2$ along linked circles as explained in §3 while $\pi^{-1}(a'b'c')$ is seen to be homeomorphic to U (c.f. Claim in §3). By the Alexander trick, $\mathbf{CP}^2$ is therefore homeomorphic to $|K|$.

In order to make this clearer, we specify an explicit triangulation of $\mathbf{CP}^2$, isomorphic to $\bar{K}$. The idea stems from a suggestion by A. Marin simplified and modified in order to resemble the construction in §3, which in some sense it dualises.

The explicit triangulation of $\mathbf{CP}^2$ by $\bar{K}$. Let

$$\begin{array}{lll} v_1 = [0, 1, \omega] & v_4 = [\omega, 0, 1] & v_7 = [1, \omega, 0] \\ v_2 = [0, 1, \omega^2] & v_5 = [\omega^2, 0, 1] & v_8 = [1, \omega^2, 0] \\ v_3 = [0, 1, 1] & v_6 = [1, 0, 1] & v_9 = [1, 1, 0] \\ v_{10} = [1, 0, 0] & v_{11} = [0, 1, 0] & v_{12} = [0, 0, 1] \end{array}$$

be the images in $\mathbf{CP}^2$ of the vertices $1, \dots, 12$ in $\bar{K}$, where $\omega = \exp 2\pi i/3$. Let $\tilde{\chi}: \Delta_2 \times Q \rightarrow \mathbf{CP}^2$ be defined by setting

$$\tilde{\chi}(t, \psi) = [\sqrt{t_0} e^{2\pi i \psi_0}, \sqrt{t_1} e^{2\pi i \psi_1}, \sqrt{t_2} e^{2\pi i \psi_2}],$$

where $t = (t_0, t_1, t_2) \in \Delta_2$, $\psi = (\psi_0, \psi_1, \psi_2) \in Q$ and

$$Q = \{(\psi_0, \psi_1, \psi_2) \in \mathbf{R}^3 \mid \psi_0 + \psi_1 + \psi_2 = 0\}.$$

The map $\tilde{\chi}$ is onto and factors through $\Delta_2 \times Q/L_0$ where L_0 is the orthogonal projection of $\mathbf{Z}^3$ to Q . Let us denote by χ the quotient map

$$\Delta_2 \times Q/L_0 \rightarrow \mathbf{CP}^2.$$

Restricted to $\Delta_2 \times Q/L_0$, χ is a homeomorphism onto the complement of the three complex lines

$$y_0 = 0, y_1 = 0, y_2 = 0,$$

where $[y_0, y_1, y_2]$ denotes the system of homogeneous coordinates in $\mathbf{CP}^2$.

In order to describe the triangulation of $\mathbf{CP}^2$ in terms of (t, ψ) -coordinates, we need to compute the projections Pr_2 onto the second factor of

$$\chi^{-1}(v_i) \subset \Delta_2 \times Q/L_0 \quad (i = 1, \dots, 9).$$

These are circles on the torus Q/L_0 . Let us now think of the hexagon H as being the Dirichlet domain of Q at the origin under the action of L_0 .

Using the translations e_0, e_1 and e_2 , one can color all the meridians and small diagonals in such a way as to obtain the following diagrams.

The following figure represents the circles $Pr_2\chi^{-1}(v_i)$ ($i = 1, \dots, 9$) viewed in H .

This picture illustrates the fact that these 9 circles decompose Q/L_0 into 18 triangles. Let us denote by $\Delta(ijk)$ the triangle with sides labelled i, j and k ($1 \leq i, j, k \leq 9$), and assume that vertices opposite to sides have been given

the order ijk . Let $C: \Delta_2 \rightarrow \Delta_2$ be the Cremona transformation of the standard 2-simplex in $\mathbf{R}^3$ defined by

$$C(t_0, t_1, t_2) = (t_1 t_2, t_2 t_0, t_0 t_1) / (t_1 t_2 + t_2 t_0 + t_0 t_1).$$

This is a one-to-one involution on Δ_2 , which shrinks each side to the opposite vertex and blows up each vertex to the opposite side. An ordering of the vertices of two triangles T_1 and T_2 defines, in an obvious and unique manner, reciprocal Cremona transformations $T_1 \rightarrow T_2$ and $T_2 \rightarrow T_1$. Let C_{692} be the Cremona transformation

$$(a' b' c') \rightarrow \Delta(692)$$

and define the 17 other C_{ikj} 's by composing C_{692} with rotations of angle π centered at the middle of the sides with $\Delta(692)$, thus defining

$$C_{673}, C_{934}, C_{247}$$

and so on using the newly introduced sides. For all $t \in (a'b'c')$ consider the decomposition of Q/L_0 into 27 parallelograms obtained by joining each $C_{ijk}(t) = p_{ijk}$ to the vertices of $\Delta(ijk)$ and then erasing the sides of $\Delta(ijk)$.

As in §3 we thus obtain a decomposition of the torus depending on t but this time the shape of the torus is fixed, while the angles of the parallelograms vary. Notice that some of the parallelograms degenerate when the p_{ijk} 's are on the boundaries of $\Delta(ijk)$'s, namely when t is on the boundary of $(a'b'c')$. Note also that when t is at a vertex, the p_{ijk} 's live on the corresponding sides with no definite location but remember that when such is the case, it turns out that χ precisely quotients Q/L_0 along circles parallel to these sides. From this fact one immediately sees that the previous construction defines an explicit homeomorphism from $|K'|$ (realized as in §3) to $\pi^{-1}(a'b'c')$ sending the parallelograms of §3 to the present one. In order to define the homeomorphism over all of $|\bar{K}|$, observe that each of the three 3-spheres bounding $\pi^{-1}(a'b'c')$ is decomposed into nine 3-simplices; extend radially the three above decompositions of $\mathbf{CP}^2$ using v_{10}, v_{11} and v_{12} as origins. This operation is completely well defined since for instance $\pi^{-1}(ab'c')$ is the standard unit 4-ball in the 2-dimensional complex plane $\mathbf{CP}^2 - \pi^{-1}(bc)$ with coordinates $(z_1/z_0, z_1/z_0)$.

No doubt the severe reader will decide that the triangulation presented in this section is self-sufficient and that the construction of §3 is needless. This is indeed true, but we like §3 since we obtained it long ago, while §4 became clear to us very recently. On the other hand as we pointed out in the introduction, it is useful to show many approaches to the same phenomenon in order to get a good intuitive grasp of it. That is why we hoped most readers would also enjoy reading §3.

Chapter II The Kühnel complex viewed as an orbifold

§5 The G -invariant torus T in K

The existence of the combinatorial torus $T \subset K$ presented in this section is the basic fact that enables one to connect Kühnel's triangulation with crystallography.

Figure 5-1 represents the union T of the G -orbits of (269) (positive orbit) and (169) (negative orbit) where opposite sides should be identified and then erased.

Figure 5-1

The topological space $|T|$ is homeomorphic to a torus. Considered as a subtorus of $\mathbf{CP}^2$, it is the set of $\chi(t, C_{ijk}(t))$ of §4 for all possible ijk 's as t varies in $(a'b'c')$. The complex T contains all those simplices of type 1-1-1 whose link is a triangle.

Notice that the links of the remaining simplices of type 1-1-1 (all belonging to the G -orbit of (147)) are hexagons, and that those simplices are in one-to-one

correspondence with the hexagons in Figure 1-1. Notice also that the intersection of T with the link $\mathcal{M}$ of vertex 9 is the combinatorial trefoil knot:

$$14 \cup 43 \cup 35 \cup 52 \cup 26 \cup 61$$

shown in Figure 2-2. Therefore T has 9 combinatorial cusps.

In §9, it will be shown that T is the combinatorial model for the embedding in $\mathbf{CP}^2$ of a complex curve of genus 1 with 9 cusps (the dual of a non-singular cubic). In §8 Remark 2, we will show the existence of a 6 fold cover of $|K|$ branched over T , which is isomorphic to the 4-torus, and on which K induces an *improper triangulation*: This means a decomposition into simplices which is not a triangulation because each simplex is not uniquely determined by its vertices, such as the case of the decomposition of the circle S^1 into two intervals.

Suitably identified with $\mathbf{C}^2$ the universal cover of this improperly triangulated 4-torus will appear to be the rectilinear triangulation $\tilde{K}$ of $\mathbf{C}^2$ defined in §8, which is invariant under the action of the complex crystallographic group Γ'' defined in §7. In $\tilde{K}$ the inverse image of T will turn out to be the union A of a family of complex lines each of which maps onto $|T|$ as its universal cover (see §8 Remark 1). Links of all simplices of type 1-1-1 in $\tilde{K}$ will be hexagons. Associated with a crystallographic subgroup Γ of Γ'' , there exists a map $Y: \mathbf{C}^2 \rightarrow \mathbf{CP}^2$ branched along A , which will produce a new explicit homeomorphism between $|K| = |\tilde{K}/\Gamma|$ and $\mathbf{CP}^2$ (cf. §9). In order to visualize the triangulation $\tilde{K}$, we construct in §6 combinatorial 6 fold cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$ branched along the trefoil knot $T \cap \mathcal{M}$. The complex $\tilde{\mathcal{M}}$ will then be realized as the boundary of a 4 dimensional polytope $\mathcal{N}$ in $\mathbf{C}^2$ namely as the link of the origin in the triangulation $\tilde{K}$. Since Γ'' is transitive on vertices of $\tilde{K}$, all links of vertices in $\tilde{K}$ are isomorphic to $\tilde{\mathcal{M}}$. The realization $\partial\mathcal{N}$ is not *strict* in the sense that the star $\mathcal{N}$ (at the origin in $\tilde{K}$) is not convex and some of its faces are unions of more than one 3-simplex of $\tilde{\mathcal{M}}$ (i.e. it has inward and also flat dihedral angles).

The detailed study of the convex hull of $\tilde{\mathcal{M}}$ (i.e. of $\mathcal{N}$) is the bulk of the proof of the Claim 1 of §8.

Observe that it would not be difficult to deform the polytope $\mathcal{N}$ so that its boundary becomes a strict realization of $\tilde{\mathcal{M}}$, in other words, so that it becomes a convex polytope whose hypersurfaces are 3-simplices decomposing its boundary accordingly to the pattern $\tilde{\mathcal{M}}$. Recall that contrary to $\tilde{\mathcal{M}}$, the Grünbaum-Brückner triangulation $\mathcal{M}$ is the only triangulation of S^3 with 8 vertices which refuses to be strictly realized in $\mathbf{R}^4$ (in the previous sense), essentially because it contains the combinatorial trefoil knot $T \cap \mathcal{M}$, which is the union of 6 edges. Indeed we have

Proposition. *In the boundary of a 4 dimensional polytope, any PL closed curve with not more than 6 edges either is unknotted or is a trefoil knot with 6 edges contained in one hyperface of the polytope.*

Proof. See Exercise 9 at the end of the following appendix.

Appendix Strangeness of Brückner's triangulation: The trefoil knot pleads guilty, by Alexis Marin

In this appendix we shall first construct a 4-dimensional convex polytope B whose boundary has a linear subdivision which is simplicially isomorphic to the Brückner triangulation $\mathcal{M}$ of the 3-sphere: One facet (i.e. one hyperface) B_+ of B containing ten of the twenty tetrahedra of $\mathcal{M}$. Then we shall prove that the Brückner triangulation is not isomorphic to the boundary of a convex polytope all of whose facets are tetrahedra. This non-polytopality of $\mathcal{M}$ is a consequence of the last proposition of §5. It can also be established more simply by observing that since in $\mathcal{M}$ the trefoil knot meets a tetrahedron τ of the triangulation along two opposite edges, the complex $\mathcal{M} - \tau$ is not combinatorially isomorphic to a linear subdivision of τ (see Exercise 2 below). Indeed if $\mathcal{M}$ were polytopal, this would be the case as one can see by using a Schlegel projection (see A §2 below for a definition).

A §1 A rectilinear construction of Brückner's triangulation of S^3

Let us look at the drawing of $\mathcal{M}$ in Figure 2-2, turn it inside out like a glove and deform the two curved triangles 123 and 546 so that they become true concentric homothetic equilateral triangles in the same plane (say, in the horizontal plane). The points 7 and 8 are still on the vertical axis of the figure

but now 7 is up and 8 is down. Figure 5-3 shows this modified presentation of Bruckner's triangulation.

Bruckner's triangulation seen from the other side

Figure 5-3

In this modified presentation, the “central core” is decomposed into the two tetrahedra (1237) and (1238), while the exterior region splits into the three infinite tetrahedra (4578), (5678) and (6478). In order to avoid intersections, the six edges into which the right handed trefoil knot 1-4-3-5-2-6 decomposes must be curved (14 above the horizontal plane, 25 below, and so on), but all the other edges of $\mathcal{M}$ can be represented by segments in $\mathbf{R}^3$ in such a way that everything is invariant by the dihedral group D_6 of order six whose non trivial elements are the rotations of order two around the three altitudes of the triangle say, (1 2 3) and the rotations of order three around the vertical axis.

At the cost of loosing the invariance under the rotations of order two all edges can be made rectilinear by lifting the big triangle (654) and slightly turning it counterclockwise so that the projections on the horizontal plane of the quadrilateral 2 1 5 4, 3 2 4 6 and 1 3 6 5, which originally were isosceles

trapezia, remain convex (see Figure 5-4). Choose the vertex 7 on the vertical axis anywhere above the center of the big triangle (654). The convex hull B_+ of the seven points $1, \dots, 7$ is triangulated by rectilinear images of ten of the twenty tetrahedra of $\mathcal{M}$: namely the three tetrahedra of the quilted annulus, the six tetrahedra of the visual cone from 7 on the top part of the quilted annulus and the tetrahedron (1237) as displayed in Braille notations in the following table (Compare with Figure 2-1):

	(a)	(b)	(c)
(e ⁻)			
(d ⁺)			

The polyhedron B_+ has ten triangular faces and its boundary is combinatorially isomorphic to the link in $\mathcal{M}$ of the vertex 8.

Figure 5-4 presents a version of B_+ which looks like a three horned hat.

Figure 5-4

Now think of the space $\mathbf{R}^3$ which contains the non-flattened version of B_+ as being the horizontal space $x_0 = 0$ in $\mathbf{R}^4$ and put the point 8 anywhere below $\mathbf{R}^3$ in the fourth direction; let then B be the cone with base B_+ and apex 8. The boundary of the convex polytope B has a linear subdivision isomorphic to $\mathcal{M}$ (ten tetrahedra where already described in the facet B_+ , the ten remaining are the cones with apex 8 and bases the faces of B_+).

Exercise 1. a) Check that the only edges of $\mathcal{M}$ of order 3 (ie. in the boundary of three triangles of $\mathcal{M}$) are the central edge 78 and the edges of the trefoil knot (the orbit under D_6 of 14).

b) Use a) to show that D_6 is the full isomorphism group of $\mathcal{M}$.

c) Define the *symmetric Brückner triangulation* $\mathcal{M}_s$ by replacing in $\mathcal{M}$ the core $(4\ 5\ 7\ 8) \cup (5\ 6\ 7\ 8) \cup (6\ 4\ 7\ 8)$ by $(4\ 5\ 6\ 7) \cup (4\ 5\ 6\ 8)$ and prove that the symmetry group of $\mathcal{M}_s$ is the group of order twelve generated by D_6 and the glove process $(1, 5)(2, 4)(3, 6)(7, 8)$.

Exercise 2. The fake Brückner triangulation $\mathcal{M}$.

Consider the erroneous copy of Figure 2-2 shown in figure 5-5 where one of the three crossings of the trefoil knot has been altered so that the edges 16 and 35 are now respectively below and up and that an unknotted trefoil knot is now displayed.

Figure 5-5

Following §2 construct then a triangulation of S^3 by replacing the lines (e^+) and (e^-) in box (a) of Figure 2-1 by those of Figure 5-6. This triangulation is the fake Brückner triangulation $\mathcal{M}$.

Figure 5-6

a) Prove that $\mathcal{M}$ is a linear subdivision of the boundary of the 4-simplex $(1\ 2\ 3\ 7\ 8)$.

b) Moving out a little the vertices 4, 5 and 6, build a convex polytope in $\mathbf{R}^4$ all of whose facets are tetrahedra and whose boundary is isomorphic to $\mathcal{M}$; The fake Brückner triangulation is *polytopal*.

c) Check that $\mathcal{M}$ is isomorphic to the polytope P_{37}^8 of Grünbaum-Streedharan's list.

A §2 The trefoil knot prevents the existence of some Schlegel diagrams for $\mathcal{M}$.

Let $\mathcal{P}$ be a $(d + 1)$ -dimensional convex polytope in $\mathbf{R}^{d+1}$; consider a point f_0 interior to a facet F of $\mathcal{P}$. If a point f outside $\mathcal{P}$ is close enough to f_0 the radial projection from f gives rise to a homeomorphism from $\overline{\partial\mathcal{P} - F}$ to F called a *Schlegel projection of $\mathcal{P}$ with base F* .

Let $\mathcal{S}$ be a triangulation of the sphere S^d and Δ be one of the d -simplex of $\mathcal{S}$. A *Schlegel diagram for $\mathcal{S}$ with base Δ* is a linear subdivision of a d -simplex Δ' which is combinatorially equivalent to $\mathcal{S} - \Delta$. If $\mathcal{S}$ is a polytopal triangulation of S^d the Schlegel projections give rise to Schlegel diagrams for $\mathcal{S}$ so that any d -simplex of $\mathcal{S}$ is the base of one of these.

Exercise 3. Using the polytope B prove that Brückner's triangulation $\mathcal{M}$ has a Schlegel diagram with base any tetrahedron of $\mathcal{M}$ except those in the quilted annulus.

Exercise 4. a) Prove that $\mathcal{M}$ has no Schlegel diagram with base, say, $(1\ 2\ 4\ 5)$. [If the simplex $\Delta' = (1'\ 2'\ 4'\ 5')$ would admit a linear subdivision isomorphic to $\mathcal{M} - (1\ 2\ 4\ 5)$ then a pillow projection of Δ' to the square $1'\ 4'\ 2'\ 5'$ (see Figure 5-7) would lead to an impossible projection of the trefoil knot $1'-4'-3'-5'-2'-6'$, since it should have only one crossing (see Figure 5-7).] Compare with [G-S].

b) Prove that the symmetric Brückner triangulation $\mathcal{M}_s$ is not polytopal, so that it is not in the Grünbaum-Streedharan list. Seek for it in the literature. [It is sometimes called the Barnette triangulation.]

Exercise 5. Let $\mathcal{M}$ and $\mathcal{M}'$ be two copies of Brückner's triangulation and let $\mathcal{S}$ be the triangulation of S^3 obtained by gluing $\mathcal{M} - (1\ 2\ 4\ 5)$ and $\mathcal{M}' - (1'\ 2'\ 4'\ 5')$ along their boundaries $\partial(1\ 2\ 4\ 5)$ and $\partial(1'\ 2'\ 4'\ 5')$ identified in the obvious way. Prove that $\mathcal{S}$ has no Schlegel diagram.

Figure 5-7

For the sake of completeness let us now prove the rather well established following fact.

Claim. The trefoil knot is knotted.

Exercise 6. First proof. Recall that the exterior of the unknot is $S^1 \times \mathbf{R}^2$ so that all its connected coverings are cyclic. In §6 Claim 1 and 2, a non cyclic connected covering of the exterior of the trefoil knot will be constructed.

Exercise 7. Second proof (actually it is the same proof). a) Check that the upstrands of a projection of the trefoil knot can be colored with three colors 0, 1, 2 so that:

- (i) More than one color is used.
- (ii) At each crossing, twice the color of the upstrand is congruent modulo 3 to the sum of the colors of the two other strands.

b) Let $n (\geq 3)$ be an integer and k a knot presented in such a way that it admits a coloring using colors $\{0, \dots, n-1\}$ satisfying conditions (i) and (ii) in $\mathbf{Z}/n\mathbf{Z}$. Check then that Reidemeister's moves $\Omega.1$, $\Omega.2$ and $\Omega.3$ depicted in Figure 5-8 do not destroy the possibility of such a coloring.

c) If the trefoil knot were trivial, using Reidemeister's moves one should be able to pass from it to a nice round unknot which can obviously not fulfill condition (i).

Coloring the trefoil

Invariance of coloring by Reidemeister's move

Figure 5-8

A §3 Knots with few segments

Claim. In $\mathbf{R}^3$ a knot k which is the union of at most six segments is either the unknot or the trefoil knot in which case it is made of six segments.

Exercise 8. Proof of the claim. Let $k = [a_1, a_2] \cup [a_2, a_3] \cup \dots \cup [a_n, a_1]$.

(a) If $n \leq 4$ the knot k is contained in the 1-skeleton of a possibly degenerated rectilinear tetrahedron and hence trivial.

b) The case $n = 5$ reduces to the case $n = 4$ since at least one of the two triangles (a_1, a_2, a_3) and (a_3, a_4, a_5) can be used to isotope two adjacent segments of k onto the third side of the triangle. In the case of the picture below only the triangle (a_3, a_4, a_5) can be used.

Figure 5-9

c) In the case $n = 6$ when the reduction to the case $n = 5$ is impossible, study the intersection of the two remaining segments with the two triangles in order to show that the knot is a trefoil.

Exercise 9. Proof of the last proposition of §5.

a) When the convex hull of the knot k is not 4-dimensional, show that either k is drawn on a 2-dimensional PL -sphere and hence trivial or that it is contained in a facet of the polyhedron P . In the last case apply Exercise 8.

b) When the convex hull is 4-dimensional show that the situation reduces to the case where the vertices of k are vertices of P .

c) Let v be a vertex of k and therefore of P , let $St(v)$ be the star of v in ∂P , let H_0 be a hyperplane such that $H_0 \cap P = \{v\}$ and let $r: \partial P - St(v) \rightarrow H$ be the radial projection from v to a hyperplane H parallel to H_0 . Using the triangle trick of Exercise 8 show that the ball pair $(r(\overline{\partial P - St(v)}), r(\overline{k - k \cap St(v)}))$ (a convex 3-dimensional compact polyhedron containing an arc made of four or less segments) is unknotted.

d) Conclude that k itself is unknotted in ∂P .

§6 The six fold cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$

A 6 fold cover $\mathcal{M}$ (with 120 3-simplices and 30 vertices) of $\mathcal{M}$ branched

along the trefoil knot $\mathcal{M} \cap T$ can be described as the union of 6 copies of a triangulation (with 20 simplices and 11 vertices) of the prism $\Delta_2 \times \Delta_1$. Three of these copies are glued to each other along their triangular faces in order to produce a solid torus with 9 square faces. The complement in S^3 of such a solid torus is again a solid torus with 9 square faces. It is in turn obtained by gluing similarly the 3 remaining copies of the prism. Figure 6-1 shows a prism of one of the tori surrounded by the 3 prisms of the other torus.

Figure 6-1

The inverse image of $\mathcal{M} \cap T$ in $\tilde{\mathcal{M}}$ is the union of three linked circles contained in the common boundary of the two solid tori. Each of these circles is the union of diagonals of three of the nine square faces and maps bijectively onto $\mathcal{M} \cap T$. We first construct the triangulation $\tilde{\mathcal{M}}$, that we realize in $\mathbb{C}^2$, and next show that the obvious projection $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ is indeed the required branched covering.

Construction of $\tilde{\mathcal{M}}$. We now proceed to describe the triangulation of the first prism $\mathcal{M}' \cong \Delta_2 \times [0, 1]$ where we denote by m_1, m_2 and m_3 the vertices of

the triangle $\Delta_2 \times \{0\}$; l_2, l_3 and l_1 the vertices of the triangle $\Delta_2 \times \{1\}$; where k_6, k_5 and k_4 are the centers of the rectangular faces

$$(m_2, m_3, l_1, l_3), (m_3, m_1, l_2, l_1), (m_1, m_2, l_3, l_2)$$

and where

$$k_7 = \left(b, \frac{1}{4}\right), \quad k_8 = \left(b, \frac{3}{4}\right)$$

(in which b stands for the barycenter of Δ_2) so that k_7 and k_8 lie on the axis of symmetry of the prism. Now $\mathcal{M}'$ is the complex with vertices

$$\begin{array}{ccccc} & l_2 & & l_3 & & l_1 \\ k_8 & & & & & \\ & & k_4 & & k_6 & & k_5 \\ k_7 & & & & & \\ & m_1 & & m_2 & & m_3 \end{array}$$

generated by the following twenty 3-simplices.

Figure 6-2

In this figure we introduce for the first time the *colored Braille notation* which should be understood as follows; every letter appearing in a given Braille cell is assumed to carry the color of the place it occupies in the cell. For instance,

$$\begin{array}{|c|c|} \hline l & \\ \hline m & k \\ \hline \end{array}$$

means (l_1, m_3, k_5, k_6) .

The three simplices appearing on the right of boxes (a), (b) and (c) are called *notches*, while the union of the three simplices appearing on the left of the same boxes is called the *central core* of $\mathcal{M}'$. This central core can also be triangulated as the union of two 3-simplices

$$\begin{array}{|c|c|} \hline k & k \\ \hline k & \\ \hline k & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline k & \\ \hline k & k \\ \hline k & \\ \hline \end{array}$$

Now $\mathcal{M}'$ minus the union of its three notches, with the central core retriangulated as explained, can be considered as the union of the two octahedra

$$\begin{array}{|c|c|} \hline m & k \\ \hline m & k \\ \hline m & k \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline l & k \\ \hline l & k \\ \hline l & k \\ \hline \end{array}$$

triangulated as cones on their boundaries with barycenters k_7 and k_8 , respectively. Thus if one now restores the original triangulation of the central core one gets a very good understanding of the way $\mathcal{M}' - \{\text{notches}\}$ is constructed. See Figure 6-3.

In order to build the 5 other prisms, it suffices to list their vertices according to a chosen order for those of $\mathcal{M}'$. Figure 6-4 does this for you.

G. Francis

Two cohedral (irregular)
Octahedra are formed
by removing 3 notches
from a triangular prism.
[In the model, windows reveal
an illuminated interior with
a ball vertex at the center of each
octahedron.]

Figure 6-3

$m_1, m_2, m_3; k_7; k_6, k_5, k_4; k_8; l_2, l_3, l_1$
 $l_1, l_2, l_3; m_7; m_6, m_5, m_4; m_8; k_2, k_3, k_1$
 $k_1, k_2, k_3; l_7; l_6, l_5, l_4; l_8; m_2, m_3, m_1$
 $k_1, m_2, l_3; k_{-7}; k_6, m_5, l_4; k_{-8}; k_2, m_3, l_1$
 $l_1, k_2, m_3; m_{-7}; l_6, k_5, m_4; m_{-8}; l_2, k_3, m_1$
 $m_1, l_2, k_3; l_{-7}; m_6, l_5, k_4; l_{-8}; m_2, l_3, k_1$

Figure 6-4

The logic of the table (Figure 6-4) is represented by the following pattern showing the vertices on the common boundary of the two solid tori as well as the vertices of the core of each torus, and thus displaying all vertices of $\tilde{\mathcal{M}}$. Please identify opposite sides of the square to obtain a torus.

Figure 6-5

Claim 1. The natural projection $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ is a 6 fold cover of $\mathcal{M}$ branched along $T \cap \mathcal{M}$, in such a way that the inverse image of each simplex of $\mathcal{M}$ is the union of 6 simplices of the same dimension except for the edges and vertices of $T \cap \mathcal{M}$ which are covered only thrice.

Proof. By assigning to each of the 30 vertices of $\tilde{\mathcal{M}}$ its color (namely the index attached to it), one readily sees that each of the 120 3-simplices of $\tilde{\mathcal{M}}$ is mapped to a 3-simplex of $\mathcal{M}$. This defines a natural projection of $\tilde{\mathcal{M}}$ to $\mathcal{M}$. Notice that the restriction of this projection to each of the six prisms, which $\tilde{\mathcal{M}}$ has been decomposed into, is a surjective map from the chosen prism to $\mathcal{M}$. For instance observe that $\mathcal{M}$ is combinatorially equivalent to the quotient of $\mathcal{M}'$ obtained by identifying together the k_i 's, l_i 's and m_i 's ($i = 1, 2, 3$) of same index. These identifications force each lateral face to be folded along

one of its diagonals and in addition the following pairs of triangles are glued together

$$l_3 k_6 l_1 \longleftrightarrow m_3 k_6 l_1,$$

$$l_3 m_2 k_6 \longleftrightarrow m_3 m_2 k_6.$$

Similarly for the two other faces. This operation induces an anti-clockwise twist of the top and the bottom triangular faces with respect to each other. In the quotient, the image of $\partial \Delta_2 \times \Delta_1$ in $\mathcal{M}$ is the Möbius strip S formed by the union of the six triangles:

$$(124), (125), (236), (234), (315), (316)$$

bounded by $\mathcal{M} \cap T$. In order to complete the quotienting procedure do not forget to identify together the simplices $l_1 l_2 l_3$ and $m_1 m_2 m_3$ (otherwise the object would not be a simplicial complex i.e. it would not be properly triangulated in the sense of § 5). The last identification gives you the Grünbaum-Brückner triangle (1 2 3) centered at infinity. The way the mapping $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ has been described makes Claim 1 obvious.

Claim 2. The mapping $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ is a normal covering whose Galois group is the symmetric group $\mathfrak{S}_3$. Moreover the alternating group A_3 acts freely on $\tilde{\mathcal{M}} \cong S^3$ so that $\tilde{\mathcal{M}}/A_3$ is the lens space $L(3, 1)$ and that the projection $\tilde{\mathcal{M}}/A_3 \rightarrow \tilde{\mathcal{M}}/\mathfrak{S}_3 \cong \mathcal{M} \cong S^3$ is the 2 fold cover of $\mathcal{M}$ branched along the right hand trefoil knot $\mathcal{M} \cap T$.

Proof. Notice that A_3 acts on each of the two solid tori, into which $\tilde{\mathcal{M}}$ has split (see Figure 6-1). The quotient of the first torus under this action can be described as being the prism $\mathcal{M}'$ with the two triangular faces (m_1, m_2, m_3) and (l_1, l_2, l_3) identified together. One is now left with an improperly triangulated solid torus $\bar{\mathcal{M}}_1$ whose boundary $\partial\bar{\mathcal{M}}_1 = \tilde{S}$ is obtained by doubling the Möbius strip S . The Deck involution $\iota: \tilde{S} \rightarrow \tilde{S}$, with respect to the projection $\tilde{S} \rightarrow S$ fixes the curve $\iota^{-1}(T \cap \mathcal{M}) \sim 2\mu + 3\lambda$:

$$1 \text{ --- } 4 \text{ --- } 3 \text{ --- } 5 \text{ --- } 2 \text{ --- } 6 \text{ --- } 1$$

and sends the meridian μ :

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 1$$

to the curve $\mu' \sim \mu + 3\lambda$:

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 1$$

where λ is the chosen longitude indicated in the picture below.

Next consider a second copy $\bar{\mathcal{M}}_2$ of $\bar{\mathcal{M}}_1$ representing the quotient under the A_3 -action of the second torus of Figure 6-1. Now $\tilde{\mathcal{M}}/A_3$ splits into $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_2$ glued along their common boundary using the involution ι . Observe that the meridian of $\bar{\mathcal{M}}_2$ is mapped to the curve μ' on $\bar{\mathcal{M}}_1$. According to [Sie],

$$\bar{\mathcal{M}}_1 \cup_{\iota} \bar{\mathcal{M}}_2$$

is the improperly triangulated lens space $L(3, 1)$ whose universal cover has three sheets and identifies with S^3 , properly triangulated by $\tilde{\mathcal{M}}$. This ends the proof of Claim 2.

The realization of $\tilde{\mathcal{M}}$ in $\mathbb{C}^2$.

The promised representation of $\tilde{\mathcal{M}}$ as the boundary of a star-polytope $\mathcal{N}$ will be completely determined by giving coordinates for its 30 vertices in $\mathbb{C}^2$. In order to obtain these coordinates we first represent the torus of all vectors $(z_1, z_2) \in \mathbb{C}^2$ satisfying $z_1 \bar{z}_1 = z_2 \bar{z}_2 = 1$ as a square horizontally parameterized by the argument of z_1 and vertically by the argument of z_2 as indicated:

Place the 18 vertices of Figure 6-5 of colors 1, ..., 6 in this square as they appear in the previous diagram. Now the cores of the tori should be treated as circles of radius $\sqrt{3}$ in each coordinate axis represented in Figure 6-5 as dashed segments parameterized by the arguments running from $-\pi$ to π from left to right for the first one and from top to bottom for the second one.

On these circles, place the 12 remaining vertices of Figure 6-5 in such a way that each of these 12 points can be obtained as the sum of two of the 18 points described above: e.g.

$$k_7 = m_1 + k_6.$$

Therefore these 12 points belong to the lattice $L[f]$ (defined in §7) generated by the 18 points of Figure 6-5. In order not to get mixed up while performing this very tedious exercise, the assistance of Figure 7-0 might be helpful. If you are patient enough to complete this homework, you will get the following table.

In §7 and §8 we will see that this set of 30 points is the intersection of the lattice $L[f]$ (constructed in §7) with the ball of radius $\sqrt{3}$ centered at the origin of $\mathbb{C}^2$ minus zero.

Now that we know the vertices of $\mathcal{N}$, it is easy to construct its boundary $\partial\mathcal{N}$ by taking the convex hulls in $\mathbb{C}^2$ of all the vertices of *each* of the 120 3-simplices triangulating $\tilde{\mathcal{M}}$. Then $\mathcal{N}$ is just the cone on $\partial\mathcal{N}$ with vertex the origin. In order to obtain the convex hull $\bar{\mathcal{N}}$ of $\mathcal{N}$ it suffices to fill 6 notches carved in $\mathcal{N}$ by adding 6 new 4-simplices to the 120 4-simplices into which $\mathcal{N}$ splits when considering the triangulation of its boundary. Here are the notch 4-simplices to be added:

$$\begin{array}{cccccc} m-k & l-m & k-l & l-m_- & m-l_- & k-k_- \\ m & l & k & k_- & l & m_- \\ m \text{---} \lceil & l \text{---} \lceil & k \text{---} \lceil & m \text{---} \lceil & k \text{---} \lceil & l \text{---} \lceil \end{array}$$

To derive the triangulation of $\partial\bar{\mathcal{N}}$ from the triangulation $\tilde{\mathcal{M}}$, it will be sufficient to replace the pair of simplices

$$\begin{array}{cc} m \text{---} \lceil & m \text{---} k \\ m & l \\ m \text{---} \lceil & m \text{---} \lceil \end{array}$$

contained in the boundary of $\begin{array}{cc} m-k \\ m & l \\ m \text{---} \lceil \end{array}$ by the three simplices

$$\begin{array}{ccc} m-k & \lceil-k & m-k \\ m & l & m \\ \lceil & \lceil & \lceil \end{array}$$

and similarly for the five pairs of simplices associated to the other notches.

Although the boundary $\partial\bar{\mathcal{N}}$ is naturally triangulated with 126 3-simplices, we will see in §8 that $\bar{\mathcal{N}}$ is a polytope with only 18 hyperfaces (called pegs). Each of these is the union of seven 3-simplices of the triangulation. It will then be clear that both $\bar{\mathcal{N}}$ and $\mathcal{N}$ have flat dihedral angles when considered as polytopes with 126 and 120 simplicial hyperfaces respectively.

There are 2 ways of triangulating the solid hexahedron

A tetrahedron is the join of two skew segments

join a diangle of segments to a given skew segment to obtain the co-hedral triangulation of the hexahedron -
join a triangle of segments to a given skew segment for the coaxial triangulation

G. Francis

§7 Some lattices and crystallographic groups in $\mathbb{C}^2$

In order to understand how the image of $\tilde{\mathcal{M}}$ in $\mathbb{C}^2$ was obtained, we need the definitions of this section, the construction of §8 and the map Y in §9.

Let

$$L(\omega) = \mathbb{Z} + \mathbb{Z}\omega$$

arguments

Figure 7-0

be the hexagonal lattice in $\mathbf{C}$.*)

In $\mathbf{C}^2$ with standard coordinates (z_1, z_2) , we define the following vectors

$$\begin{aligned} e_1 &= (1, 0), & e_2 &= (0, 1), \\ f_0 &= (-1, 1), & f_1 &= (-\omega, \omega^2), & f_2 &= (-\omega^2, \omega), \\ h_0 &= f_1 - f_2 & h_1 &= f_2 - f_0 & h_2 &= f_0 - f_1 \\ &= (\omega^2 - \omega)(1, 1), & &= (\omega^2 - \omega)(\omega, \omega^2), & &= (\omega^2 - \omega)(\omega^2, \omega). \end{aligned}$$

Note that

$$f_0 + f_1 + f_2 = h_0 + h_1 + h_2 = 0.$$

Let $F \subset \mathbf{C}^2$ be the real plane given by

$$F = \sum_{j=0}^2 \mathbf{R} f_j \quad (= \sum_{j=0}^2 \mathbf{R} h_j).$$

We can define the following lattices in $\mathbf{C}^2$:

$$\begin{aligned} L[e] &= L(\omega) e_1 + L(\omega) e_2, \\ L[f] &= \sum_{j=0}^2 L(\omega) f_j, \quad L[h] = \sum_{j=0}^2 L(\omega) h_j, \end{aligned}$$

which have inclusion relations whose indices are given by the marks attached below the inclusion symbols:

$$L[h] \subset_9 L[f] \subset_3 L[e].$$

Let us consider the following elements and finite subgroups of the 2×2 unitary group $U(2)$. The matrices act, as usual, on points of $\mathbf{C}^2$ considered as columns:

$$\sigma = \begin{pmatrix} \omega & \\ & \omega^2 \end{pmatrix}, \quad \tau = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}, \quad \tilde{\gamma} = \begin{pmatrix} \omega^2 & \\ & \omega^2 \end{pmatrix},$$

$$G_6 = \langle \sigma, \tau \rangle,$$

*) In §12 the lattice considered will be $\mathbf{Z} + \mathbf{Z}\tau$ where τ is any complex number such that $\text{Im } \tau > 0$, not necessarily equal to $\exp 2\pi i/6$, thus losing the symmetry γ but still obtaining a triangulation combinatorially equal to $\tilde{K}$.

$$= \left\{ \begin{pmatrix} \omega & \\ & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & \\ & \omega \end{pmatrix}, -\begin{pmatrix} & \omega \\ \omega^2 & \end{pmatrix}, -\begin{pmatrix} & \omega^2 \\ \omega & \end{pmatrix}, \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}, 1 \right\},$$

$$G_{18} = \langle G_6, \tilde{\gamma} \rangle.$$

Here $\langle a, b, \dots \rangle$ stands for the group generated by $a, b, \dots$. Notice that G_6 acts on the set $\{f_0, f_1, f_2\}$ as the symmetric group and $G_6 \trianglelefteq_3 G_{18}$, that is to say, G_6 is a normal subgroup of G_{18} of index 3.

We are now in a position to define the following *two dimensional complex crystallographic groups* (that is to say, cocompact discrete subgroups of $\mathbb{C}^2 \rtimes U(2)$, presented as semi-direct products of their *point groups with their lattices*:

$$\Gamma = L[h] \rtimes G_6,$$

$$\Gamma'' = L[f] \rtimes G_{18}.$$

It is known ([Y2]) and easy to show that Γ is generated by a finite number of complex reflections of order 2, and that the set of points in $\mathbb{C}^2$ fixed by some non-trivial element of Γ is the union A of the following lines

$$L[f] + \mathbb{C}f_j \quad j = 0, 1, 2,$$

which intersect in threes at each element of $L[f]$ and intersect nowhere else.

Identifying vectors with corresponding translations, we define the two following operations:

$$\tilde{\alpha} = f_0, \quad \tilde{\beta} = (\omega - \omega^2)f_0.$$

Then we have

$$\Gamma'' = \langle \Gamma, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \rangle \quad (= \langle \Gamma, \tilde{\alpha}, \tilde{\gamma} \rangle).$$

We next proceed to assign Γ -invariant colors 1, 2, ..., 9 to the points in $L[f]$ and 10, 11, 12 to the points in $L[e]$ not belonging to $L[f]$. The quotient $L[f]/L[h]$ (which identifies with $L[f]$ modulo Γ) consists of the following 9 classes $L_1, \dots, L_9$:

$$\begin{array}{lll} L_4 = 2\omega f_0 + L[h] & L_7 = (1 + 2\omega)f_0 + L[h] & L_1 = (2 + 2\omega)f_0 + L[h] \\ L_2 = \omega f_0 + L[h] & L_5 = (1 + \omega)f_0 + L[h] & L_8 = (2 + \omega)f_0 + L[h] \\ L_9 = L[h] & L_3 = f_0 + L[h] & L_6 = 2f_0 + L[h] \end{array}$$

In addition, let

$$\begin{aligned} L_{10} &= \Gamma e_1 = G_6 e_1 + L[h] \\ L_{11} &= \tilde{\alpha} L_{10}, \quad L_{12} = \tilde{\alpha} L_{11}. \end{aligned}$$

Since $L_{10} \cup L_{11} \cup L_{12} = L[e] - L[f]$, the coloring is now completed.

Let us define

$$\tilde{\delta}: (z_1, z_2) \mapsto (-\overline{z_2}, -\overline{z_1}) \text{ (conjugation with respect to } F),$$

$$\tilde{\varepsilon}: (z_1, z_2) \mapsto (\overline{z_2}, \overline{z_1}) \quad \text{(conjugation with respect to } iF).$$

Then one can readily check that $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ and $\tilde{\varepsilon}$ act on the colors $1, \dots, 12$ exactly as $\alpha, \beta, \gamma, \delta$ and ε acted on vertices in §1:

$$\tilde{\eta} L_i = L_{\eta(i)} \quad i = 1, \dots, 12$$

where $\eta = \alpha, \beta, \gamma, \delta$ or ε . In particular, by sending $\tilde{\alpha} \mapsto \alpha, \tilde{\beta} \mapsto \beta, \tilde{\gamma} \mapsto \gamma$, we get

$$\Gamma''/\Gamma \cong G.$$

In the sequel the colors introduced here will enable one to speak of the type and the color of simplices with vertices in $L[f]$.

Recall that as decided in §1 and §3, these twelve colors are organized into the following families

$$\{1, 2, 3, 10\}, \{4, 5, 6, 11\} \text{ and } \{7, 8, 9, 12\}$$

respectively considered as blue, white and red. If one insists on bearing in mind actual colors, rather than numbers, we suggest substituting for the array

$$\begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 10 & 11 & 12 \end{array}$$

the following palette:

sky blue	white	pink
royal blue	gray	sparkling red
navy blue	black	scarlet
green	yellow	orange

§8 The rectilinear triangulation $\tilde{K}$ of C^2

In C^2 consider the real hyperplane

$$\begin{aligned} \Pi &= F + Cf_0 \quad (F = \sum_{j=0}^2 \mathbf{R}f_j) \\ &= \{(z_1, z_2) \in C^2 \mid \operatorname{Re}(z_1 + z_2) = 0\}, \end{aligned}$$

and the collection of hyperplanes

$$\Sigma = \Gamma'' \Pi.$$

Since

$$|\Sigma| = \bigcup_{\gamma \in \Gamma''} \gamma \Pi$$

is Γ'' -invariant, the intersection pattern of hyperplanes in Σ is the same at each point of $L[f]$. Here is the intersection pattern at the origin.

A. The 9 hyperplanes of Σ passing through the origin are

$$\Pi_{jk} : \operatorname{Re}(\omega^j z_1 + \omega^k z_2) = 0 \quad j, k \in \mathbf{Z}/3\mathbf{Z}.$$

B. In $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ let $(j, k) \neq (j', k')$.

- (i) If $k - j = k' - j'$ then $\Pi_{jk} \cap \Pi_{j'k'}$ is the complex line Cf_{k-j} .
- (ii) If $k + j = k' + j'$ then $\Pi_{jk} \cap \Pi_{j'k'}$ is the real plane $\omega^{k+j}F$.
- (iii) If $j = j'$ (or $k = k'$) then $\Pi_{jk} \cap \Pi_{j'k'}$ is the real plane $H(j; k, k')$ spanned by p_{jk} and $p_{j'k'}$ (or the real plane $V(j, j'; k)$ spanned by p_{jk} and $p_{j'k}$), where

$$\begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} l_1 & k_2 & m_3 \\ l_2 & k_3 & m_1 \\ l_3 & k_1 & m_2 \end{pmatrix}.$$

The nine vectors appearing in the second array have been defined in the list

concluding §6 (see also Figure 6-5).

C. Along each of the three complex lines Cf_j and each of the three real planes $\omega^j F_j$ ($j = 0, 1, 2$), three Π_{jk} 's intersect.

D. Along each of the nine H 's and each of the nine V 's defined in (iii) above, only two Π_{jk} 's intersect.

E. Along each of the six real lines spanned by $k_7, l_7, m_7, k_{-7}, l_{-7}$ and m_{-7} respectively (recall that these are defined at the end of §6), three Π_{jk} 's intersect.

F. Along each of the nine real lines spanned by the p_{jk} 's defined in (iii), five Π_{jk} 's intersect.

Figure 8-1

Claim 1. The connected components of the open set $\mathbf{C}^2 - |\Sigma|$ are of two types: either 4-simplices or 4-cells with 6 vertices and 9 simplicial hyperfaces. The latter type will be referred to as cells of type 3-3 (the colors of vertices of such cells will indeed appear later to fill two columns of the array

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \text{ of §1).}$$

Proof. We first prove the claim near the origin, by studying the convex hull $\tilde{\mathcal{N}}$ of the 30 vertices of $\tilde{\mathcal{M}}$ listed in §6 (See Subclaim below). Remember that the closest points to the origin in $L[f]$ are the 18 points at distance $\sqrt{2}$ from the origin listed in §6. If we take the convex hull $\mathcal{W}$ of these 18 points, we get a polytope with 12 octahedral and 18 tetrahedral hyperfaces. Observe that the barycenters of these octahedra are the midpoints of the segments joining k_9 to the 12 elements of $L[f]$ at distance $\sqrt{3}$ from the origin, and that the hyperplanes carried by the octahedra are the perpendicular bisectors of these segments. In order to understand how the boundary of this polytope is organized, we proceed as follows. First notice that 6 of the 12 octahedra are glued together in a hexagonal pattern, in such a way that one octahedron touches its two neighbors along two of its opposite faces so that their union is a solid torus shown in Figure 8-1. Although it does not live in the boundary of the same polytope it is helpful to have in mind that the present picture is Figure 6-1 with the notches deleted.

Secondly notice that one can think of the union of the six remaining octahedra as a second solid torus linking the first one in S^3 . Now the complement in S^3 of the union of these two solid tori is the union of 18 tetrahedra touching each other along an edge. Each tetrahedron is connected to 4 other tetrahedra along all of its edges except two opposite ones. In order to understand this patchwork of tetrahedra, one must think of a tetrahedron as a very flat object almost equal to its projection to the square.

The 18 squares thus obtained are matched together in order to produce a quilted torus as shown in Figure 8-2. We remark that the group G_{18} acts transitively on the 18 squares.

The Quilted Torus

Figure 8-2

The 18 hyperplanes carrying the tetrahedra are in Σ , while the 12 hyperplanes carrying the octahedra are not. For instance, the tetrahedron $k_5 m_1 k_4 l_2$ is carried by the hyperplane

$$\operatorname{Re}(z_1 + z_2) = 3/2$$

which is parallel to $\Pi = \Pi_{00}$ and belongs to Σ . But the octahedron $m_1 m_2 m_3 k_4 k_5 k_6$ is carried by the hyperplane

$$\operatorname{Im}(\omega z_2) = \sqrt{3}/2$$

which is not in Σ . Besides the tetrahedron $k_5 m_1 k_4 l_2$ the hyperplane $\operatorname{Re}(z_1 + z_2) = 3/2$ happens also to contain

$$k_7, k_8, m_{-7} \text{ and } m_{-8}.$$

This remark proves that $\bar{\mathcal{N}}$ is a polytope with 18 hyperfaces carried by those hyperplanes of Σ belonging to the G_{18} -orbit of $\operatorname{Re}(z_1 + z_2) = 3/2$. All these faces have the same shape and will be referred to as *pegs*. Check that these 18 hyperplanes are the closest hyperplanes of Σ to the origin not passing through the origin and that, up to similarity, the polytope $\mathcal{W}$ (the convex hull of the 18 points in Figure 6-5) previously considered is dual to $\bar{\mathcal{N}}$. We now describe the peg carried by $\operatorname{Re}(z_1 + z_2) = 3/2$, which is one of the two hypersurfaces parallel to Π . It is the convex hull

$$\overline{k_5 \ m_1 \ k_4 \ l_2 \ k_7 \ k_8 \ l_{-7} \ m_{-8}}$$

which is a polyhedron with 8 vertices, 10 edges and 8 faces. Four of the faces (called *lateral faces*) are rhombi and the others (called *wedge faces*) are isosceles triangles (see below).

It can be considered as the intersection of a prism based on the square of side $\sqrt{6}/2$ with the tetrahedron whose faces are obtained by producing the four wedge faces of the peg.

The 8 hyperplanes Π_{ij} ($i, j \neq (0, 0)$) meet the peg and split it into the seven following tetrahedra.

Observe that contrary to the lateral faces, the wedge faces are not intersections of the peg with the Π_{ij} 's but only with hyperplanes carrying the neighbouring pegs. So that five of the seven 3-simplices—the saturated ones—have all their faces contained in some Π_{ij} 's, while two of them (situated on the top and the bottom of the previous picture)—the wedge simplices—have only two faces satisfying this property.

Each of the segments $\overline{l_{-7}m_{-8}}$ and $\overline{k_7k_8}$ is the intersection of three of the 18 pegs limiting $\mathcal{N}$. The following picture indicates how the three pegs meeting along $\overline{l_{-7}m_{-8}}$ match together.

Two Pegs

Three Pegs from above

Let us define a *cap* to be either one of the two unions of three 3-simplices

$$\begin{array}{|c|c|} \hline k & k \\ \hline k & k \\ \hline \end{array} \cup \begin{array}{|c|c|} \hline & k \\ \hline k & k \\ \hline k & \end{array} \cup \begin{array}{|c|c|} \hline & k \\ \hline & k \\ \hline k & \end{array}$$

and

$$\begin{array}{|c|c|} \hline m & l_- \\ \hline l & m_- \\ \hline \end{array} \cup \begin{array}{|c|c|} \hline & l_- \\ \hline l & m_- \\ \hline k & \end{array} \cup \begin{array}{|c|c|} \hline m & l_- \\ \hline & m_- \\ \hline k & \end{array}$$

or one of their images under the action of G_6 . Notice that the convex hull $\overline{\{k_9\} \cup C}$ (where C is any cap) is a 3-3 cell on the boundary of which the cap is sitting.

The following assertion should now be clear.

Subclaim. The collection Σ splits $\bar{\mathcal{N}}$ into 90 4-simplices of type 2-2-1 (namely all the cones on saturated simplices of $\partial\bar{\mathcal{N}}$ with vertex at the origin) and 12 cells of type 3-3 (namely the convex hull of k_9 and each of the 12 caps just considered).

In order to finish the proof of Claim 1, apply $L[f]$ to the $|\Sigma|$ -decomposition of the convex hull $\bar{\mathcal{N}}$.

Finishing the construction of $\tilde{K}$. The 2-2-1 simplices of $\tilde{K}$ are the simplices of $\mathbb{C}^2 - |\Sigma|$. The 3-2 simplices of $\tilde{K}$ are obtained by linearly subdividing each 3-3 cell into three 3-2 simplices. Since this must be done in a Γ'' -invariant fashion, we need only to split a given 3-3 cell. The splitting can be achieved in two different ways, so that a choice must be made. In fact the choice has already been dictated to us in [K-B] where it has been decreed that K should

contain $\begin{array}{|c|} \hline \bullet \bullet \\ \bullet \bullet \\ \hline \bullet \end{array}$. Had they wished, the authors could have chosen $\begin{array}{|c|} \hline \bullet \bullet \\ \bullet \bullet \\ \hline \bullet \end{array}$

in place of $\begin{array}{|c|} \hline \bullet \bullet \\ \bullet \bullet \\ \hline \bullet \end{array}$. In order to obey them, and to complete the definition of $\tilde{K}$, it suffices to set

$$k_4 k_5 k_6 k_7 k_8 k_9 = \begin{array}{|c|} \hline k \ k \\ k \ k \\ \hline k \end{array} \cup \begin{array}{|c|} \hline k \\ k \ k \\ \hline k \ k \end{array} \cup \begin{array}{|c|} \hline k \ k \\ k \\ \hline k \ k \end{array}.$$

Observe that the open set $\mathbb{C}^2 - |\Sigma|$ is $\tilde{\delta}$ and $\tilde{\varepsilon}$ -invariant but that the need to triangulate 3-3 cells forced us to exhibit a triangulation which was $\tilde{\delta}$ -invariant but not $\tilde{\varepsilon}$ -invariant. The choice we had to make was in fact between the triangulations $\tilde{K}$ and $\tilde{\varepsilon}\tilde{K}$. Note also that all 1-simplices of type 1-1 in $\tilde{K}$ are of length $\sqrt{2}$ while those of type 2-0-0 are of length $\sqrt{3}$, and that all 4-simplices of types 2-2-1 and of 3-2 have volume $3/32$.

Claim 2. As an abstract complex, $\tilde{K}/\Gamma$ is isomorphic to K .

Proof. Let $c_1, \dots, c_5$ be the colors of a given simplex in $\tilde{K}$. There exists an element g in Γ'' which maps the vertex of color c_1 to the origin k_9 . Let $\bar{g}$

be the image of g in $G = \Gamma''/\Gamma$. Now $(\bar{g}(c_1), \dots, \bar{g}(c_5))$ is a simplex in K and therefore also $(c_1, \dots, c_5)$.

Remarks.

(1) The inverse image of K'' in $\tilde{K}$ is the union of all the 3–3 cells in $\mathbf{C}^2 - |\Sigma|$. We observe that the set of barycenters of these cells is $L[e] - L[f]$ and that the colors correspond to their images 10, 11, 12 in $|K|$. The inverse image of the subdivision $\bar{K}$ of K is obtained by considering the subdivision generated by $\tilde{K}$ and $\tilde{\varepsilon}\tilde{K}$. The inverse image of $|T|$ is the union A of all the axes of reflections in Γ . Figure 8–4 shows how the vertices contained in the typical axis of reflection are colored.

Figure 8–4

(2) The quotient $|\tilde{K}|/L[h]$ is homeomorphic to the torus $(S^1)^4$ decomposed into 6×36 4-simplices with 9 vertices. This decomposition is an improper triangulation. This 4-torus is a 6 fold cover of $|K|$ branched over $|T|$. In order to get a properly triangulated 4-torus, one has to consider some 9 fold cover of $\tilde{K}/L[h]$, for instance $\tilde{K}/3L[f']$ where

$$L[f'] = L(\omega)f'_0 + L[\omega]f'_1 + L[\omega]f'_2$$

with

$$f'_0 = (1, 1), \quad f'_1 = (\omega, \omega^2), \quad f'_2 = (\omega^2, \omega)$$

(see Chapter V). The triangulation has 81 vertices and $9 \times 6 \times 36 = 1944$ 4-simplices. The 4-torus $\tilde{K}/3L[f]$ would have worked equally well but it will turn out in Chapter V that the Dirichlet domain of $3L[f']$ at the origin is the convex hull of all points in $L[f]$ at distance not exceeding 3. This remark provides a very good grasp on $\tilde{K}/3L[f']$. In order to understand this quotient the Kühnel-Banchoff coloring, used throughout this paper since § 1, is no longer satisfactory. This explains why Chapter V sets up a much more logical system of 81 colors that will prove very efficient. Nevertheless we decided not to use this powerful tool, devised by Marin, in the presentation of our material in order to supply the reader with a smooth transition between [K-B] and Chapter V.

(3) If one introduces in $\mathbf{C}^2$ new coordinates by using the basis $(f_0, -f'_0)$ where $f_0 = k_3 = (-1, 1)$ and $-f'_0 = (-1, -1)$, then it becomes convenient to think of this reparameterized and renormalized $\mathbf{C}^2$ as $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$. The identification goes as follows:

$$f_0 = 1 \otimes 1, \quad if_0 = i \otimes 1, \quad if'_0 = 1 \otimes i, \quad -f'_0 = i \otimes i.$$

Then the lattice $L[f]$ identifies with $L(\omega) \otimes_{\mathbf{Z}} L(\omega)$ where, as in § 7, $L(\omega)$ is the hexagonal lattice. Now the set of the eighteen vertices of the quilted torus is the image by the canonical bilinear map of $\langle -\omega \rangle \times \langle -\omega \rangle$ in $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ where

$\langle -\omega \rangle$ is the group of sixth roots of unity in $\mathbb{C}$. The picture in the previous page illustrates this fact by showing how the restriction to $S^1 \times S^1$ of the canonical bilinear map $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ covers twice a subtorus in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. The central square represents some rotated version of the quilted torus (Figure 8–2).

Recall that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a real vector space equipped with a ring structure and with two complex structures (multiplications on the left and on the right) for both of which $(1 \otimes 1, i \otimes i)$ is a complex basis. The complex structure used in §6, 7 and 8 is the left complex structure. Notice that $\tilde{\gamma}$ and σ are the multiplications by ω^2 on the left and on the right, respectively. Moreover $\tilde{\delta}$ and τ are induced by the complex conjugations on the first and on the second factor of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, respectively. In the Epilogue of Chapter V Marin describes the trip which eventually lead him to this very basic remark.

§9 An explicit triangulation of $\mathbb{CP}^2$ by K

Let L be a lattice in $\mathbb{C}$ and let φ be an embedding of the torus $\mathbb{C}/L$ into a complex projective plane $M^\vee$ as a non-singular cubic curve C such that $\varphi(0)$ is a point of inflexion of C . Let M be the complex projective plane dual to $M^\vee$, that is, the set of lines in $M^\vee$. Let N be the hyperplane of $\mathbb{C}^3$ (with coordinates u_0, u_1 and u_2) defined by

$$u_0 + u_1 + u_2 = 0.$$

We shall introduce a map

$$Y: N \rightarrow M.$$

In order that the three points $\varphi(u_0)$, $\varphi(u_1)$ and $\varphi(u_2)$ be collinear it is necessary and sufficient that (u_0, u_1, u_2) should belong to N (Abel's theorem). Thus if at least two of the three points are distinct, one can define Y geometrically by mapping (u_0, u_1, u_2) to the line through $\varphi(u_j)$ ($j = 0, 1, 2$). Since Y is defined outside a set of co-dimension two, Y extends uniquely to a holomorphic map on the whole plane N .

By this definition, the following claims are clear.

Claim 1. The map Y is invariant under translation by the lattice

$L \oplus L \oplus L \cap N$ and under the symmetric group permuting u_0, u_1 and u_2 .

Claim 2. The map Y is just the quotient by the crystallographic group acting on N generated by the lattice above and the above symmetric group.

Claim 3. The set of critical points of Y is the union of lines defined by

$$(9.1) \quad u_j \equiv u_k \pmod{L} \quad (j, k = 0, 1, 2, j \neq k).$$

Claim 4. The set of critical values of Y is the dual curve $C^\vee \subset M$ of $C \subset M^\vee$, where $C^\vee$ is defined as the set of lines tangent to C .

Recall that, corresponding to the 9 inflection points of C (Which are the images of the 3-torsion points of $\mathbf{C}/L$), the curve $C^\vee$ has 9 cusps.

In order to fit into the context of the previous sections, we let $L = L(\omega)$ and make everything explicit as follows. Let $\wp(u)$ be the Weierstrass function belonging to the lattice $L(\omega)$:

$$\wp(u) = \frac{1}{u^2} + \sum_{\substack{\zeta \in L(\omega) \\ \zeta \neq 0}} \left(\frac{1}{(u - \zeta)^2} - \frac{1}{\zeta^2} \right)$$

and let E be the positive real number $\wp(1/2)$. The function $\wp$ and its derivative $\wp'$ are related by the following well-known formula:

$$(\wp')^2 - 4\wp^3 + 4E^3 = 0.$$

If we let

$$\xi_0 = \xi_0(u) = -\wp'(u) + a,$$

$$\xi_1 = \xi_1(u) = \wp'(u) + a,$$

$$\xi_2 = \xi_2(u) = b\wp(u),$$

where

$$a = \wp'(1/3) = -2\sqrt{3}E\sqrt{E}, \quad b = \sqrt[3]{16}\sqrt{3}\sqrt{E},$$

then one can easily check that the above formula becomes

$$(9.2) \quad (\xi_0)^3 + (\xi_1)^3 + (\xi_2)^3 = 0.$$

Let $[\xi_0, \xi_1, \xi_2]$ be homogeneous coordinates on $M^\vee$ and let $C \subset M^\vee$ be the non-singular curve defined by (9.2). Define the map $\varphi: \mathbf{C} \rightarrow M^\vee$ by

$$\varphi(u) = [\xi_0(u), \xi_1(u), \xi_2(u)].$$

The map φ defines the embedding

$$\mathbf{C}/L(\omega) \xrightarrow{\sim} C \hookrightarrow M.$$

Let $[y_0, y_1, y_2]$ be homogeneous coordinates on M . Then the map $Y: N \rightarrow M$ is given explicitly as follows:

$$\begin{aligned} y_0 &= \det \begin{pmatrix} \xi_1(u_1) & \xi_2(u_1) \\ \xi_1(u_2) & \xi_2(u_2) \end{pmatrix}, & y_1 &= \det \begin{pmatrix} \xi_2(u_1) & \xi_0(u_1) \\ \xi_2(u_2) & \xi_0(u_2) \end{pmatrix}, \\ y_2 &= \det \begin{pmatrix} \xi_0(u_1) & \xi_1(u_1) \\ \xi_0(u_2) & \xi_1(u_2) \end{pmatrix}, \end{aligned}$$

and the dual curve $C^\vee$ is given by

$$(y_0)^6 + (y_1)^6 + (y_2)^6 - 2((y_0 y_1)^3 + (y_1 y_2)^3 + (y_2 y_0)^3) = 0$$

Notice that the cusps of $C^\vee$ are the points $v_1, \dots, v_9$ defined at the end of §4.

With the notations of §7, identify N with $\mathbf{C}^2$ by setting

$$\begin{aligned} e_1 &= \frac{1}{3}(\omega, 1, \omega^2), \\ e_2 &= -\frac{1}{3}(\omega^2, 1, \omega). \end{aligned}$$

We then have

$$\begin{aligned} h_0 &= (1, 0, -1), & h_1 &= (-1, 1, 0), & h_2 &= (0, -1, 1), \\ 3f_0 &= (1, -2, 1), & 3f_1 &= (1, 1, -2), & 3f_2 &= (-2, 1, 1). \end{aligned}$$

Now one has

$$L(\omega) \oplus L(\omega) \oplus L(\omega) \cap N = \sum_{j=0}^2 L(\omega) h_j \quad (= L[h]),$$

while the critical set of Y (cf. 9.1) becomes the union A of all the lines of the form

$$\mathbf{C}f_j + \zeta \quad (j = 0, 1, 2, \zeta \in L[f]).$$

Using the identification, one concludes from Claim 2 that the map Y is

invariant under the action on $\mathbf{C}^2$ of the crystallographic group $L[h] \rtimes G_6$ (the group Γ defined in §7) and that Y/Γ identifies $\mathbf{C}^2/\Gamma$ with $\mathbf{CP}^2$.

Furthermore, one has

$$Y(L_i) = v_i \quad i = 1, \dots, 12.$$

Put

$$\hat{\alpha} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \hat{\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

and denote by $\hat{\delta}$ the complex conjugation map

$$[y_0, y_1, y_2] \rightarrow [\overline{y_0}, \overline{y_1}, \overline{y_2}]$$

of $M(\cong \mathbf{CP}^2)$ and by $\hat{\varepsilon}$ the composition of $\hat{\delta}$ and the mapping represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then have

$$\hat{\eta} \circ Y = Y \circ \tilde{\eta}$$

so that

$$\tilde{\eta} v_i = v_{\eta(i)} \quad j = 1, \dots, 12,$$

where $\eta = \alpha, \beta, \gamma, \delta, \varepsilon$.

How does $\mathbf{RP}^2$ fit into the picture. Let $M_{\mathbf{R}} \cong \mathbf{RP}^2$ be the subset of $M \cong \mathbf{CP}^2$ fixed by the complex conjugation $\hat{\delta}$ defined above. We exhibit the inverse image $Y^{-1}(M_{\mathbf{R}})$ of $M_{\mathbf{R}}$ in the complex plane N . On $N \cong \mathbf{C}^2$ define the following two complex conjugations which fix the real planes

$$E = \sum_{j=1}^2 \mathbf{R} e_j$$

and

$$F = \sum_{j=0}^2 \mathbf{R} f_j$$

respectively, by setting

$$\tilde{\delta}_E: \begin{cases} (u_0, u_1, u_2) \mapsto (\overline{u_2}, \overline{u_1}, \overline{u_0}) \\ (z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2}) \end{cases}$$

$$\tilde{\delta}_F = \tilde{\delta}: \begin{cases} (u_0, u_1, u_2) \mapsto (\overline{u_0}, \overline{u_1}, \overline{u_2}) \\ (z_1, z_2) \mapsto (-\overline{z_2}, -\overline{z_1}). \end{cases}$$

Observe that the Y is covariant with respect not only to $\tilde{\delta}_E$ and $\hat{\delta}$ but also to $\tilde{\delta}_F$ and $\hat{\delta}$. This means that the following relations hold.

$$Y \circ \tilde{\delta}_E = \hat{\delta} \circ Y,$$

$$Y \circ \tilde{\delta}_F = \hat{\delta} \circ Y.$$

In Γ , let Γ_E and Γ_F be the isotropy subgroups of E and F , respectively. These are real crystallographic subgroups of the complex crystallographic group Γ . Since one has

$$\Gamma_E = \{g \in \Gamma \mid g\hat{\delta}_E = \hat{\delta}_E g\}$$

and

$$\Gamma_F = \{g \in \Gamma \mid g\tilde{\delta}_F = \tilde{\delta}_F g\},$$

it follows that Γ_E is the semi-direct extension of a two dimensional lattice $\mathbf{Z} \oplus \mathbf{Z}$ by a group of order 2 generated by a real reflection and that Γ_F is the affine Weyl group of type A_1 (i.e. the semi-direct extension of a two dimensional lattice by the symmetric group $\mathfrak{S}_3$). Figure 9-1 shows fundamental domains D_E and D_F of Γ_E and Γ_F in E and F respectively, on which colors of points in $L[e] \cap D_E$ and $L[e] \cap D_F$ are shown.

Since E/Γ_E is the Möbius strip obtained by identifying the vertical and horizontal sides of D_E (following the indicated color code) and since F/Γ_F is an equilateral triangle, one can conclude that

$$E/\Gamma_E \cup_{\partial(936)} F/\Gamma_F \cong \mathbf{RP}^2.$$

Figure 9-1

Figure 9-2

Moreover $M_{\mathbf{R}}$ splits into $Y(E) \cup Y(F)$ so that

$$Y^{-1}(\mathbf{RP}^2) = \left(\bigcup_{g \in \Gamma} gE \right) \cup \left(\bigcup_{g \in \Gamma} gF \right).$$

Figure 9-2 shows the intersection of $\tilde{K}$ with E . In the picture, $\hat{12}$ is the color attached to the midpoint of a segment of length $\sqrt{3}$ whose endpoints are colored by 1 and 2. Similarly for $\hat{45}$ and $\hat{78}$. Figures 9-1 and 9-2 tell us that the triangulation $\tilde{K}$ induces on $M_{\mathbf{R}}$ the Möbius decahedral triangulation of $\mathbf{RP}^2$ obtained by quotienting the boundary of the regular icosahedron by the antipodal map (cf. [Mö]). The six vertices of the Möbius decahedron map to

$$3, 6, 9, \hat{12}, \hat{45} \text{ and } \hat{78}.$$

Notice that the use of the map Y forces us to define a rectilinear embedding of the decahedron in $|K|$ which is precisely the embedding constructed in [K-B] for quite different reasons.

§ 10 The Dirichlet domain P of Γ with respect to $e_1 = k_{10}$

Notice that the isotropy group Γ_{10} of k_{10} in Γ'' is the abelian group generated by the following two elements of order 3

$$\sigma_{10} = e_1 \sigma(-e_1) \quad \text{and} \quad \tilde{\gamma}_{10} = e_1 \tilde{\gamma}(-e_1),$$

while the isotropy group of k_{10} in Γ is trivial. Let $\bar{\Gamma}_{10}$ be the group generated by Γ_{10} and by the two involutions

$$\tilde{\delta}_{10} = \tilde{\delta}\tau \quad \text{and} \quad \tilde{\varepsilon}_{10} = e_1 \tilde{\varepsilon}(-e_1).$$

We now construct a $\bar{\Gamma}_{10}$ -invariant polytope P and show that P is the Dirichlet domain of Γ with respect to k_{10} . In the orbit $L_{10} = \Gamma k_{10}$ of k_{10} under the action of Γ the closest points to k_{10} are at distance $\sqrt{2}$. There are nine of them listed below:

$$\begin{aligned} (1 - \omega, -1) &= \tau_{67}(k_{10}) & (1 - \omega^2, -1) &= \tau_{68}(k_{10}) \\ (1 - \omega, -\omega) &= \tau_{57}(k_{10}) & (1 - \omega^2, -\omega) &= \tau_{58}(k_{10}) \\ (1 - \omega, -\omega^2) &= \tau_{47}(k_{10}) & (1 - \omega^2, -\omega^2) &= \tau_{48}(k_{10}) \\ (0, -1) &= \tau_{69}(k_{10}) \\ (0, -\omega) &= \tau_{59}(k_{10}) \\ (0, -\omega^2) &= \tau_{49}(k_{10}) \end{aligned}$$

The τ_{ij} 's are 2 fold reflections in Γ with the following axes of reflections:

$$\begin{aligned} \tau_{67} &: D(2, -f_0) & \tau_{68} &: D(1, -f_0) \\ \tau_{57} &: D(0, -\omega^2 f_1) & \tau_{58} &: D(2, -\omega^2 f_1) \\ \tau_{47} &: D(1, -\omega f_2) & \tau_{48} &: D(0, -\omega f_2) \\ & \tau_{69} &: D(0, -f_0) \\ & \tau_{59} &: D(1, -\omega^2 f_1) \\ & \tau_{49} &: D(2, -\omega f_2) \end{aligned}$$

where

$$D(i, \zeta) = \mathbf{C}f_i + \zeta \quad (\zeta \in L[f]).$$

Next we find the following twelve points at distance $\sqrt{3}$ from k_{10} :

$$\begin{aligned} (1, \omega^2 - 1) &= \sigma_6(k_{10}) & (2 - \omega, 0) &= \sigma_7(k_{10}) \\ (1, \omega - 1) &= \sigma_6^2(k_{10}) & (-2\omega, 0) &= \sigma_7^2(k_{10}) \\ (1, 1 - \omega) &= \sigma_5(k_{10}) & (-2\omega^2, 0) &= \sigma_8(k_{10}) \\ (1, \omega^2 - \omega) &= \sigma_5^2(k_{10}) & (2 - \omega^2, 0) &= \sigma_8^2(k_{10}) \\ (1, \omega - \omega^2) &= \sigma_4(k_{10}) & (\omega^2, 0) &= \sigma_9(k_{10}) \\ (1, 1 - \omega^2) &= \sigma_4^2(k_{10}) & (\omega, 0) &= \sigma_9^2(k_{10}) \end{aligned}$$

where the σ_i 's are the rotations of order 3 in Γ given as follows:

$$\begin{aligned} \sigma_6 &= \sigma(-f_0) & \sigma_7 &= \sigma^{-1}(\omega^2 f_2 - f_0) \\ \sigma_5 &= \sigma(-\omega^2 f_1) & \sigma_8 &= \sigma^{-1}(-\omega f_2 - \omega^2 f_0) \\ \sigma_4 &= \sigma(-\omega f_0) & \sigma_9 &= \sigma^{-1} \end{aligned}$$

By definition, the Dirichlet domain of Γ relative to k_{10} is contained in the intersection P of half spaces containing k_{10} . Furthermore, the hyperplanes bounding this domain are the perpendicular bisectors of the line segments joining k_{10} to the 21 points at distance not greater than $\sqrt{3}$. Let

$$\Pi = \mathbf{C}f_0 + F \text{ (as in § 8)} \quad \text{and} \quad \Pi' = \mathbf{C}e_1 + \mathbf{R}e_2$$

be the real hyperplanes satisfying

$$\operatorname{Re}(z_1 + z_2) = 0 \quad \text{and} \quad \operatorname{Im}(z_2) = 0,$$

respectively. As indicated in the following table, the 21 hyperplanes considered

are the transforms of Π (for the nine first) and of $\Pi'_9 = \tilde{\gamma}^2 \tau \Pi'$ (for the twelve last) by elements of $\bar{\Gamma}_{10}$.

$$\begin{array}{ll}
 \tau_{67}: \tilde{\gamma}_{10} \sigma_{10}^2(\Pi) & \tau_{68}: \tilde{\gamma}_{10}^2 \sigma_{10}(\Pi) \\
 \tau_{57}: \tilde{\gamma}_{10}^2(\Pi) & \tau_{58}: \sigma_{10}^2(\Pi) \\
 \tau_{47}: \sigma_{10}(\Pi) & \tau_{48}: \tilde{\gamma}_{10}(\Pi) \\
 \tau_{69}: & \Pi \\
 \tau_{59}: \tilde{\gamma}_{10} \sigma_{10}(\Pi) & \\
 \tau_{49}: \tilde{\gamma}_{10}^2 \sigma_{10}^2(\Pi) & \\
 \sigma_6: \tilde{\varepsilon}_{10} \tilde{\delta}_{10}(\Pi'_9) & \sigma_7: \tilde{\gamma}_{10}^2(\Pi'_9) \\
 \sigma_6^2: \tilde{\varepsilon}_{10}(\Pi'_9) & \sigma_7^2: \tilde{\gamma}_{10}^2 \tilde{\delta}_{10}(\Pi'_9) \\
 \sigma_5: \tilde{\gamma}_{10}^2 \tilde{\varepsilon}_{10} \tilde{\delta}_{10}(\Pi'_9) & \sigma_8: \tilde{\gamma}_{10}(\Pi'_9) \\
 \sigma_5^2: \tilde{\gamma}_{10}^2 \tilde{\varepsilon}_{10}(\Pi'_9) & \sigma_8^2: \tilde{\gamma}_{10} \tilde{\delta}_{10}(\Pi'_9) \\
 \sigma_4: \tilde{\gamma}_{10} \tilde{\varepsilon}_{10} \tilde{\delta}_{10}(\Pi'_9) & \sigma_9: \Pi'_9 \\
 \sigma_4^2: \tilde{\gamma}_{10} \tilde{\varepsilon}_{10}(\Pi'_9) & \sigma_9^2: \tilde{\delta}_{10}(\Pi'_9)
 \end{array}$$

In fact as we now intend to show, P is indeed the Dirichlet domain we are looking for. In order to see this, consider P as the convex hull of its vertices, thus enabling one to compute its volume. The vertices of P are the 15 points of $L[f]$ at distance not greater than $\sqrt{2}$ of k_{10} , namely six at distance 1 and nine at distance $\sqrt{2}$ as tabulated below.

$$\begin{array}{l}
 k_j (4 \leq j \leq 9), \quad l_j (1 \leq j \leq 3), \quad m_j (1 \leq j \leq 3), \\
 n_1 = (2, \omega), \quad n_2 = (2, \omega^2), \quad n_3 = (2, 1)
 \end{array}$$

(Notice that the indexing is compatible with the coloring defined in §1 and §7). One can check that the set of 15 vertices of P are invariant under the group $\bar{\Gamma}_{10}$. The boundary of P is the union of the nine octahedra and the twelve tetrahedra shown below. The table also indicates how the τ_{ij} 's and σ_k 's act on the faces.

Nine Octahedra

$$\tau_{67} \hookrightarrow (m_3, n_2, n_1, m_2, k_6, k_7) \quad \tau_{68} \hookrightarrow (n_2, l_1, l_3, n_1, k_6, k_8)$$

$$\tau_{57} \hookrightarrow (m_1, n_3, n_2, m_3, k_5, k_7) \quad \tau_{58} \hookrightarrow (n_3, l_2, l_1, n_2, k_5, k_8)$$

$$\tau_{47} \hookrightarrow (m_2, n_1, n_3, m_1, k_4, k_7) \quad \tau_{48} \hookrightarrow (n_1, l_3, l_2, n_3, k_4, k_8)$$

$$\tau_{69} \hookrightarrow (l_1, m_3, m_2, l_3, k_6, k_9)$$

$$\tau_{59} \hookrightarrow (l_2, m_1, m_3, l_1, k_5, k_9)$$

$$\tau_{49} \hookrightarrow (l_3, m_2, m_1, l_2, k_4, k_9)$$

The mirror of the reflection
 τ_{67}

(Note that axis of the reflection τ_{ij} is the complex line joining k_i and k_j for $4 \leq i \leq 6$, $7 \leq j \leq 9$.)

Twelve Tetrahedra

$$\sigma_6 \left(\begin{array}{c} l \text{ --- } k \\ \uparrow n \\ m \text{ --- } k \\ \uparrow n \\ n \text{ --- } k \\ \uparrow m \\ m \text{ --- } k \\ \uparrow l \end{array} \right)$$

$$\sigma_7 \left(\begin{array}{c} n \text{ --- } k \\ \uparrow n \\ n \text{ --- } k \\ \uparrow m \\ m \text{ --- } k \\ \uparrow m \\ m \text{ --- } k \end{array} \right)$$

(Note that σ_i fixes k_i ($4 \leq i \leq 9$).)

The convex hulls of k_{10} and each of the hyperfaces of the polytope decompose P into nine hyperpyramids of height $\sqrt{2}/2$ with octahedral bases each of volume $\sqrt{2}$ and twelve hyperpyramids of height $\sqrt{3}/2$ with tetrahedral bases each of volume $\sqrt{3}/4$. The volume of P is therefore

$$\begin{aligned} V(P) &= \frac{1}{4} \left\{ 9 \times \frac{\sqrt{2}}{2} \times \sqrt{2} + 12 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{4} \right\} \\ &= 27/8 \\ &= V(\mathbf{C}^2/\Gamma), \quad (\text{cf. } \S 8) \end{aligned}$$

which shows that P is the Dirichlet domain of Γ with respect to k_{10} . Using the Poincaré theorem, one readily checks from this fact that Γ is generated by the nine τ_{ij} ($4 \leq i \leq 6, 7 \leq j \leq 9$) subject to the relations

$$\tau_{48}\tau_{47}\tau_{57}\tau_{59}\tau_{69}\tau_{68} = 1$$

$$\tau_{i9}\tau_{i8} = \tau_{i8}\tau_{i7}$$

$$\tau_{4j}\tau_{5j} = \tau_{5j}\tau_{6j}$$

and

$$\tau_{ij}^2 = 1 \quad (\text{for } 4 \leq i \leq 6, 7 \leq j \leq 9).$$

Compare this presentation with the one given in [Y2].

The polytope P is rectilinearly triangulated by the 36 4-simplices each of volume $3/32$ listed below.

Decomposition of the central core:

Teeth:

Bridges:

Notches:

The construction of P by adding layers of simplices to the central core is exactly analogous to the following sequence of operations. Take a regular dodecahedron and stellate it in order to obtain the small starred dodecahedron (due to Kepler). Next, bridge the teeth of the starred solid to get the great Poinset dodecahedron and finally fill in the craters so as to end up with the convex hull of the starred Kepler polytope, which is a regular icosahedron.

In the previous list, the first 33 simplices are in $\tilde{K}$ but the last three are in $\tilde{\varepsilon}\tilde{K}$. In order to get a fundamental domain P' (which is unfortunately not convex) for $\tilde{K}$, it suffices to replace the last three (undesired) simplices by the three following ones:

$$\begin{array}{ccc} \begin{array}{|c|} \hline k' \\ \hline n \quad k \\ \hline m \quad k \\ \hline \end{array} & \begin{array}{|c|} \hline n \quad k \\ \hline l \quad k' \\ \hline l \quad k \\ \hline \end{array} & \begin{array}{|c|} \hline m \quad k \\ \hline l \quad k \\ \hline l \quad k' \\ \hline \end{array} \end{array}$$

where

$$\begin{aligned} k'_4 &= -k_4 + k_5 + k_6 \\ k'_5 &= k_4 - k_5 + k_6 \\ k'_6 &= k_4 + k_5 - k_6. \end{aligned}$$

The starred polytope which is the union of the central core and the nine teeth shown in the above table is a fundamental domain for the group $\langle \Gamma, \tilde{\alpha} \rangle$ giving rise to a quotient of $\mathbf{CP}^2$ with three singular points (vertices of cones with bases isomorphic to the lens space $L(3, 1)$ considered in §6). On the other hand the quotient $\mathbf{C}^2 / \langle \Gamma, \tilde{\beta}, \tilde{\gamma} \rangle$ is bi-holomorphic to $\mathbf{CP}^2$ as can be seen by raising to the power 3 the three coordinates of the map Y of §9. The map

$$(y_0, y_1, y_2) \rightarrow (\dot{y}_0, \dot{y}_1, \dot{y}_2) = (y_0^3, y_1^3, y_2^3)$$

is a nine fold cover of $\mathbf{CP}^2$ by itself branching along the lines $\dot{y}_i = 0$ ($i = 0, 1, 2$). It maps the curve $C^\vee$ to the conic

$$\dot{y}_0^2 + \dot{y}_1^2 + \dot{y}_2^2 - 2(\dot{y}_0\dot{y}_1 + \dot{y}_1\dot{y}_2 + \dot{y}_2\dot{y}_0) = 0$$

which is tangent to the three lines $\dot{y}_i = 0$ at the three images of the nine cusps of $C^\vee$ (See [Y2]). By considering the following diagram

$$\begin{array}{ccc} \mathbf{CP}^2 \ni (y_0, y_1, y_2) & \longrightarrow & (y_0^3, y_1^3, y_2^3) \in \mathbf{CP}^2 \\ p \downarrow & & \downarrow \tilde{p} \\ \mathcal{A}_2 \ni (|y_0|^2, |y_1|^2, |y_2|^2) & \longrightarrow & (|y_0|^6, |y_1|^6, |y_2|^6) \in \dot{\mathcal{A}}_2 \end{array}$$

one is lead to a curved version $\dot{\mathcal{A}}_2$ of $\mathcal{A}_2$. The inverse image under $\tilde{p}$ of the central triangle of $\dot{\mathcal{A}}_2$ decomposes into three 2–2–1 simplices while each corner triangle is the image of one third of a 3–2 simplex. A fundamental domain of $\langle \Gamma, \tilde{\beta}, \tilde{\gamma} \rangle$ in $\mathbf{C}^2$ can easily be found by noticing that, say, $\overline{l_1 k_5 k_6 k_8 k_9 k_{10}}$ is a fundamental domain for $\Gamma'' = \langle \Gamma, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \rangle$.

The Marin hyperprisms MaP_1 and MaP_2 in K . In order to understand what follows, notice that if one adds to the central core of P the three teeth listed on the main diagonal of the three array on which they are presented one gets a triangulation MaP_1 (using six 4-simplices of a subpolytope of $\tilde{K}$) rectilinearly isomorphic to the hyperprism $\Delta_2 \times \Delta_2$. By letting Γ'' act on this prism, one sees that any 3–3 cell of $\tilde{K}$ gives rise to three such prisms intersecting at the considered 3–3 cell. Notice also that the star in $\tilde{K}$ of any 2-simplex belonging to the Γ'' -orbit of say $(l_3 k_6 k_9)$ is again a triangulation MaP_2 (using six 4-simplices of a subpolytope of $\tilde{K}$) rectilinearly isomorphic to the hyperprism $\Delta_2 \times \Delta_2$, and that MaP_1 and MaP_2 are not combinatorially equivalent.

First considered by Marin, these triangulated subhyperprisms of $\tilde{K}$ play a key roll in Chapter III. Indeed the union MaT of two adjacent hyperprisms of type MaP_1 and MaP_2 tessellates $\mathbb{C}^2$ in such a way that the union MaD (shown in Figure 11–1) of three adjacent copies of MaT is a fundamental domain of $\tilde{K}$ rectilinearly isomorphic to the product of Δ_2 with a hexagon as explained in §11 and Chapter III.

§11 The prismatic Marin fundamental domain MaD

In this section we plug into our display of $\tilde{K}$ the fundamental domain described in Chapter III (due to A. Marin) and we prove with our techniques that it is indeed a fundamental domain for the action of Γ both in $\mathbb{C}^2$ and in $\tilde{K}$.

Claim. The set MaD of vectors in $\mathbb{C}^2$ of the form $z + w$ where

$$z \in \overline{(k_9 m_1 k_5)} \quad \text{and} \quad w \in \overline{(k_4 l_2 m_4 k_2 l_4 m_2)}$$

is a fundamental domain for Γ both in $\mathbb{C}^2$ and in $\tilde{K}$.

Dictionary Observe that $\overline{(k_9 m_1 k_5)}$, $\overline{(k_9 k_4 l_2)}$, $\overline{(k_4 l_2 m_4 k_2 l_4 m_2)}$ and MaD respectively stand for Marin's $\mathcal{A}'$, $\mathcal{A}$, $\mathcal{H}'$ and $\mathcal{A}' \times \mathcal{H}$. Indeed one has

$$\begin{aligned} m_j &\longrightarrow h_j & (j = 0, 1, 2), \\ s_0 &\longrightarrow k_9, & s'_1 \longrightarrow m_1, & s'_2 \longrightarrow k_5, \\ & & s_1 \longrightarrow k_4, & s_2 \longrightarrow l_2, \end{aligned}$$

where the symbols on the left are Marin's notation (see Chapter III).

Description of MaD . The polytope MaD is the convexhull of the 21 following vetices of $\tilde{K}$:

k_9	m_1	k_5
$k_9 + k_4 = k_4$	$m_1 + k_4 = p_8$	$k_5 + k_4 = n_3$
$k_9 + l_2 = l_2$	$m_1 + l_2 = k'_6$	$k_5 + l_2 = p_7$
$k_9 + m_4 = m_4$	$m_1 + m_4 = m_{-8}$	$k_5 + m_4 = p_3$
$k_9 + k_2 = k_2$	$m_1 + k_2 = l_6$	$k_5 + k_2 = m_{-7}$
$k_9 + l_4 = l_4$	$m_1 + l_4 = l_8$	$k_5 + l_4 = m_3$
$k_9 + m_2 = m_2$	$m_1 + m_2 = p_6$	$k_5 + m_2 = k_7$

where p_3, p_6, p_7 and p_8 appear here for the first time. The polytope MaD is invariant under the action of $m_1\sigma$. The action of $m_1\sigma$ on these 21 vertices is given as follows:

$k_9 \mapsto m_1$	$m_1 \mapsto k_5$	$k_5 \mapsto k_9$
$k_4 \mapsto m_{-8}$	$p_8 \mapsto p_3$	$n_3 \mapsto m_4$
$l_2 \mapsto l_6$	$k'_6 \mapsto m_{-7}$	$p_7 \mapsto k_2$
$m_4 \mapsto l_8$	$m_{-8} \mapsto m_3$	$p_3 \mapsto l_4$
$k_2 \mapsto p_6$	$l_6 \mapsto k_7$	$m_{-7} \mapsto m_2$
$l_4 \mapsto p_8$	$l_8 \mapsto n_3$	$m_3 \mapsto k_4$
$m_2 \mapsto k'_6$	$p_6 \mapsto p_7$	$k_7 \mapsto l_2$

The boundary ∂MaD of this polytope decomposes into six triangular prisms whose union is a solid torus with a complementary torus which decomposes into three hexagonal prisms. These two solid tori are glued to each other along their common boundary, which is a hollow torus T_0 , decomposed into 6×3 quadrilaterals. The picture on the next page illustrates ∂MaD . In this picture (displaying a skewed stereographic projection of S^3), the vertex k_9 is at infinity and all the edges of ∂MaD are shown except $l_6 k_9$ and $k_9 k_3$. Once $\partial MaD - \{k_9\}$ has been identified with $S^3 - \{\infty\} \cong \mathbf{R}^3$, the boundary of the 2-simplex $(k_9 m_1 k_5)$ becomes the vertical axis of $\mathbf{R}^3$, while (although zigzagging) the core of the hexagonal solid torus (union of the six triangular prisms) works like a horizontal circle in $\mathbf{R}^3$ whose axis is $\partial(k_9 m_1 k_5)$. On the picture the action of $m_1\sigma$ is a rotation through $2\pi/3$ around the $\partial(k_9 m_1 k_5)$ -axis combined with a rotation fixing the core of the torus

and moving $\partial(k_9 m_1 k_5)$ through an angle of $2\pi/3$.

∂MaD

The mirror of the reflection $\tau = \omega^2 h_0$

Proof of the claim. Translations of Γ permute the triangular prisms in ∂MaD as indicated now.

Here we are abusing the Braille notation to indicate the triangular prisms. On each of the 3 hexagonal prisms in ∂MaD , a reflection in Γ acts as follows,

$$\begin{aligned}
 \sigma^{-1}\tau\sigma(=\tau_2) & : (k_4, l_2, m_4, k_2, l_4, m_2) \mapsto (k_4, m_2, l_4, k_2, m_4, l_2) \\
 & \quad (p_8, k'_6, m_{-8}, l_6, l_8, p_6) \mapsto (p_8, p_6, l_8, l_6, m_{-8}, k'_6) \\
 \sigma\tau\sigma^{-1}(=\tau_1) & : (n_3, p_7, p_3, m_{-7}, m_3, k_7) \mapsto (p_3, p_7, n_3, k_7, m_3, m_{-7}) \\
 & \quad (k_4, l_2, m_4, k_2, l_4, m_2) \mapsto (m_4, l_2, k_4, m_2, l_4, k_2) \\
 \tau - \omega^2 h_0 & : (n_3, p_7, p_3, m_{-7}, m_3, k_7) \mapsto (m_3, m_{-7}, p_3, p_7, n_3, k_7) \\
 (= \tau_0 + m_0 = \tau'_0) & (p_8, k'_6, m_{-8}, l_6, l_8, p_6) \mapsto (l_8, l_6, m_{-8}, k'_6, p_8, p_6)
 \end{aligned}$$

Here Marin's notation is given in parentheses. (See the picture on the previous page.)

Since the three translations and the three reflections just introduced generate the group Γ (see Chapter III), one sees that MaD is a fundamental domain of Γ in $\mathbb{C}^2$.

The fundamental domain MaD naturally splits into the six following Marin hyperprisms (for the definition of the Marin hyperprisms see the end of §10):

	Type
$\Pi_0 = \Delta' \times \Delta = (k_9, m_1, k_5; k_4, p_8, n_3; l_2, k'_6, p_7)$	MaP_1
$\Pi_1 = (k_9, m_1, k_5; m_4, m_{-8}, p_3; l_2, k'_6, p_7)$	MaP_2
$\Pi_2 = (k_9, m_1, k_5; m_4, m_{-8}, p_3; k_2, l_6, m_{-7})$	MaP_1
$\Pi_3 = (k_9, m_1, k_5; l_4, l_8, m_3; k_2, l_6, m_{-7})$	MaP_2
$\Pi_4 = (k_9, m_1, k_5; l_4, l_8, m_3; m_2, p_6, k_7)$	MaP_1
$\Pi_5 = (k_9, m_1, k_5; k_4, p_8, n_3; m_2, p_6, k_7)$	MaP_2
(cf. end of §10)	

In Chapter III, Marin's Braille cell (which differs from ours because it has to reflect the product structure in each prism) gives rise to the following array when painted with our colors:

$$\begin{array}{ccc}
 9 & 1 & 5 \\
 4 & 8 & 3 \\
 2 & 6 & 7 .
 \end{array}$$

According to our conventions, the table in Chapter III becomes as follows.

The triangulation of MaD

Figure 11-1

One can now easily check that the 36 4-simplices tabulated above are in $\tilde{K}$. If we replace letters by dots in the cells of the previous table we get the Kühnel list of §1. This complete the proof of the claim.

Images of the Marin hyperprisms in $|K| = \mathbf{CP}^2$. The following figure shows how the images in $|K|$ of the six Marin hyperprisms intersect the torus pictured on Figure 3-1.

The triangulation induced by $\tilde{K}$ on the boundaries of the 6 Marin hyperprisms. In Chapter III it will be shown that the triangulation of any hyperprism of type MaP_2 is determined by its restriction to the boundary and that the restriction to the boundary of the triangulation of a hyperprism of type MaP_1 extends to the hyperprism in two ways the original one giving rise to $\tilde{K}$ and the other one to $\varepsilon\tilde{K}$. Each boundary of a Marin hyperprism splits into two complementary solid tori, whose common boundary T_1 is the union of 9 quadrilaterals. Each solid torus splits in turn into 3 triangular prisms. In order to determine the triangulation of the whole boundary of a Marin hyperprism, it suffices to know what the triangulation does on T_1 . Here are the splitting of the boundaries of two of the Marin hyperprisms together with

the triangulations induced by $\tilde{K}$ on the corresponding T_1 's. To obtain the boundaries of the four others, apply first once and then twice the operation $m_1\sigma$.

Observe that four hyperfaces of a Marin hyperprism are visible (i.e. are contained in ∂MaD), while each of the two others is shared by another Marin hyperprism. Therefore inside ∂MaD , one has 6 invisible prisms which can be thought of as walls separating two adjacent Marin hyperprisms. The following figure shows the triangulation induced by $\tilde{K}$ on the torus T_0 considered above (see the picture of ∂MaD).

REFERENCES

- [B] M. Brückner Ueber die Ableitung der allgemeinen Polytope und die nach Isomorphismus verschiedenen Typen der allgemeinen Achtzelle (Oktatope). *Verhandel. Koninkl. Akad. Wetenschap.* (Eerste Sectie), Vol. 10. No. 1 (1909).
- [B-K] U. Brehm and W. Kühnel, A 5-neighborly triangulation of the quaternionic projective plane with 15 vertices, Univ. Duisburg-preprint.
- [E-K] J. Eells and N. H. Kuiper, Manifolds which are like projective planes, *Publ. IHES* 14 (1962)
- [G1] B. Grünbaum, *Convex polytopes*, Wiley-Interscience (1967)
- [G2] ———, Polytopes, graphs and complexes, *Bull. AMS* (1970) 1131–1201.
- [G-S] B. Grünbaum and V. P. Sreedharan, An enumeration of simplicial 4-polytopes with 8 vertices, *J. Comb. Th.* 2 (1967) 437–465.
- [Jä] B. Morin, The Kühnel triangulation of $\mathbb{CP}^2$. Note taken by K. Jänich (1981), unpublished.
- [K] N. Kuiper, Geometry in Total Absolute Curvature theory, in *Anniversary of Oberwolfach 1984, Perspectives in Mathematics*, Birkhäuser Verlag Basel, p. 377–392.
- [K-B] W. Kühnel and T. F. Banchoff, The 9-vertex complex projective plane, *Math. Intelligencer* 5–3 (1983) 11–22.
- [K-L] W. Kühnel and G. Lassmann, The unique 3-neighborly 4-manifold with few vertices. *J. Comb. Th. (A)* 35 (1983).
- [Mi] J. Milnor, On the 3-dimensional Brieskorn Manifold $M(p,q,r)$ in *Knots, Groups and 3-manifolds* ed. L. P. Neuwirth, *Ann. of Math. Studies* 84.
- [Mö] A. Möbius, *Gesammelte Werke* (~ 1865)
- [Moi] E. E. Moise, Affine structures on 3-manifolds V, *Ann. of Math.* 56 (1952) 96–114.
- [R-S] C. P. Rourke and B. I. Sanderson, *Introduction to piecewise linear topology*, Springer Erg. Math. Bd 69.

- [S] R. J. Stern, Instantons and the topology of 4-manifolds, *Math. Intelligencer* 5-3 (1983) 39-44.
- [Sie] L. C. Siebenman, Exercices sur les noeuds rationels, Preprint. Orsay (1975).
- [Wh1] J. H. C. Whitehead, Simplicial spaces nuclei and m -groups. *Proc. London Math. Soc.* 45 (1939) 243-327.
- [Wh2] ———, On C^1 -complexes, *Ann. of Math.* 41, (1940) 809-824.
- [Y1] S. Tokunaga and M. Yoshida, Complex crystallographic groups I, *J. Math. Soc. Japan* 34 (1982) 581-593.
- [Y2] J. Kaneko, S. Tokunaga and M. Yoshida, Complex crystallographic groups II, *J. Math. Soc. Japan* 34 (1982) 594-605.

B. MORIN
Univ. Louis Pasteur
7, rue René Descartes
67084 Strasbourg
FRANCE

M. YOSHIDA
Kyushu Univ. 33
Fukuoka 812
JAPAN