ON p -ADIC DEDEKIND SUMS (II)

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§ 1. Introduction

The higher-order Dedekind sums are defined by

$$S_{m+1}^{(r)}(h, k) = \sum_{a=0}^{k-1} \bar{B}_{m+1-r} \left(\frac{a}{k} \right) \bar{B}_r \left(\frac{ha}{k} \right), \quad 0 \leq r \leq m+1,$$

where h is an integer, k and m are positive integers, and $\bar{B}_n(x)$, $n \geq 0$, are the n -th periodic Bernoulli functions. Since $S_{m+1}^{(r)}(h, k) = 0$ for even m , we shall assume in what follows that m is odd. For $m = 1$, we have $S_2^{(1)}(h, k) = s(h, k)$, where $s(h, k)$ is the ordinary Dedekind sum. Carlitz [1] expressed the numbers $S_{m+1}^{(r)}(h, k)$, $0 \leq r \leq m+1$, by $c_r(h, k)$ for a fixed odd integer $m \geq 1$.

Let $\mathbf{Z}_p$ denote the ring of rational p -adic integers, for a prime number p . Let r be a fixed integer ≥ 1 . Put $e = p - 1$ if $p > 2$ and $e = 2$ if $p = 2$. As a generalization of a result of Rosen-Snyder [8], we have constructed in [5] a p -adic analytic function $S_p(s; r, h, k)$ of $s \in \mathbf{Z}_p$ which satisfies

$$S_p(m; r, h, k) = k^m S_{m+1}^{(r)}(h, k) - p^{m-r} k^m S_{m+1}^{(r)}(ph, k)$$

for all integers m such that $m \geq r$ and $m+1 \equiv 0 \pmod{e}$. In [5], we also discussed the properties of $S_p(s; r, h, k)$ as a continuous function on $\mathbf{Z}_p$. The purpose of the present paper is to investigate these properties more closely. At the same time, we shall extend the argument to the case that $m+1 \equiv \alpha \pmod{e}$, where α is an even positive integer $\leq e$.

In the next section we define the function $S_{p,\alpha}(s; r, h, k)$, which is analytic in $\mathbf{Z}_p$ and satisfies the above equation for $m+1 \equiv \alpha \pmod{e}$. In sections 3–6, we describe the properties of the functions $\bar{S}_{p,\alpha}(s; r, h, k)$ and $T_{p,\alpha}(s; r, h, p^v)$, which are introduced to define $S_{p,\alpha}(s; r, h, k)$. In sections 7, 8, we state the results for $S_{p,\alpha}(s; r, h, k)$ and some p -adic properties of $S_{m+1}^{(r)}(h, k)$ derived from our considerations about $S_{p,\alpha}(s; r, h, k)$.

Let $\mathbf{Q}_p$, $\mathbf{C}_p$, $\mathfrak{o}$ and $\mathbf{Z}$ denote the rational p -adic number field, the completion of an algebraic closure of $\mathbf{Q}_p$, the ring of integers in $\mathbf{C}_p$ and the ring of rational integers, respectively. Throughout, we fix p and consider all algebraic numbers to be contained in $\mathbf{C}_p$. Let $||$ denote the absolute value on $\mathbf{C}_p$ normalized so that $|p| = p^{-1}$. We also define $\text{ord}_p(x)$, for a rational number x , by $|x| = p^{-\text{ord}_p(x)}$, and put $q = p$ or $q = 4$ according as $p > 2$ or $p = 2$.

§2. Definition of $S_{p,\alpha}(s; r, h, k)$

We define the Bernoulli numbers and polynomials by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad B_n(X) = \sum_{i=0}^n \binom{n}{i} B_i X^{n-i}, \quad n \geq 0,$$

respectively. For a real number x , we have $\bar{B}_n(x) = B_n(\{x\})$, except for the case $x \in \mathbf{Z}$ and $n = 1$, when we have $\bar{B}_n(x) = 0$. Here, $\{x\}$ denotes the smallest real number $t \geq 0$ such that $x - t \in \mathbf{Z}$. For $1 \leq r \leq m$, we get

$$(1) \quad S_{m+1}^{(r)}(h, k) = \sum_{a=0}^{k-1} B_{m+1-r} \left(\frac{a}{k} \right) B_r \left(\left\{ \frac{ha}{k} \right\} \right), \quad \text{if } m \geq 3,$$

and

$$S_2^{(1)}(h, k) + \frac{1}{4} = \sum_{a=0}^{k-1} B_1 \left(\frac{a}{k} \right) B_1 \left(\left\{ \frac{ha}{k} \right\} \right).$$

We define the numbers $E_n(\rho)$, $n \geq 0$, for a root of unity $\rho \neq 1$, by

$$\frac{\rho}{e^t - \rho} = \sum_{n=0}^{\infty} E_n(\rho) \frac{t^n}{n!},$$

and put $E_n(1) = \frac{B_{n+1}}{n+1}$, $n \geq 0$. If $\rho \neq 1$, then $(1 - \rho)^{n+1} E_n(\rho)$, $n \geq 0$, are polynomials of ρ with coefficients in $\mathbf{Z}$ ([10], [12]). We have

$$(2) \quad \sum_{\xi^k=1} E_n(\rho \xi) = k^{n+1} E_n(\rho^k), \quad n \geq 0,$$

for any root of unity ρ and integer $k \geq 1$, and

$$(3) \quad k^n B_n \left(\left\{ \frac{a}{k} \right\} \right) = n \sum_{\xi^k=1} E_{n-1}(\xi) \xi^a, \quad n \geq 1,$$

for any integers a and $k \geq 1$ (see [1] or [5]). By (1) and (3), we obtain the following formula of Carlitz [1]: for $1 \leq r \leq m$,

$$(4) \quad k^m S_{m+1}^{(r)}(h, k) = (m+1-r)r \sum_{\zeta^k=1} E_{m-r}(\zeta^h) E_{r-1}(\zeta^{-1}), \quad \text{if } m \geq 3,$$

and

$$k \left(S_2^{(1)}(h, k) + \frac{1}{4} \right) = \sum_{\zeta^k=1} E_0(\zeta^h) E_0(\zeta^{-1}).$$

It is known [1, Theorem 3] that for all integers $d \geq 1$

$$(5) \quad S_{m+1}^{(r)}(dh, dk) = d^{r-m} S_{m+1}^{(r)}(h, k).$$

Hence, in what follows, we shall assume that $(h, k) = 1$ for simplicity. But we remark here that the formulas (1) and (4) are proved without this restriction. We shall often use later the formula

$$(5)' \quad S_{m+1}^{(r)}(h, k) = S_{m+1}^{(m+1-r)}(h^*, k), \quad hh^* \equiv 1 \pmod{k}.$$

Now suppose $\rho^{p^N} \neq 1$ for all natural numbers N . Then, we have $|E_n(\rho)| \leq 1$ for all $n \geq 0$. Further, in this case, we can define a finitely additive ρ -valued measure μ_ρ on $\mathbf{Z}_p$ by

$$\mu_\rho(a + p^N \mathbf{Z}_p) = \frac{\rho^{p^N - a}}{1 - \rho^{p^N}}, \quad 0 \leq a \leq p^N - 1.$$

And we know by [10], [11] that

$$(6) \quad \int_{\mathbf{Z}_p} x^n d\mu_\rho(x) = E_n(\rho), \quad n \geq 0,$$

and

$$(7) \quad \int_{\mathbf{Z}_p^*} x^n d\mu_\rho(x) = E_n(\rho) - p^n E_n(\rho^p), \quad n \geq 0,$$

where $\mathbf{Z}_p^*$ denotes the group of units in $\mathbf{Z}_p$.

Let α be an even integer with $0 < \alpha \leq e$, and let r be a positive integer. For each root of unity ρ not of p power order, we define

$$G_{p,\alpha}(s; r, \rho) = \int_{\mathbf{Z}_p^*} \omega^{\alpha-1}(x) \langle x \rangle^s \frac{1}{x^r} d\mu_\rho(x), \quad s \in \mathbf{Z}_p,$$

where ω is the Teichmüller character mod q , and $\langle x \rangle = \omega^{-1}(x)x$ for $x \in \mathbf{Z}_p^*$. Let $\exp$ and $\log$ denote the p -adic exponential and logarithm functions, respectively [2], [13]. Then, since $\langle x \rangle^s = \exp(s \log x)$ for all $x \in \mathbf{Z}_p^*$ and $s \in \mathbf{Z}_p$, the function $G_{p,\alpha}(s; r, \rho)$ is expanded as

$$G_{p,\alpha}(s; r, \rho) = \sum_{n=0}^{\infty} c_{n,r}(\rho)(s+1-r)^n$$

with

$$c_{n,r}(\rho) = \int_{\mathbf{Z}_p^*} \omega^{\alpha-r}(x) \frac{(\log x)^n}{n!} \frac{1}{x} d\mu_\rho(x),$$

$$|c_{n,r}(\rho)| \leq \left| \frac{q^n}{n!} \right| \leq (q^{-1} p^{\frac{1}{p-1}})^n.$$

Hence $G_{p,\alpha}(s; r, \rho)$ is an $\mathfrak{o}$ -valued analytic function on $\mathbf{Z}_p$. Further, it follows from the definition of $G_{p,\alpha}(s; r, \rho)$ and (7) that

$$\begin{aligned} (8) \quad G_{p,\alpha}(m; r, \rho) &= \int_{\mathbf{Z}_p^*} x^{m-r} d\mu_\rho(x) \\ &= E_{m-r}(\rho) - p^{m-r} E_{m-r}(\rho^p) \end{aligned}$$

for all integers m such that $m+1 \equiv \alpha \pmod{e}$ and $m \geq r$.

Now, let h and k be integers with $k = k_0 p^v$, $(k_0, p) = 1$, $k_0 \geq 1$, $v \geq 0$ and $(h, k) = 1$. Denote by $p^{\bar{v}}$ the least common multiple of p^v and q , and let $c = 1 + p^{\bar{v}}$. In these situations, we consider the functions

$$\bar{S}_{p,\alpha}(s; r, h, k) = (s+1-r)r \sum_{\zeta^k=1, \zeta^{p^v} \neq 1} G_{p,\alpha}(s; r, \zeta^h) E_{r-1}(\zeta^{-1})$$

and

$$T_{p,\alpha}(s; r, h, p^v) = r \sum_{\zeta^{p^v}=1} \sum_{\eta^c=1, \eta \neq 1} G_{p,\alpha}(s; r, \zeta^h \eta) E_{r-1}(\zeta^{-1}).$$

Then, by (8) and (2), we see

$$\begin{aligned} (9) \quad \bar{S}_{p,\alpha}(m; r, h, k) &= (m+1-r)r \sum_{\zeta^k=1, \zeta^{p^v} \neq 1} (E_{m-r}(\zeta^h) - p^{m-r} E_{m-r}(\zeta^{ph})) E_{r-1}(\zeta^{-1}), \end{aligned}$$

and

$$(10) \quad T_{p,\alpha}(m; r, h, p^v) = (c^{m+1-r} - 1)r \sum_{\zeta^{p^v}=1} (E_{m-r}(\zeta^h) - p^{m-r} E_{m-r}(\zeta^{ph})) E_{r-1}(\zeta^{-1}),$$

for all integers $m \geq r$ with $m+1 \equiv \alpha \pmod{e}$. Clearly they are $\mathbf{Q}_p$ -valued analytic functions on $\mathbf{Z}_p$. In particular $\bar{S}_{p,\alpha}(s; r, h, k)$ takes the values in $r\mathbf{Z}_p$ and $\bar{S}_{p,\alpha}(r-1; r, h, k) = 0$.

Now, the analytic function

$$U_{p,r}^{(v)}(s) = \sum_{n=0}^{\infty} B_n \frac{(\log c)^{n-1}}{n!} (s+1-r)^n \quad (c = 1 + p^{\bar{v}})$$

satisfies

$$(11) \quad U_{p,r}^{(v)}(m) = \frac{m+1-r}{c^{m+1-r}-1}$$

for all integers $m \geq r$. We therefore define the function $S_{p,\alpha}(s; r, h, k)$ by

$$S_{p,\alpha}(s; r, h, k) = \bar{S}_{p,\alpha}(s; r, h, k) + U_{p,r}^{(v)}(s) T_{p,\alpha}(s; r, h, p^v).$$

Then, combining (9), (10), (11), and recalling the formula (4), we obtain the following

THEOREM 1. *For any integers $r \geq 1$, $h, k \geq 1$, with $(h, k) = 1$, and even integer α , $0 < \alpha \leq e$, there exists a $\mathbf{Q}_p$ -valued analytic function $S_{p,\alpha}(s; r, h, k)$ on $\mathbf{Z}_p$, which satisfies*

$$S_{p,\alpha}(m; r, h, k) = k^m S_{m+1}^{(r)}(h, k) - p^{m-r} k^m S_{m+1}^{(r)}(ph, k)$$

for all integers m such that $m+1 \equiv \alpha \pmod{e}$ and $m \geq r$.

REMARK. we have for $k = p^v$

$$(12) \quad S_{p,\alpha}(s; r, h, p^v) = U_{p,r}^{(v)}(s) T_{p,\alpha}(s; r, h, p^v).$$

Hence, for $k = k_0 p^v$, $(k_0, p) = 1$, $v \geq 0$,

$$(13) \quad S_{p,\alpha}(s; r, h, k) = \bar{S}_{p,\alpha}(s; r, h, k) + S_{p,\alpha}(s; r, h, p^v).$$

By virtue of the fundamental properties of Dedekind sums: $S_{m+1}^{(r)}(-h, k) =$

$(-1)^r S_{m+1}^{(r)}(h, k)$ and $S_{m+1}^{(r)}(h, k) = S_{m+1}^{(r)}(h', k)$, $h \equiv h' \pmod{k}$, we see

$$S_{m+1}^{(r)}(h, 1) = S_{m+1}^{(r)}(h, 2) = 0 \quad \text{for } r \text{ odd, } 1 \leq r \leq m.$$

Hence, if r is odd, we have

$$(14) \quad S_{p,\alpha}(s; r, h, 1) = S_{p,\alpha}(s; r, h, 2) \equiv 0, \quad s \in \mathbf{Z}_p.$$

§3. The function $\bar{S}_{p,\alpha}(s; r, h, k)$

By the facts that we explained in the preceding section, the function $\bar{S}_{p,\alpha}(s; r, h, k)$ is expanded as

$$\bar{S}_{p,\alpha}(s; r, h, k) = \sum_{n=1}^{\infty} \bar{a}_n (s+1-r)^n, \quad \bar{a}_n \in r\mathbf{Z}_p.$$

If $(k, q) = 1$ or 2 , and r is odd, then by (13) and (14) we have

$$(15) \quad S_{p,\alpha}(s; r, h, k) = \bar{S}_{p,\alpha}(s; r, h, k).$$

The goal of this section is to prove the following

PROPOSITION 1. *Let $k = k_0 p^v$, $(k_0, p) = 1$ and $k_0 > 1$. Then,*

$$(16) \quad |\bar{a}_n| \leq \left| \frac{r q^{n-1} p^v}{(n-1)!} \right|, \quad n \geq 1.$$

In the case $p = 2$, if either $v \geq 1$ or $r > 1$,

$$(17) \quad |\bar{a}_n| \leq \left| \frac{r q^{n-1} 2^{v+1}}{(n-1)!} \right|, \quad n \geq 1.$$

The following Lemmas are easily verified by a straightforward calculation of the left hand side of each expression.

LEMMA 1. *Let ζ be a root of unity of order not a power of p . Then, for any $\mathbf{C}_p$ -valued continuous function f on $a + p^v \mathbf{Z}_p$, with $0 \leq a < p^v$, we have*

$$\int_{a+p^v \mathbf{Z}_p} f(x) d\mu_{\zeta}(x) = \zeta^{-a} \int_{\mathbf{Z}_p} f(a + p^v x) d\mu_{\zeta p^v}(x).$$

LEMMA 2. *Let ζ be as above and let f be continuous on $\mathbf{Z}_p$. Then,*

$$\int_{\mathbf{Z}_p} f(x) d\mu_{\zeta}(x) = -\zeta \int_{\mathbf{Z}_p} f(-1-x) d\mu_{\zeta^{-1}}(x).$$

Now it follows from the definition that

$$\begin{aligned}
 (18) \quad \bar{a}_n &= r \sum_{\zeta^k=1, \zeta^{p^v} \neq 1} c_{n-1,r}(\zeta^h) E_{r-1}(\zeta^{-1}) \\
 &= r \sum_{\zeta^{k_0}=1, \zeta \neq 1} \sum_{\xi^{p^v}=1} c_{n-1,r}(\zeta^h \xi^h) E_{r-1}(\zeta^{-1} \xi^{-1}) \\
 &= r \sum_{\zeta^{k_0}=1, \zeta \neq 1} \sum_{\xi^{p^v}=1} \int_{\mathbf{Z}_p} \omega^{\alpha-r}(x) \frac{(\log x)^{n-1}}{(n-1)!} \frac{1}{x} d\mu_{\zeta^h \xi^h}(x) E_{r-1}(\zeta^{-1} \xi^{-1}).
 \end{aligned}$$

For each integer a with $0 < a < p^{\bar{v}}$, $(a, p) = 1$, let

$$(19) \quad I_n^{(v)}(a; \zeta) = \zeta^{-ha} \int_{\mathbf{Z}_p} \frac{(\log(a + p^{\bar{v}}x))^{n-1}}{(a + p^{\bar{v}}x)(n-1)!} d\mu_{\zeta^{hp^{\bar{v}}}}(x), \quad n \geq 1, v \geq 0.$$

Then, applying Lemma 1 for the function $f(x) = \omega^{\alpha-r}(x) \frac{(\log x)^{n-1}}{x}$ defined on each subset $a + p^{\bar{v}}\mathbf{Z}_p$, $0 < a < p^{\bar{v}}$, $(a, p) = 1$, we have

$$(20) \quad \bar{a}_n = r \sum_{a=1}^{p^{\bar{v}}-1} \omega^{\alpha-r}(a) \sum_{\zeta^{k_0}=1, \zeta \neq 1} I_n^{(v)}(a; \zeta) \sum_{\xi^{p^v}=1} E_{r-1}(\zeta^{-1} \xi^{-1}) \xi^{-ha}, \quad n \geq 1,$$

where $\sum^*$ means to take sum over all integers prime to p in the given range. Clearly we have

$$(21) \quad |I_n^{(v)}(a; \zeta)| \leq \left| \frac{q^{n-1}}{(n-1)!} \right|,$$

for all $n \geq 1$ and $v \geq 0$, in the formula (20).

Now, consider the polynomials $E_n(X; \rho)$, $n \geq 0$, defined for each root of unity $\rho \neq 1$ by

$$E_n(X; \rho) = \sum_{i=0}^n \binom{n}{i} E_i(\rho) X^{n-i}.$$

Then we obtain, for all integers $k \geq 1$ such that $\rho^k \neq 1$,

$$(22) \quad \sum_{\xi^k=1} E_n(\rho \xi) \xi^a = k^{n+1} \rho^{-a} E_n\left(\frac{a}{k}; \rho^k\right), \quad 0 \leq a < k, \quad n \geq 0.$$

Indeed, we see from the identity $\sum_{\xi^k=1} \frac{\xi^{a+1}}{X - \xi} = \frac{kX^a}{X^k - 1}$ ($0 \leq a < k$) that

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\sum_{\xi^k=1} E_n(\rho \xi) \xi^a \right) \frac{t^n}{n!} &= \sum_{\xi^k=1} \frac{\rho \xi^{a+1}}{e^t - \rho \xi} = k \rho^{-a} \frac{\rho^k e^{at}}{e^{kt} - \rho^k} \\ &= k \rho^{-a} \sum_{n=0}^{\infty} E_n \left(\frac{a}{k}; \rho^k \right) \frac{(kt)^n}{n!}. \end{aligned}$$

For an integer x , let $\{x\}_{p^v}$ denote the integer y such that $0 \leq y < p^v$ and $y \equiv x \pmod{p^v}$. Then, by applying (22) for $k = p^v$ and $\rho = \zeta^{-1}$, we have

$$\begin{aligned} \sum_{\xi^{p^v}=1} E_{r-1}(\zeta^{-1} \xi^{-1}) \xi^{-ha} &= (p^v)^r \zeta^{\{ha\}_{p^v}} E_{r-1} \left(\frac{\{ha\}_{p^v}}{p^v}; \zeta^{-p^v} \right) \\ &= p^v \zeta^{\{ha\}_{p^v}} \sum_{i=0}^{r-1} \binom{r-1}{i} E_i(\zeta^{-p^v}) \{ha\}_{p^v}^{r-1-i} (p^v)^i, \end{aligned}$$

which implies that

$$(23) \quad \left| \sum_{\xi^{p^v}=1} E_{r-1}(\zeta^{-1} \xi^{-1}) \xi^{-ha} \right| \leq |p^v|,$$

for all $v \geq 0$, in the formula (20), because $|E_i(\zeta^{-p^v})| \leq 1$, $0 \leq i \leq r-1$. Hence we have $|\bar{a}_n| \leq \left| \frac{rq^{n-1}p^v}{(n-1)!} \right|$ for all $n \geq 1$ and $v \geq 0$. Thus we have proved the first assertion of Proposition 1.

Next, by Lemma 2, we have

$$(24) \quad \int_{\mathbf{Z}_p} \frac{(\log(p^{\bar{v}} - a + p^{\bar{v}}x))^n}{p^{\bar{v}} - a + p^{\bar{v}}x} d\mu_{\zeta}(x) = \zeta \int_{\mathbf{Z}_p} \frac{(\log(a + p^{\bar{v}}x))^n}{a + p^{\bar{v}}x} d\mu_{\zeta^{-1}}(x),$$

for all $n \geq 0$, $v \geq 0$ and $0 < a < p^{\bar{v}}$ with $(a, p) = 1$. Hence, it follows in the formula (20) that

$$(25) \quad I_n^{(v)}(p^{\bar{v}} - a; \zeta) = I_n^{(v)}(a; \zeta^{-1}), \quad n \geq 1, v \geq 0.$$

On the other hand, we see by definition that

$$(26) \quad E_0(\rho^{-1}) = -1 - E_0(\rho) \quad \text{and} \quad E_{r-1}(\rho^{-1}) = (-1)^r E_{r-1}(\rho), \quad r > 1,$$

for every root of unity ρ . Hence, we have

$$\sum_{\xi^{p^v}=1} E_{r-1}(\zeta^{-1} \xi^{-1}) \xi^{-h(p^{\bar{v}}-a)} = \sum_{\xi^{p^v}=1} E_{r-1}(\zeta^{-1} \xi^{-1}) \xi^{ha}$$

$$= \begin{cases} (-1)^r \sum_{\xi^{p^v}=1} E_{r-1}(\xi \zeta) \xi^{ha} & r > 1, \\ - \sum_{\xi^{p^v}=1} (1 + E_0(\xi \zeta)) \xi^{ha} & r = 1. \end{cases}$$

Suppose $v \geq 1$ or $r > 1$. Then we consequently have

$$(27) \quad \sum_{\xi^{p^v}=1} E_{r-1}(\zeta^{-1} \xi^{-1}) \xi^{-h(p^v-a)} = (-1)^r \sum_{\xi^{p^v}=1} E_{r-1}(\zeta \xi^{-1}) \xi^{-ha}.$$

Since $\omega^{\alpha-r}(p^v-a) = (-1)^r \omega^{\alpha-r}(a)$, the terms in (20) are invariant under the change of indices from a to p^v-a , if $v \geq 1$ or $r > 1$. For $p = 2$, we thus obtain the formula

$$(28) \quad \bar{a}_n = 2r \sum_{\substack{a=1 \\ a \equiv 1 \pmod{4}}}^{2^v-1} \sum_{\substack{\zeta^{k_0}=1, \zeta \neq 1}} I_n^{(v)}(a; \zeta) \sum_{\xi^{2^v}=1} E_{r-1}(\zeta^{-1} \xi^{-1}) \xi^{-ha}, \quad n \geq 1,$$

under the condition that $v \geq 1$ or $r > 1$. Combining this with (21) and (23), we have proved the second assertion of Proposition 1.

§4. The function $T_{p,\alpha}(s; r, h, p^v)$

In this section we examine the properties of the expansion

$$T_{p,\alpha}(s; r, h, p^v) = \sum_{n=0}^{\infty} t_n(s+1-r)^n,$$

with $(h, p^v) = 1$, in general.

Let B_{n,ω^r} , $n \geq 0$, be the generalized Bernoulli numbers, attached to the character ω^r , defined by

$$\sum_{a=0}^{q-1} \frac{\omega^r(a) t e^{at}}{e^{qt} - 1} = \sum_{n=0}^{\infty} B_{n,\omega^r} \frac{t^n}{n!}.$$

Then, we have

$$(29) \quad B_{n,\omega^r} = (kq)^{n-1} \sum_{a=0}^{kq-1} \omega^r(a) B_n\left(\frac{a}{kq}\right), \quad n \geq 0,$$

and

$$(30) \quad \sum_{a=0}^{kq-1} \omega^r(a) a^n = \sum_{j=1}^{n+1} \binom{n}{j-1} \frac{1}{j} B_{n+1-j,\omega^r}(kq)^j, \quad n \geq 0,$$

for any integer $k \geq 1$. For the principal character 1 we have $B_{n,1} = B_n$, $n \geq 0$. The von Staudt-Clausen Theorem is generalized as

$$(31) \quad \left| \frac{B_{n,\omega^r}}{n} \right| \leq 1 \quad \text{for } n \not\equiv -r \pmod{e}, n \geq 2,$$

$$\text{and } B_{n,\omega^{-n}} \equiv -\frac{1}{p} \pmod{p^0} \quad \text{for } n \geq 1$$

(cf. e.g. [7], [9]).

We define for integers a , $0 < a < p^{\bar{v}}$, $(a, p) = 1$,

$$(32) \quad A_n^{(v)}(a) = \sum_{\eta^c=1, \eta \neq 1} \eta^a \int_{\mathbf{Z}_p} \frac{(\log(a + p^{\bar{v}}x))^n}{a + p^{\bar{v}}x} d\mu_{\eta}(x), \quad n \geq 0, v \geq 0,$$

where $c = 1 + p^{\bar{v}}$ as before. Then $A_n^{(v)}(a) = A_n^{(\bar{v})}(a)$ and we have $|A_n^{(v)}(a)| \leq |q^n|$, $n \geq 0$, $v \geq 0$.

PROPOSITION 2. Let $v \geq 0$ and $(h, p^v) = 1$. Then,

$$(33) \quad t_n = \frac{1}{n!} \sum_{a=1}^{p^{\bar{v}}-1} \omega^{\alpha-r}(a) A_n^{(v)}(a) (p^v)^r B_r \left(\left\{ \frac{ha}{p^v} \right\} \right), \quad n \geq 0.$$

If $p = 2$, and if either $v \geq 1$ or $r > 1$, then

$$(34) \quad t_n = \frac{2}{n!} \sum_{\substack{a=1 \\ a \equiv 1 \pmod{4}}}^{2^{\bar{v}}-1} A_n^{(v)}(a) (2^v)^r B_r \left(\left\{ \frac{ha}{2^v} \right\} \right), \quad n \geq 0.$$

PROOF. By definition, we have

$$\begin{aligned} t_n &= r \sum_{\zeta^{p^v}=1} \sum_{\eta^c=1, \eta \neq 1} c_{n,r}(\zeta^h \eta) E_{r-1}(\zeta^{-1}) \\ &= r \sum_{\zeta^{p^v}=1} \sum_{\eta^c=1, \eta \neq 1} \int_{\mathbf{Z}_p^*} \omega^{\alpha-r}(x) \frac{(\log x)^n}{n!} \frac{1}{x} d\mu_{\zeta^h \eta}(x) E_{r-1}(\zeta^{-1}). \end{aligned}$$

Applying Lemma 1 for $f(x) = \omega^{\alpha-r}(x) \frac{(\log x)^n}{x}$ on $a + p^{\bar{v}}\mathbf{Z}_p$, ($0 < a < p^{\bar{v}}$, $(a, p) = 1$), and using (3), we obtain

$$t_n = \frac{1}{n!} \sum_{a=1}^{p^{\bar{v}}-1} \omega^{\alpha-r}(a) A_n^{(v)}(a) r \sum_{\zeta^{p^v}=1} \zeta^{-ha} E_{r-1}(\zeta^{-1})$$

$$= \frac{1}{n!} \sum_{a=1}^{p^{\bar{v}}-1} \omega^{a-r}(a) A_n^{(v)}(a) (p^v)^r B_r \left(\left\{ \frac{ha}{p^v} \right\} \right), \quad n \geq 0.$$

Thus the first assertion is proved. Next, by (24), we see

$$A_n^{(v)}(p^{\bar{v}} - a) = A_n^{(v)}(a), \quad n \geq 0,$$

for all $v \geq 0$ and $0 < a < p^{\bar{v}}$, $(a, p) = 1$. Further we have

$$B_r \left(\left\{ \frac{h(p^{\bar{v}} - a)}{p^v} \right\} \right) = B_r \left(\left\{ \frac{-ha}{p^v} \right\} \right) = B_r \left(1 - \left\{ \frac{ha}{p^v} \right\} \right) = (-1)^r B_r \left(\left\{ \frac{ha}{p^v} \right\} \right), \quad r \geq 1,$$

for $v \geq 1$ and for all integers h and a prime to p . This also is true for $v = 0$, if $r > 1$. Since $\omega^{a-r}(p^{\bar{v}} - a) = (-1)^r \omega^{a-r}(a)$, we can combine the terms for a and $p^{\bar{v}} - a$ in (33) to obtain the formula (34) for $p = 2$. This completes the proof of Proposition 2.

REMARK. The facts described in the proof of the second half of Proposition 2 also show, for any prime p , that $T_{p,\alpha}(s; r, h, 1) = 0$ for odd r .

For $p = 2$, we see by (34) and a reduction using (29) that

$$(34)' \quad t_n = \frac{2}{n!} A_n^{(1)}(1) B_r, \quad n \geq 0, \quad \text{for } v = 0 \text{ and } r \text{ even},$$

$$(34)'' \quad t_n = \frac{4}{n!} A_n^{(1)}(1) (1 - 2^{r-1}) B_r, \quad n \geq 0, \quad \text{for } v = 1,$$

$$(34)''' \quad t_n = \begin{cases} \frac{4}{n!} A_n^{(1)}(1) (1 - 2^{r-1}) B_r, & n \geq 0, \quad \text{for } v = 2 \text{ and } r \text{ even}, \\ \frac{4}{n!} A_n^{(1)}(1) \omega(h) B_{r,\omega}, & n \geq 0, \quad \text{for } v = 2 \text{ and } r \text{ odd}. \end{cases}$$

By (34)', we see that $T_{2,2}(s; r, h, 2) = 0$ for odd r . By (34)'' and (34)', we also see that $T_{2,2}(s; r, h, 2) = T_{2,2}(s; r, h, 4)$ for even r .

The fact that $T_{p,\alpha}(s; r, h, p^v) = 0$ for odd r and $v < \bar{v}$ can also be deduced from (12) and (14) in section 2.

By Proposition 2 we can obtain the following estimates of $|t_n|$.

PROPOSITION 3. Assume that r is even, if $v < \bar{v}$. Then,

$$(i) \quad |t_n| \leq \left| \frac{2q^n p^\theta}{n!} \right|, \quad n \geq 0, \quad \text{for } p > 2, v = 0, 1, \\ \text{or } p = 2, v = 0, 1, 2,$$

where $\theta = \text{ord}_p(B_r)$ for $v = 0$, $\theta = 0$ for $v \geq 1$.

$$(ii) \quad |t_n| = \left| \frac{2q^n}{n!} \right|, \quad n \geq 1, \quad \text{for } p = 2, v = 3.$$

$$(iii) \quad n!t_n \equiv -h^r p^{v-1} q^n B_n \pmod{p^{v-1} q^n}, \quad n \geq 0, \quad \text{for } p = 2, v \geq 4, \\ \text{or } p > 2, v \geq 2 \text{ and } \alpha = e.$$

$$(iv) \quad |t_n| \leq \left| \frac{p^{v-1+n}}{n!} \right|, \quad n \geq 0, \quad \text{for } p \geq 5, v \geq 2 \text{ and } \alpha < e.$$

PROOF. By the von Staudt-Clausen Theorem, we see that

$$(35) \quad (p^v)^r B_r \left(\left\{ \frac{ha}{p^v} \right\} \right) \equiv h^r a^r \pmod{p^{v+1} q^{-1}} \quad \text{for } v \geq 1.$$

Since $|A_n^{(v)}(a)| \leq |q^n|$, the assertion (i) follows from the formulas (33), (35) and (34)'-(34)'''. Hence, we may assume in the rest of the proof that $v \geq 2$ if $p > 2$, $v \geq 3$ if $p = 2$.

We let $0 < a < p^v$, $(a, p) = 1$ and $n \geq 0$. Then,

$$\frac{(\log(a + p^v x))^n}{a + p^v x} \equiv \frac{(\log a)^n}{a} \pmod{p^v q^{n-1}}, \quad \text{for all } x \in \mathbb{Z}_p.$$

Substituting this into (32), and using (6), we have

$$A_n^{(v)}(a) \equiv \frac{(\log a)^n}{a} \sum_{\eta^c = 1, \eta \neq 1} E_0(\eta) \eta^a \pmod{p^v q^{n-1}}.$$

We see from (3) and the von Staudt-Clausen Theorem

$$\sum_{\eta^c = 1, \eta \neq 1} E_0(\eta) \eta^a \equiv a \pmod{p^{v-1}}.$$

Therefore we obtain

$$(36) \quad A_n^{(v)}(a) \equiv (\log a)^n \pmod{p^v q^{n-1}}.$$

Now suppose $p > 2$ ($v \geq 2$). Then it follows from (33), (35) and (36) that

$$\begin{aligned} n!t_n &\equiv \sum_{a=1}^{p^v-1} \omega^{a-r}(a) h^r a^r (\log a)^n \pmod{p^{v+n-1}} \\ &= h^r \sum_{a=1}^{p^v-1} \omega^a(a) \langle a \rangle^r (\log a)^n. \end{aligned}$$

For each a , $0 < a < p^v$, $(a, p) = 1$, there exist the unique integers a_1 and i such that $a \equiv \omega(a_1)(1+p)^i \pmod{p^v}$, $1 \leq a_1 \leq p-1$, $0 \leq i \leq p^{v-1}-1$. Since $\omega(a) = \omega(a_1)$, $\langle a \rangle \equiv (1+p)^i \pmod{p^v}$ and $(\log a)^n \equiv i^n (\log(1+p))^n \pmod{p^{v+n-1}}$, we see

$$n!t_n \equiv h^r (\log(1+p))^n \sum_{a_1=1}^{p-1} \omega^a(a_1) \sum_{i=0}^{p^{v-1}-1} i^n (1+p)^{ir} \pmod{p^{v+n-1}}.$$

The assertion (iv) is immediate from this, so that we let $\alpha = e$. By (30) and the von Staudt-Clausen Theorem we know $\sum_{i=0}^{p-1} i^n (1+p)^{ir} \equiv \sum_{i=0}^{p-1} i^n \equiv pB_n \pmod{p}$. Then we see by induction on v that

$$\sum_{i=0}^{p^{v-1}-1} i^n (1+p)^{ir} \equiv p^{v-1} B_n \pmod{p^{v-1}}$$

for all $v \geq 2$. Hence we have the assertion (iii) and Proposition 3 is proved for $p > 2$.

Next let $p = 2$ ($v \geq 3$). Then, it follows from (34), (35) and (36) that

$$n!t_n \equiv 2 \sum_{\substack{a=1 \\ a \equiv 1 \pmod{4}}}^{2^v-1} h^r a^r (\log a)^n \pmod{2^{v-1}q^n}.$$

By letting $a \equiv (1+q)^i \pmod{2^v}$, $0 \leq i \leq 2^{v-2}-1$, we have

$$n!t_n \equiv 2h^r (\log(1+q))^n \sum_{i=0}^{2^{v-2}-1} i^n (1+q)^{ir} \pmod{2^{v-1}q^n}.$$

This becomes $n!t_n \equiv 2q^n \pmod{q^{n+1}}$ for $v = 3$, $n \geq 1$, so (ii) is proved. Next we have $\sum_{i=0}^{q-1} i^n (1+q)^{ir} \equiv qB_n \pmod{q}$, $n \geq 0$, as in the case $p > 2$, and

$$\sum_{i=0}^{2^{v-2}-1} i^n (1+q)^{ir} \equiv 2^{v-2} B_n \pmod{2^{v-2}}, \quad n \geq 0, v \geq 4,$$

by induction on v . This gives the assertion (iii) for $p = 2$, and completes the

proof of Proposition 3.

§5. $T_{p,\alpha}(r-1; r, h, p^v)$

The value $T_{p,\alpha}(r-1; r, h, p^v)$ ($= t_0$) can be calculated following the theory of Γ -transform (e.g. in [5]). We give here a different proof to

PROPOSITION 4. (i) If $v = 0$, then

$$t_0 = \begin{cases} \left(1 - \frac{1}{p}\right) B_r \log(1+q) & \text{for } r \equiv \alpha \pmod{e}, \\ 0 & \text{for } r \not\equiv \alpha \pmod{e}. \end{cases}$$

(ii) If $p = 2$ and $v = 1$, then

$$t_0 = \begin{cases} (1 - 2^{r-1}) B_r \log(1+q) & \text{for } r \text{ even}, \\ 0 & \text{for } r \text{ odd}. \end{cases}$$

(iii) If $p > 2$, $v \geq 1$ or $p = 2$, $v \geq 2$, then

$$t_0 = \begin{cases} (1 - p^{r-1}) B_r \log(1+p^v) & \text{for } r \equiv \alpha \pmod{e}, \\ \omega^{r-\alpha}(h) B_{r, \omega^{\alpha-r}} \log(1+p^v) & \text{for } r \not\equiv \alpha \pmod{e}. \end{cases}$$

PROOF. We use the following

LEMMA 3. Let k be an integer ≥ 1 divisible by p . Then, for any integer a with $0 \leq a < k$ and $(a, p) = 1$,

$$\sum_{\zeta^{1+k}=1, \zeta \neq 1} \zeta^a \int_{\mathbf{Z}_p} \frac{1}{a+kx} d\mu_\zeta(x) = \frac{1}{k} \log(1+k).$$

By this Lemma, we get

$$A_0^{(v)}(a) = \frac{1}{p^v} \log(1+p^v) \quad \text{for } 0 < a < p^v, (a, p) = 1.$$

Hence, if $v = 0$, we have from the formula (33)

$$t_0 = \sum_{a=1}^{q-1} \omega^{\alpha-r}(a) A_0^{(1)}(a) B_r = \sum_{a=1}^{q-1} \omega^{\alpha-r}(a) \frac{1}{q} \log(1+q) B_r.$$

This gives (i) immediately. If $p = 2$, $v = 1$, we get similarly

$$t_0 = \sum_{a=1}^{q-1} \omega^{-r}(a) \frac{1}{q} \log(1+q) {}_2rB_r\left(\frac{1}{2}\right) = \sum_{a=1}^{q-1} \omega^{-r}(a) \frac{1}{q} \log(1+q) {}_2(1-2^{r-1})B_r.$$

Hence (ii) is proved. If $p > 2$, $v \geq 1$ or $p = 2$, $v \geq 2$, then

$$\begin{aligned} t_0 &= \sum_{a=1}^{p^v-1} \omega^{\alpha-r}(a) \frac{1}{p^v} \log(1+p^v) (p^v)^r B_r\left(\left\{\frac{ha}{p^v}\right\}\right) \\ &= \omega^{r-\alpha}(h)(p^v)^{r-1} \sum_{a=1}^{p^v-1} \omega^{\alpha-r}(a) B_r\left(\frac{a}{p^v}\right) \log(1+p^v). \end{aligned}$$

Hence, if $r \not\equiv \alpha \pmod{e}$, (iii) is immediate from (29). For $r \equiv \alpha \pmod{e}$, it is also easy to derive the result from the above and (29).

PROOF OF LEMMA 3. Put $T =$ the left-hand side of the equation. Since

$$\frac{1}{a+kx} = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \frac{k^n}{a^n} x^n \text{ for } x \in \mathbf{Z}_p, \text{ it follows from (6) and (3) that}$$

$$\begin{aligned} T &= \sum_{n=0}^{\infty} (-1)^n \frac{k^n}{a^{n+1}} \sum_{\zeta^1+k=1, \zeta \neq 1} E_n(\zeta) \zeta^a \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{k^{n-1}}{na^n} \left((1+k)^n B_n\left(\frac{a}{1+k}\right) - B_n \right). \end{aligned}$$

Since

$$(1+k)^n B_n\left(\frac{a}{1+k}\right) = (1+k)^n \sum_{i=0}^n \binom{n}{i} B_i \left(\frac{a}{1+k}\right)^{n-i} = \sum_{j=0}^n \sum_{i=j}^n \binom{n}{i} \binom{i}{j} B_i a^{n-i} k^j,$$

we have

$$(1+k)^n B_n\left(\frac{a}{1+k}\right) - B_n = \sum_{j=0}^n \alpha_j^{(n)} k^j,$$

where

$$\alpha_0^{(n)} = \sum_{i=0}^{n-1} \binom{n}{i} B_i a^{n-i}, \text{ and } \alpha_j^{(n)} = \sum_{i=j}^n \binom{n}{i} \binom{i}{j} B_i a^{n-i}, \quad 1 \leq j \leq n.$$

Noting that $|\alpha_j^{(n)}| \leq \left|\frac{1}{p}\right|$ for all n and j , we see

$$\begin{aligned}
T &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{na^n} \sum_{j=0}^n \alpha_j^{(n)} k^{n+j-1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{na^n} \sum_{j=n}^{2n} \alpha_{j-n}^{(n)} k^{j-1} \\
&= \sum_{j=1}^{\infty} \left(\sum_{\substack{j \leq n \leq j \\ 2}} (-1)^{n-1} \frac{1}{na^n} \alpha_{j-n}^{(n)} \right) k^{j-1} \\
&= \sum_{j=1}^{\infty} \left(\sum_{0 \leq n \leq \frac{j}{2}} (-1)^{j-n-1} \frac{1}{(j-n)a^{j-n}} \alpha_n^{(j-n)} \right) k^{j-1} \\
&= \sum_{j=1}^{\infty} (-1)^{j-1} \left(\frac{1}{ja^j} \alpha_0^{(j)} + \sum_{1 \leq n \leq \frac{j}{2}} (-1)^n \frac{1}{(j-n)a^{j-n}} \alpha_n^{(j-n)} \right) k^{j-1}.
\end{aligned}$$

Substituting into the above

$$\frac{1}{ja^j} \alpha_0^{(j)} = \frac{1}{j} + \sum_{i=1}^{j-1} \binom{j-1}{i-1} \frac{1}{i} B_i a^{-i}$$

and

$$\begin{aligned}
&\sum_{1 \leq n \leq \frac{j}{2}} (-1)^n \frac{1}{(j-n)a^{j-n}} \alpha_n^{(j-n)} \\
&= \sum_{1 \leq n \leq \frac{j}{2}} (-1)^n \frac{1}{(j-n)a^{j-n}} \sum_{i=n}^{j-n} \binom{j-n}{i} \binom{i}{n} B_i a^{j-n-i} \\
&= \sum_{i=1}^{j-1} \sum_{n=1}^i (-1)^n \binom{i}{n} \binom{j-1-n}{i-1} \frac{1}{i} B_i a^{-i},
\end{aligned}$$

we obtain

$$T = \sum_{j=1}^{\infty} (-1)^{j-1} \left(\frac{1}{j} + \sum_{i=1}^{j-1} \sum_{n=0}^i (-1)^n \binom{i}{n} \binom{j-1-n}{i-1} \frac{1}{i} B_i a^{-i} \right) k^{j-1}.$$

But, since $\binom{j-1-x}{i-1}$ is a polynomial of x of degree $i-1$, we have

$$\sum_{n=0}^i (-1)^n \binom{i}{n} \binom{j-1-n}{i-1} = 0,$$

so that

$$T = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j} k^{j-1} = \frac{1}{k} \log(1+k),$$

as claimed. This completes the proof of Lemma 3.

§ 6. The coefficient t_1

In this section we consider the estimate of $|t_1|$, in the case $p > 2$, more precisely.

Let $p > 2$ and assume that r is even if $v = 0$. By the formula (33) we have

$$(37) \quad t_1 = \sum_{a=1}^{p^{\bar{v}}-1} \omega^{a-r}(a) A_1^{(\bar{v})}(a) (p^v)^r B_r \left(\left\{ \frac{ha}{p^v} \right\} \right) \quad \text{for } v \geq 0.$$

Now, for each integer a with $0 < a < p^{\bar{v}}$ and $(a, p) = 1$, we see

$$\frac{\log(a + p^{\bar{v}}x)}{a + p^{\bar{v}}x} \equiv \frac{\log a}{a} + \frac{p^{\bar{v}}}{a^2} x \pmod{p^{\bar{v}+1}}, \quad x \in \mathbf{Z}_p.$$

Hence, by the definition of $A_1^{(\bar{v})}(a)$ and (6), we have

$$A_1^{(\bar{v})}(a) \equiv \sum_{\eta^c=1, \eta \neq 1} \left(\frac{\log a}{a} E_0(\eta) + \frac{p^{\bar{v}}}{a^2} E_1(\eta) \right) \eta^a \pmod{p^{\bar{v}+1}}.$$

Here $c = 1 + p^{\bar{v}}$ as before. Since p is odd, we see from (3) that

$$\sum_{\eta^c=1, \eta \neq 1} E_0(\eta) \eta^a \equiv a \pmod{p^{\bar{v}}}$$

and

$$\sum_{\eta^c=1, \eta \neq 1} E_1(\eta) \eta^a \equiv \frac{1}{2} a^2 - \frac{1}{2} a + \frac{p^{\bar{v}}}{6} \pmod{p}.$$

Therefore we have

$$(38) \quad A_1^{(\bar{v})}(a) \equiv \begin{cases} \log a + 3 \left(1 - \frac{1}{2} a \right) \pmod{3^2}, & \text{if } p = 3, \bar{v} = 1, \\ \log a + \frac{p^{\bar{v}}}{2} \left(1 - \frac{1}{a} \right) \pmod{p^{\bar{v}+1}}, & \text{otherwise.} \end{cases}$$

If $v \geq 1$ ($v \geq 2$ for $p = 3$), then we obtain from (37), (35) and (38) that

$$(39) \quad t_1 \equiv h^r \sum_{a=1}^{p^v-1} \omega^a(a) \langle a \rangle^r \left(\log a + \frac{p^v}{2} \left(1 - \frac{1}{a} \right) \right) \pmod{p^{v+1}},$$

in which we have

$$(40) \quad \sum_{a=1}^{p^v-1} \omega^\alpha(a) \langle a \rangle^r \left(1 - \frac{1}{a}\right) \equiv \sum_{a=1}^{p^v-1} (\omega^\alpha(a) - \omega^{\alpha-1}(a)) \pmod{p}$$

$$\equiv \begin{cases} -1 & \pmod{p}, \quad v=1, \alpha=p-1, \\ 0 & \pmod{p}, \quad \text{otherwise.} \end{cases}$$

Now let us consider the sequence $L^{(v)}$, $v \geq 1$, defined for each integer $r \geq 1$ and even integer α with $0 < \alpha \leq p-1$ by

$$L^{(v)} = L_{a,r}^{(v)} = \sum_{a=1}^{p^v-1} \omega^\alpha(a) \langle a \rangle^r \log a.$$

LEMMA 4. (i) If $p \geq 5$, then $L^{(v+1)} \equiv pL^{(v)} \pmod{p^{v+2}}$, $v \geq 1$. (ii) If $p = 3$, then $L^{(2)} \equiv 3L^{(1)} + (1+r)3^2 \pmod{3^3}$ and

$$L^{(v+1)} \equiv 3L^{(v)} \pmod{3^{v+2}}, \quad v \geq 2.$$

PROOF. For $v \geq 1$, we have

$$L^{(v+1)} = \sum_{a=1}^{p^{v+1}-1} \omega^\alpha(a) \sum_{b=0}^{p-1} \langle a + p^v b \rangle^r \log(a + p^v b).$$

Since

$$\begin{aligned} \langle a + p^v b \rangle^r &= (\langle a \rangle + \omega^{-1}(a)p^v b)^r \\ &\equiv \langle a \rangle^r + r \langle a \rangle^{r-1} \omega^{-1}(a) p^v b \pmod{p^{v+1}} \end{aligned}$$

and

$$\begin{aligned} \log(a + p^v b) &= \log a + \log \left(1 + p^v \frac{b}{a}\right) \\ &\equiv \log a + p^v \frac{b}{a} - p^{2v} \frac{b^2}{2a^2} + p^{3v} \frac{b^3}{3a^3} \pmod{p^{v+2}}, \end{aligned}$$

we obtain

$$\begin{aligned} &\langle a + p^v b \rangle^r \log(a + p^v b) \\ &\equiv \langle a \rangle^r \log a + \langle a \rangle^r p^v \frac{b}{a} - \langle a \rangle^r p^{2v} \frac{b^2}{2a^2} + \langle a \rangle^r p^{3v} \frac{b^3}{3a^3} \end{aligned}$$

$$+ r\langle a \rangle^{r-1} \omega^{-1}(a) p^v b \log a + r\langle a \rangle^{r-1} \omega^{-1}(a) p^{2v} \frac{b^2}{a} \pmod{p^{v+2}}.$$

First, let $p \geq 5$. Then, since $\sum_{b=0}^{p-1} b \equiv \sum_{b=0}^{p-1} b^2 \equiv 0 \pmod{p}$ and $3v \geq v+2$ for $v \geq 1$, it follows that

$$\sum_{b=0}^{p-1} \langle a + p^v b \rangle^r \log(a + p^v b) \equiv p \langle a \rangle^r \log a + p^v \frac{p(p-1)}{2} \frac{1}{a} \pmod{p^{v+2}}.$$

Since $\sum_{a=1}^{p^v-1} \omega^a(a) \frac{1}{a} \equiv 0 \pmod{p}$, we obtain (i). Next, let $p = 3$. Then we see

$$\sum_{b=0}^2 \langle a + 3^v b \rangle^r \log(a + 3^v b) \equiv 3 \langle a \rangle^r \log a + 3^{v+1} a - (1+r) 3^{2v} \pmod{3^{v+2}},$$

for $v \geq 1$. If we take sum over $1 \leq a \leq 3^v - 1$, $(a, 3) = 1$, we obtain (ii). Hence, Lemma 4 is proved.

Now assume that $p \geq 5$ and let

$$W_{p,\alpha} = \begin{cases} \frac{B_{p-1}}{p-1} - \frac{1}{p} & \text{for } \alpha = p-1, \\ \frac{B_\alpha}{\alpha} & \text{for } \alpha < p-1. \end{cases}$$

Obviously, $W_{p,\alpha}$ are p -integral rational numbers.

PROPOSITION 5. Let $p \geq 5$, and let r be even if $v = 0$. Then,

$$(i) \quad t_1 \equiv \begin{cases} p \left(W_{p,p-1} - \frac{1}{2} \right) B_r \pmod{p^{2+\theta}} & \text{for } r \equiv \alpha \pmod{p-1}, v = 0, \\ p W_{p,\alpha_1} B_r \pmod{p^{2+\theta}} & \text{for } r \not\equiv \alpha \pmod{p-1}, v = 0, \end{cases}$$

where $\theta = \text{ord}_p(B_r)$, and α_1 denotes the integer satisfying $0 < \alpha_1 \leq p-1$ and $\alpha_1 \equiv \alpha - r \pmod{p-1}$.

$$(ii) \quad t_1 \equiv \begin{cases} h^r p \left(W_{p,p-1} - \frac{1}{2} \right) \pmod{p^2} & \text{for } \alpha = p-1, v = 1, \\ h^r p W_{p,\alpha} \pmod{p^2} & \text{for } \alpha < p-1, v = 1. \end{cases}$$

$$(iii) \quad t_1 \equiv h^r p^v W_{p,\alpha} \pmod{p^{v+1}} \quad \text{for } v \geq 2.$$

PROOF. For any even integer α with $0 < \alpha \leq p-1$, we have

$$\begin{aligned} \sum_{a=1}^{p-1} \omega^\alpha(a) \log a &= \frac{1}{p-1} \sum_{a=1}^{p-1} \omega^\alpha(a) \log a^{p-1} \\ &\equiv \frac{1}{p-1} \sum_{a=1}^{p-1} \omega^\alpha(a) (a^{p-1} - 1) \pmod{p^2}. \end{aligned}$$

Now, since $B_{p-2,\omega^\alpha} = 0$ for $p \geq 5$, it follows from (30) and (31) that

$$\sum_{a=1}^{p-1} \omega^\alpha(a) a^{p-1} \equiv p B_{p-1,\omega^\alpha} \pmod{p^2}.$$

If $\alpha = p-1$ we therefore obtain $\sum_{a=1}^{p-1} \omega^\alpha(a) \log a \equiv \frac{p}{p-1} B_{p-1} - 1 = p W_{p,p-1} \pmod{p^2}$. If $\alpha < p-1$ then $\frac{B_{p-1,\omega^\alpha}}{p-1} \equiv \frac{B_\alpha}{\alpha} = W_{p,\alpha} \pmod{p}$ by Kummer's congruence for generalized Bernoulli numbers [2], [9], [13]. Hence, in both cases, we have

$$(41) \quad \sum_{a=1}^{p-1} \omega^\alpha(a) \log a \equiv p W_{p,\alpha} \pmod{p^2}.$$

Let $v = 0$. Then, it follows from (37), (38), and (41) that

$$\begin{aligned} t_1 &\equiv \sum_{a=1}^{p-1} \omega^{\alpha-r}(a) \left(\log a + \frac{p}{2} \left(1 - \frac{1}{a} \right) \right) B_r \\ &\equiv p W_{p,\alpha_1} B_r + \frac{p}{2} \sum_{a=1}^{p-1} (\omega^{\alpha-r}(a) - \omega^{\alpha-r-1}(a)) B_r \pmod{p^{2+\theta}}. \end{aligned}$$

This gives (i) as easily be seen.

Let $v \geq 1$. Then we see from (39) and (40) that

$$t_1 \equiv \begin{cases} h^r \left(L_{\alpha,r}^{(1)} - \frac{p}{2} \right) \pmod{p^2}, & \text{if } v = 1, \alpha = p-1, \\ h^r L_{\alpha,r}^{(v)} \pmod{p^{v+1}}, & \text{if } v \geq 2 \text{ or } \alpha < p-1. \end{cases}$$

Now we have $L_{\alpha,r}^{(1)} \equiv \sum_{a=1}^{p-1} \omega^\alpha(a) \log a \equiv pW_{p,\alpha} \pmod{p^2}$ by (41). Hence we have

(ii). For $v \geq 2$, we see by Lemma 4(i) that $L_{\alpha,r}^{(v)} \equiv p^{v-1} L_{\alpha,r}^{(1)} \pmod{p^{v+1}}$. Hence, $L_{\alpha,r}^{(v)} \equiv p^v W_{p,\alpha} \pmod{p^{v+1}}$. This gives (iii). Hence we have proved Proposition 5.

Next, let $p = 3$, so that $\alpha = 2$. For $v = 0$ or 1 , we see from (37), (38) and (35) that

$$t_1 \equiv \begin{cases} \sum_{a=1}^2 \left(\log a + 3 \left(1 - \frac{1}{2}a \right) \right) B_r \pmod{3}, & v = 0 \text{ and } r \text{ even,} \\ h^r \sum_{a=1}^2 \left(\log a + 3 \left(1 - \frac{1}{2}a \right) \right) \pmod{3^2}, & v = 1. \end{cases}$$

Since $\log 1 + 3 \left(1 - \frac{1}{2} \right) \equiv \frac{3}{2} \equiv -3 \pmod{3^2}$ and $\log 2 + 3 \left(1 - \frac{2}{2} \right) \equiv \log 2 \equiv -3 \pmod{3^2}$, we have $t_1 \equiv -6B_r \equiv 2 \equiv -1 \pmod{3}$ for $v = 0$ and $t_1 \equiv -6h^r \equiv 3h^r \pmod{3^2}$ for $v = 1$.

Let $v \geq 2$. Then we have by (39) and (40)

$$t_1 \equiv h^r L_{2,r}^{(v)} \pmod{3^{v+1}}.$$

Since $L_{2,r}^{(1)} = \log 2 \equiv -3 \pmod{3^2}$, it follows from Lemma 4(ii) that $L_{2,r}^{(2)} \equiv -3^2 + (1+r)3^2 \equiv 3^2 r \pmod{3^3}$ for $v = 2$ and $L_{2,r}^{(v)} \equiv 3^{v-2} L_{2,r}^{(2)} \equiv 3^v r \pmod{3^{v+1}}$ for $v \geq 3$. Hence, we have $t_1 \equiv 3^v r h^r \pmod{3^{v+1}}$ for $v \geq 2$. Therefore we have obtained the following

PROPOSITION 6. *For $p = 3$, we have*

$$\begin{aligned} t_1 &\equiv -1 \pmod{3}, & \text{for } v = 0 \text{ and } r \text{ even,} \\ t_1 &\equiv 3h^r \pmod{3^2}, & \text{for } v = 1, \\ t_1 &\equiv 3^v r h^r \pmod{3^{v+1}}, & \text{for } v \geq 2. \end{aligned}$$

§7. $S_{p,\alpha}(s; r, h, k)$ for $p > 2$

The function $S_{p,\alpha}(s; r, h, k)$ is expanded generally as

$$S_{p,\alpha}(s; r, h, k) = \sum_{n=0}^{\infty} a_n (s+1-r)^n.$$

For the special case $k = p^v$, we shall write

$$S_{p,\alpha}(s; r, h, p^v) = \sum_{n=0}^{\infty} a'_n (s+1-r)^n.$$

Expressing $S_{p,\alpha}(s; r, h, k)$ as the sum of $\bar{S}_{p,\alpha}(s; r, h, k)$ and $S_{p,\alpha}(s; r, h, p^v)$, by (13), we have

$$a_0 = a'_0, \quad a_n = \bar{a}_n + a'_n, \quad n \geq 1,$$

and

$$a'_n = \sum_{i=0}^n t_{n-i} u_i, \quad n \geq 0,$$

where

$$u_n = B_n \frac{(\log(1+p^v))^{n-1}}{n!}, \quad n \geq 0.$$

In this section, we assume that $p > 2$. By Proposition 3, we first obtain the following

PROPOSITION 7. *Let $p > 2$, and let r be even if $v = 0$.*

(i) *If $v = 0$ or 1, then*

$$|a'_n| \leq |p^{n-1+\theta}|, \quad 0 \leq n < p-1, \quad \text{and} \quad |a'_n| \leq \left| \frac{p^{n-2+\theta}}{n!} \right|, \quad p-1 \leq n,$$

where $\theta = \text{ord}_p(B_r)$ if $v = 0$ and $\theta = 0$ if $v = 1$. Further, for $p = 3$,

$$|a'_3| \leq |3^{1+\theta}|.$$

(ii) *If $v \geq 2$, then*

$$n!a'_n \equiv -h^r p^{n-1} B_n \pmod{p^{n-1}}, \quad n \geq 0, \quad \text{for } \alpha = p-1,$$

$$|a'_n| \leq \left| \frac{p^{n-1}}{n!} \right|, \quad n \geq 0, \quad \text{for } \alpha < p-1.$$

PROOF. (i) If $v = 0$ or 1, then we have $|t_n| \leq \left| \frac{p^{n+\theta}}{n!} \right|$, $n \geq 0$, by Proposition

3(i). On the other hand, $|u_n| \leq \left| \frac{p^{n-2}}{n!} \right|$, $n \geq 0$. Hence, $|t_{n-i} u_i| \leq \left| \frac{p^{n-i+\theta}}{(n-i)!} \frac{p^{i-2}}{i!} \right|$

$\leq \left| \frac{p^{n-2+\theta}}{n!} \right|$ for $0 \leq i \leq n$, so that $|a'_n| \leq \left| \frac{p^{n-2+\theta}}{n!} \right|$ for all $n \geq 0$. If $n < p-1$ then, since $|u_n| \leq |p^{n-1}|$, we have $|a'_n| \leq |p^{n-1+\theta}|$. Moreover, since $u_3 = 0$, we get $|a'_3| \leq |3^{1+\theta}|$ for $p = 3$.

(ii) If $v \geq 2$ then $u_0 \equiv \frac{1}{p^v} \pmod{p^{1-v}}$, $|u_1| = 1$ and $|u_n| \leq \left| \frac{p^{v(n-1)-1}}{n!} \right|$, $n \geq 2$. Hence, by Proposition 3 (iii), (iv), we see that $n!t_n u_0 \equiv -h^r p^{n-1} B_n \pmod{p^{n-1}}$ for $\alpha = p-1$, and $|t_n u_0| \leq \left| \frac{p^{n-1}}{n!} \right|$ for $\alpha < p-1$. In the next place, we have $|t_{n-1} u_1| \leq \left| \frac{p^{n-1+v-2}}{(n-1)!} \right| \leq \left| \frac{p^{n-1}}{n!} \right|$ and $|t_{n-i} u_i| \leq \left| \frac{p^{n-i+v-2}}{(n-i)!} \frac{p^{v(i-1)-1}}{i!} \right| \leq \left| \frac{p^{n+(v-1)i-3}}{n!} \right| \leq \left| \frac{p^{n-1}}{n!} \right|$ for $2 \leq i \leq n$. Therefore we obtain the results.

First, we consider the case $p \geq 5$. Let $k = k_0 p^v$ with $(k_0, p) = 1$, $v \geq 0$, and let $(h, k) = 1$, as before.

THEOREM 2. Let $p \geq 5$ and let $k \not\equiv 0 \pmod{p}$. Then, $S_{p,\alpha}(s; r, h, k) = \sum_{n=0}^{\infty} a_n (s+1-r)^n$ has the following properties.

(i) If $r \equiv 0 \pmod{p-1}$, then

$$a_0 = \left(1 - \frac{1}{p}\right) B_r \text{ for } \alpha = p-1, \quad a_0 = 0 \text{ for } \alpha < p-1,$$

$$\text{and } a_1 \equiv -\frac{1}{p} W_{p,\alpha} \pmod{p^0}, \quad |a_2| \leq 1, \quad |a_n| \leq \left| \frac{p^{n-3}}{n!} \right|, \quad n \geq 3.$$

(ii) If $r \not\equiv 0 \pmod{p-1}$, then

$$a_0 = \left(1 - \frac{1}{p}\right) B_r \text{ for } r \equiv \alpha \pmod{p-1}, \quad a_0 = 0 \text{ for } r \not\equiv \alpha \pmod{p-1},$$

$$\text{and } |a_1| \leq |r|, \quad |a_2| \leq |rp|, \quad |a_n| \leq \left| \frac{rp^{n-2}}{n!} \right|, \quad n \geq 3.$$

PROOF. The assertions for a_0 are obvious by Proposition 4 (i), so we let $n \geq 1$. (i) Let $r \equiv 0 \pmod{p-1}$. Since $p \geq 5$ and $v = 0$, Proposition 7(i) shows that $|a'_2| \leq 1$ and $|a'_n| \leq \left| \frac{p^{n-3}}{n!} \right|$, $n \geq 3$. Comparing them with (16), we

obtain the assertions for $n \geq 2$. Hence we have to prove the congruence for a_1 . Since $|\bar{a}_1| \leq 1$, it suffices to show that $a'_1 \equiv -\frac{1}{p}W_{p,\alpha} \pmod{p^0}$. First let α

$= p-1$. Then, we see $t_0 \equiv -B_r \equiv \frac{1}{p} \pmod{p^0}$ by Proposition 4(i), and $t_1 \equiv p\left(W_{p,p-1} - \frac{1}{2}\right)B_r \equiv -\left(W_{p,p-1} - \frac{1}{2}\right) \pmod{p}$ by Proposition 5(i). Since $u_0 \equiv \frac{1}{p} \pmod{p^0}$ and $u_1 = -\frac{1}{2}$, we have $a'_1 = t_0u_1 + t_1u_0 \equiv -\frac{1}{2p} - \frac{1}{p}\left(W_{p,p-1} - \frac{1}{2}\right) \equiv -\frac{1}{p}W_{p,p-1} \pmod{p^0}$. Next, let $\alpha < p-1$. Then $t_0 = 0$ and $t_1 \equiv -W_{p,\alpha}$

$\pmod{p}$ by Propositions 4, 5. Hence we have $a'_1 = t_1u_0 \equiv -\frac{1}{p}W_{p,\alpha} \pmod{p^0}$.

(ii) Let $r \not\equiv 0 \pmod{p-1}$. If r is odd then $a'_n = 0$ for all $n \geq 0$. If r is even then we have $|a'_1| \leq |r|$, $|a'_2| \leq |rp|$ and $|a'_n| \leq \left| \frac{rp^{n-2}}{n!} \right|$, $n \geq 3$, by Proposition 7(i). Hence, comparing them with (16) we obtain the assertions.

THEOREM 3. *Let $p \geq 5$, and let $k \equiv 0 \pmod{p}$. Then, $S_{p,\alpha}(s; r, h, k) = \sum_{n=0}^{\infty} a_n(s+1-r)^n$ has the following properties.*

$$a_0 \equiv \begin{cases} -\frac{h^r}{p} & \pmod{p^0}, \quad \alpha = p-1, \\ h^{r-\alpha}rW_{p,\alpha} & \pmod{p}, \quad \alpha < p-1, \end{cases}$$

$$\text{and } a_1 \equiv h^rW_{p,\alpha} \pmod{p}, \quad |a_2| \leq |p|, \quad |a_n| \leq \left| \frac{p^{n-2}}{n!} \right|, \quad n \geq 3.$$

PROOF. For $p \geq 5$ and $v \geq 1$, Proposition 7 implies that $|a'_2| \leq |p|$, and $|a'_n| \leq \left| \frac{p^{n-2}}{n!} \right|$, $n \geq 3$. Comparing them with (16) we get the assertions for $n \geq 2$, so we must only show the assertions for a_0 and a_1 . Further, since $|\bar{a}_1| \leq |p|$ by (16), we may assume that $k = p^v$.

Suppose $\alpha = p-1$. Then, it follows from Proposition 4(iii) and the von Staudt-Clausen Theorem (31) that $t_0 = (1 - p^{r-1})B_r \log(1 + p^v) \equiv -p^{v-1} \pmod{p^v}$ for $r \equiv 0 \pmod{p-1}$, and $t_0 = \omega^r(h)B_{r,\omega^{-r}} \log(1 + p^v) \equiv -h^r p^{v-1} \pmod{p^v}$

for $r \not\equiv 0 \pmod{p-1}$. Namely, $t_0 \equiv -h^r p^{v-1} \pmod{p^v}$ for $v \geq 1$ in both cases. Next, we know by Proposition 5 that $t_1 \equiv h^r p \left(W_{p,p-1} - \frac{1}{2} \right) \pmod{p^2}$ for $v = 1$, and $t_1 \equiv h^r p^v W_{p,p-1} \pmod{p^{v+1}}$ for $v \geq 2$. Since $u_0 \equiv \frac{1}{p^v} \pmod{p^{1-v}}$ and $u_1 = -\frac{1}{2}$, we see that $a_0 \equiv -\frac{h^r}{p} \pmod{p^0}$ and $a_1 \equiv h^r W_{p,p-1} \pmod{p}$.

Suppose $\alpha < p-1$. Then, by Kummer's congruence, we see $B_r \equiv r W_{p,\alpha} \pmod{p}$ for $r \equiv \alpha \pmod{p-1}$, and $B_{r,\omega^\alpha-r} \equiv r W_{p,\alpha} \pmod{p}$ for $r \not\equiv \alpha \pmod{p-1}$. Hence, by Proposition 4(iii) we have $t_0 \equiv h^{r-\alpha} p^v r W_{p,\alpha} \pmod{p^{v+1}}$. For t_1 , we know by Proposition 5(ii), (iii) that $t_1 \equiv h^r p^v W_{p,\alpha} \pmod{p^{v+1}}$. Since $u_0 \equiv \frac{1}{p^v} \pmod{p^{1-v}}$ and $u_1 = -\frac{1}{2}$, we obtain $a_0 \equiv h^{r-\alpha} r W_{p,\alpha} \pmod{p}$, and $a_1 \equiv h^r W_{p,\alpha} \pmod{p}$. This completes the proof of Theorem 3.

We shall give several results related with Theorems 2 and 3.

1. Assume that $k \not\equiv 0 \pmod{p}$ and $r \equiv 0 \pmod{p-1}$, or $k \equiv 0 \pmod{p}$. Then, from Theorems 2(i) and 3, we see that if p divides the numerator of $W_{p,\alpha}$, then

$$|S_{p,\alpha}(s; r, h, k) - S_{p,\alpha}(t; r, h, k)| \leq |p^{1+\theta}| |s - t|, \quad s, t \in \mathbb{Z}_p,$$

and otherwise,

$$|S_{p,\alpha}(s; r, h, k) - S_{p,\alpha}(t; r, h, k)| = |p^\theta| |s - t|, \quad s, t \in \mathbb{Z}_p,$$

where $\theta = -1$ if $k \not\equiv 0 \pmod{p}$ and $\theta = 0$ if $k \equiv 0 \pmod{p}$. We note that $W_{p,\alpha} = \frac{B_\alpha}{\alpha}$ if $\alpha < p-1$, and $W_{p,p-1} \equiv -\omega_p \pmod{p}$, where ω_p is the so-called Wilson quotient, $((p-1)! + 1)/p$ ([3]).

2. Assume that $r \equiv 0 \pmod{p-1}$. Let m be an integer $\geq r$ such that $m+1 \equiv 0 \pmod{p-1}$. Then, by Theorem 2(i), we have

$$S_{p,p-1}(m; r, 1, 1) \equiv \left(1 - \frac{1}{p}\right) B_r - \frac{1}{p} W_{p,p-1}(m+1-r) \pmod{p^{\text{ord}_p(m+1-r)}}.$$

On the other hand, we see from Theorem 1 that

$$S_{p,p-1}(m; r, 1, 1) = (1 - p^{m-r})B_{m+1-r}B_r \equiv B_{m+1-r}B_r \pmod{p^{m-r-2}}.$$

Since $m+1-r \geq p-1 \geq 4$ and $\text{ord}_p\left(\frac{p^{j-4}}{j}\right) \geq 0$ for $j \geq 4$, we have

$$B_{m+1-r}B_r \equiv \left(1 - \frac{1}{p}\right)B_r - \frac{1}{p}W_{p,p-1}(m+1-r) \pmod{p^{\text{ord}_p(m+1-r)}}.$$

Dividing by $B_r \equiv -\frac{1}{p} \pmod{p^0}$ and rewriting $m+1-r$ by n , we obtain

$$(42) \quad B_n \equiv 1 - \frac{1}{p} + W_{p,p-1}n \pmod{p^{1+\text{ord}_p(n)}},$$

for all integers $n \geq p-1$ such that $n \equiv 0 \pmod{p-1}$. This is a formula of Johnson [3].

3. Assume that $k \not\equiv 0 \pmod{p}$ and $r \not\equiv 0 \pmod{p-1}$. Let m be an odd integer with $m \geq r$.

If $m+1 \not\equiv r \pmod{p-1}$, then we see from (4) that

$$(43) \quad |S_{m+1}^{(r)}(h, k)| \leq |(m+1-r)r|.$$

Choose α with $m+1 \equiv \alpha \pmod{p-1}$ (therefore $r \not\equiv \alpha \pmod{p-1}$), and consider the function $S_{p,\alpha}(s; r, h, k)$. Then by Theorem 2(ii) we have

$$S_{p,\alpha}(m; r, h, k) \equiv a_1(m+1-r) \pmod{p^{1+\text{ord}_p((m+1-r)r)}}.$$

On the other hand, we see by Theorem 1 and (43) that

$$S_{p,\alpha}(m; r, h, k) \equiv k^m S_{m+1}^{(r)}(h, k) \pmod{p^{m-r+\text{ord}_p((m+1-r)r)}}.$$

Hence, assuming $m > r$, we obtain for this function $S_{p,\alpha}(s; r, h, k)$

$$(44) \quad \frac{1}{(m+1-r)r} k^m S_{m+1}^{(r)}(h, k) \equiv \frac{a_1}{r} \pmod{p}.$$

If $m+1 \equiv r \pmod{p-1}$, then we see from (4) that

$$(45) \quad |S_{m+1}^{(r)}(h, k)| \leq \left| \frac{r}{p} \right|.$$

Choose α so that $m+1 \equiv \alpha \pmod{p-1}$ (hence $r \equiv \alpha \pmod{p-1}$). Then, for

the function $S_{p,\alpha}(s; r, h, k)$, we see

$$S_{p,\alpha}(m; r, h, k) \equiv \left(1 - \frac{1}{p}\right) B_r + a_1(m+1-r) \pmod{p^{1+\text{ord}_p((m+1-r)r)}},$$

by Theorem 2(ii), and

$$S_{p,\alpha}(m; r, h, k) \equiv k^m S_{m+1}^{(r)}(h, k) \pmod{p^{m-r-1+\text{ord}_p(r)}},$$

by Theorem 1 and (45). Since $m+1-r \geq p-1 \geq 4$ and $\text{ord}_p\left(\frac{p^{j-3}}{j}\right) \geq 0$ for $j \geq 4$, we have $m-r-2 \geq \text{ord}_p(m+1-r)$, and

$$(46) \quad \frac{1}{(m+1-r)r} \left(k^m S_{m+1}^{(r)}(h, k) - \left(1 - \frac{1}{p}\right) B_r \right) \equiv \frac{a_1}{r} \pmod{p}.$$

Combining the results above for a_1 with Theorem 2(ii), we obtain the following

THEOREM 4. *Let $p \geq 5$, $k \not\equiv 0 \pmod{p}$ and $r \not\equiv 0 \pmod{p-1}$, and let h and α be as above.*

(i) *Let $r \not\equiv \alpha \pmod{p-1}$. If $\left| \frac{1}{(m+1-r)r} k^m S_{m+1}^{(r)}(h, k) \right| \leq |p|$ holds for some integer m such that $m+1 \equiv \alpha \pmod{p-1}$ and $m > r$, then*

$$(47) \quad |S_{p,\alpha}(s; r, h, k) - S_{p,\alpha}(t; r, h, k)| \leq |rp| |s - t|, \quad s, t \in \mathbb{Z}_p.$$

Otherwise,

$$(48) \quad |S_{p,\alpha}(s; r, h, k) - S_{p,\alpha}(t; r, h, k)| = |r| |s - t|, \quad s, t \in \mathbb{Z}_p.$$

(ii) *Let $r \equiv \alpha \pmod{p-1}$. If $\left| \frac{1}{(m+1-r)r} \left(k^m S_{m+1}^{(r)}(h, k) - \left(1 - \frac{1}{p}\right) B_r \right) \right| \leq |p|$ holds for some integer m such that $m+1 \equiv \alpha \pmod{p-1}$ and $m > r$, then (47) holds. Otherwise, (48) holds.*

4. Assume that $k \not\equiv 0 \pmod{p}$ and r is odd. Then, for $S_{p,\alpha}(s; r, h, k)$, we have $a_0 = 0$, and

$$a_1 = \bar{a}_1 = r \sum_{\zeta^k=1, \zeta \neq 1} c_{0,r}(\zeta^h) E_{r-1}(\zeta^{-1}).$$

Let α_1 be an integer such that $0 < \alpha_1 \leq p-1$, $\alpha_1 \equiv \alpha - r \pmod{p-1}$ and let r_1 be an integer such that $0 < r_1 \leq p-1$, $r_1 \equiv r \pmod{p-1}$. Then, for each $\zeta \neq 1$, $\zeta^k = 1$,

$$\begin{aligned} c_{0,r}(\zeta) &= \int_{\mathbf{Z}_p^*} \omega^{\alpha-r}(x) x^{-1} d\mu_\zeta(x) \equiv \int_{\mathbf{Z}_p^*} x^{\alpha_1-1} d\mu_\zeta(x) \pmod{p} \\ &= E_{\alpha_1-1}(\zeta) - p^{\alpha_1-1} E_{\alpha_1-1}(\zeta^p), \end{aligned}$$

and if $r > 1$, then

$$\begin{aligned} E_{r-1}(\zeta) &\equiv \int_{\mathbf{Z}_p^*} x^{r-1} d\mu_\zeta(x) \equiv \int_{\mathbf{Z}_p^*} x^{r_1-1} d\mu_\zeta(x) \pmod{p} \\ &= E_{r_1-1}(\zeta) - p^{r_1-1} E_{r_1-1}(\zeta^p). \end{aligned}$$

By the above and (4), we obtain the following congruences:

a°) Let $r = 1$ and $\alpha = 2$. Then, since $\alpha_1 = 1$ and $r = 1$,

$$\begin{aligned} a_1 &\equiv \sum_{\zeta^k=1, \zeta \neq 1} (E_0(\zeta^h) - E_0(\zeta^{ph})) E_0(\zeta^{-1}) \pmod{p} \\ &= kS_2^{(1)}(h, k) - kS_2^{(1)}(ph, k). \end{aligned}$$

b°) Let $r \equiv 1 \pmod{p-1}$, $r > 1$ and $\alpha = 2$. Then, since $\alpha_1 = 1$, $r_1 = 1$, and $r > 1$,

$$\begin{aligned} \frac{a_1}{r} &\equiv \sum_{\zeta^k=1, \zeta \neq 1} (E_0(\zeta^h) - E_0(\zeta^{ph}))(E_0(\zeta^{-1}) - E_0(\zeta^{-p})) \pmod{p} \\ &= kS_2^{(1)}(h, k) - kS_2^{(1)}(ph, k) + kS_2^{(1)}(h^*, k) - kS_2^{(1)}(ph^*, k), \end{aligned}$$

where h^* denotes an integer such that $hh^* \equiv 1 \pmod{k}$.

c°) Let $r \equiv 1 \pmod{p-1}$, $r > 1$ and $4 \leq \alpha \leq p-1$. Then, since $\alpha_1 = \alpha - 1$, $r_1 = 1$ and $r > 1$,

$$\begin{aligned} \frac{a_1}{r} &\equiv \sum_{\zeta^k=1, \zeta \neq 1} E_{\alpha-2}(\zeta^h)((E_0(\zeta^{-1}) - E_0(\zeta^{-p}))) \pmod{p} \\ &= \frac{1}{\alpha-1} k^{\alpha-1} S_{\alpha}^{(\alpha-1)}(h^*, k) - \frac{1}{\alpha-1} k^{\alpha-1} S_{\alpha}^{(\alpha-1)}(ph^*, k), \end{aligned}$$

where $hh^* \equiv 1 \pmod{k}$.

d°) Let $r \equiv \alpha - 1 \pmod{p-1}$, $4 \leq \alpha \leq p-1$. Then, since $\alpha_1 = 1$ and $r_1 = \alpha - 1$,

$$\begin{aligned} \frac{a_1}{r} &\equiv \sum_{\zeta^k=1, \zeta \neq 1} (E_0(\zeta^h) - E_0(\zeta^{ph})) E_{\alpha-2}(\zeta^{-1}) \pmod{p} \\ &= \frac{1}{\alpha-1} k^{\alpha-1} S_{\alpha}^{(\alpha-1)}(h, k) - \frac{1}{\alpha-1} k^{\alpha-1} S_{\alpha}^{(\alpha-1)}(ph, k). \end{aligned}$$

Combining the above with (44) we obtain the following

COROLLARY 1. *Let $p \geq 5$, and let $k \not\equiv 0 \pmod{p}$. Let r and m be odd integers such that $1 \leq r < m$. Suppose that $r \equiv m \pmod{p-1}$ or $1 < r \equiv 1 \pmod{p-1}$. Then, it holds that*

$$\frac{1}{(m+1-r)r} S_{m+1}^{(r)}(h, k) \equiv 0 \pmod{p} \quad \text{if } p \equiv 1 \pmod{k}.$$

PROOF. Let r, m and k satisfy the above conditions. Choose α so that $m+1 \equiv \alpha \pmod{p-1}$. Then, by the relation (44), it suffices to show that r and α satisfy one of the assumptions of a°)-d°).

Let $r \equiv 1 \pmod{p-1}$ and $m \equiv 1 \pmod{p-1}$. Then, $r \equiv 1 \pmod{p-1}$ and $\alpha = 2$, so the assumption of a°) or b°) is satisfied. Let $r \equiv 1 \pmod{p-1}$ and $m \not\equiv 1 \pmod{p-1}$. Then, $r \equiv 1 \pmod{p-1}$ and $4 \leq \alpha \leq p-1$, so, if we exclude the case $r = 1$, the assumption of c°) is satisfied. Let $r \not\equiv 1 \pmod{p-1}$ and $r \equiv m \pmod{p-1}$. Then, since $1 < r_1 < p-1$ (r is odd) and $\alpha \equiv m+1 \equiv r_1+1 \pmod{p-1}$ ($\leq p-1$), we have $4 \leq \alpha \leq p-1$ and $r \equiv m \equiv \alpha-1 \pmod{p-1}$. Hence, the assumption of d°) is satisfied. Therefore we have proved Corollary 1.

5. We shall apply Theorem 3 to $k^m S_{m+1}^{(r)}(h, k)$, $1 \leq r \leq m$, with $k \equiv 0 \pmod{p}$. For $k \equiv 0 \pmod{p}$, we can prove by (1) and (35) that

$$(49) \quad k^m S_{m+1}^{(r)}(h, k) \equiv h^r B_{m+1} \pmod{p^0}, \quad 1 \leq r \leq m.$$

We shall derive some refinement of this. But we do not use (49) in the following arguments.

COROLLARY 2. *Let $p \geq 5$ and $k \equiv 0 \pmod{p}$. Then, for odd integer $m \geq 1$ satisfying $m+1 \not\equiv 0 \pmod{p-1}$, we have*

$$k^m S_{m+1}^{(r)}(h, k) \equiv h^r \frac{B_{m+1}}{m+1} (r(h^{-(m+1)} - 1) + (m+1)) \pmod{p}, \quad 1 \leq r \leq m.$$

PROOF. Choosing α so that $m+1 \equiv \alpha \pmod{p-1}$, and applying Theorem 3, we see that

$$\begin{aligned} S_{p,\alpha}(m; r, h, k) &\equiv h^{r-\alpha} r W_{p,\alpha} + h^r W_{p,\alpha}(m+1-r) \pmod{p} \\ &\equiv h^r W_{p,\alpha}(r h^{-\alpha} + (m+1-r)) \pmod{p} \\ &\equiv h^r \frac{B_{m+1}}{m+1} (r(h^{-(m+1)} - 1) + (m+1)) \pmod{p}. \end{aligned}$$

On the other hand we have by Theorem 1 and (5)

$$S_{p,\alpha}(m; r, h, k) = k^m S_{m+1}^{(r)}(h, k) - p^m \left(\frac{k}{p}\right)^{m-r} S_{m+1}^{(r)}\left(h, \frac{k}{p}\right).$$

Since $|S_{m+1}^{(r)}(h, k_0)| \leq \left|\frac{1}{p}\right|$ by the assumption and (4), we see from the above and induction on $\text{ord}_p(k) \geq 1$ that $|k^m S_{m+1}^{(r)}(h, k)| \leq 1$ and

$$S_{p,\alpha}(s; r, h, k) \equiv k^m S_{m+1}^{(r)}(h, k) \pmod{p}.$$

(Note also that $|S_2^{(1)}(h, k_0)| \leq 1$). Therefore we get the Corollary.

COROLLARY 3. *Let $p \geq 5$ and $k \equiv 0 \pmod{p}$. Then, for odd integer $m \geq 1$ satisfying $m+1 \equiv 0 \pmod{p-1}$, we have*

$$k^m S_{m+1}^{(r)}(h, k) \equiv h^{pr} \left(1 - \frac{1}{p} + W_{p,p-1}(m+1)\right) \pmod{p}, \quad 1 \leq r \leq m.$$

PROOF. For integer m with $m+1 \equiv 0 \pmod{p-1}$ and $m \geq r$, we obtain from Theorem 3 that

$$S_{p,p-1}(m; r, h, k) \equiv a_0 + h^r W_{p,p-1}(m+1-r) \pmod{p},$$

where $a_0 = (1 - p^{r-1})B_r$ or $a_0 = \omega^r(h)B_{r,\omega^{-r}}$ according as $r \equiv 0 \pmod{p-1}$ or not, by Proposition 4. On the other hand we see

$$S_{p,p-1}(m; r, h, k) = k^m S_{m+1}^{(r)}(h, k) - p^m \left(\frac{k}{p}\right)^m S_{m+1}^{(r)}\left(h, \frac{k}{p}\right),$$

by Theorem 1. In this case we have $|S_{m+1}^{(r)}(h, k_0)| \leq \left| \frac{1}{p^2} \right|$ by (4). But, since $m \geq 3$, we see from the above and induction on $\text{ord}_p(k) \geq 1$ that $|k^m S_{m+1}^{(r)}(h, k)| = \left| \frac{1}{p} \right|$ and

$$S_{p,p-1}(m; r, h, k) \equiv k^m S_{m+1}^{(r)}(h, k) \pmod{p}.$$

Now it can be shown in the same manner as in [3] that

$$B_{r, \omega^{-r}} \equiv 1 - \frac{1}{p} + W_{p,p-1} r \pmod{p^{1+\text{ord}_p(r)}} \text{ for all } r \geq 1.$$

One easily see $\omega(h) \equiv h^p \pmod{p^2}$. Putting everything together, we see

$$k^m S_{m+1}^{(r)}(h, k) \equiv h^{pr} \left(1 - \frac{1}{p} + W_{p,p-1} r \right) + h^{pr} W_{p,p-1} (m+1-r) \pmod{p}.$$

This proves Corollary 3.

We next consider the case $p = 3$. Then $\alpha = 2$, so we simply write the function $S_{3,2}(s; r, h, k)$, as $S_3(s; r, h, k)$.

THEOREM 5. *Let $p = 3$ and $k = k_0 3^v$, $(k_0, 3) = 1$, $v \geq 0$. Then $S_3(s; r, h, k) = \sum_{n=0}^{\infty} a_n (s+1-r)^n$ possesses the following properties:*

(i) *If $(k, 3) = 1$ and r is even, then*

$$a_0 = \frac{2}{3} B_r, \quad |a_1| \leq 1, \quad a_2 \equiv \frac{1}{3} \pmod{3^0}$$

$$|a_3| \leq 1 \quad \text{and} \quad |a_n| \leq \left| \frac{3^{n-3}}{n!} \right|, \quad n \geq 4.$$

(ii) *If $k \equiv 0 \pmod{3}$, then*

$$a_0 = \begin{cases} (1 - 3^{r-1}) B_r & \text{for } r \text{ even,} \\ \omega(h) B_{r, \omega^{-r}} & \text{for } r \text{ odd,} \end{cases} \quad a_1 \equiv \begin{cases} 0 \pmod{3} & v = 1, \\ rh^r \pmod{3} & v \geq 2, \end{cases}$$

$$a_2 \equiv -h^r \pmod{3}, \quad |a_3| \leq |3| \quad \text{and} \quad |a_n| \leq \left| \frac{3^{n-2}}{n!} \right|, \quad n \geq 4.$$

PROOF. The value a_0 is given by $a_0 = t_0 u_0$, and the results for a_n , $n \geq 3$,

follow from (16) and Proposition 7. Therefore we must only prove the assertions for a_1 and a_2 . Since $|\bar{a}_1| \leq |3^v|$ and $|\bar{a}_2| \leq |3^{v+1}|$ by (16), we may assume that $k = 3^v$, $v \geq 0$.

(i) If $v = 0$ and r is even, then we see from Propositions 4, 6 and 3 that $t_0 = \left(1 - \frac{1}{3}\right)B_r \log(1+3) \equiv \frac{1}{3} \pmod{3^0}$, $t_1 \equiv -1 \pmod{3}$ and $t_2 \equiv 0 \pmod{3}$.

On the other hand, $u_0 = \frac{1}{\log(1+3)} \equiv \frac{1}{3} \pmod{3^0}$, $u_1 = -\frac{1}{2} \equiv 1 \pmod{3}$ and $u_2 = \frac{\log(1+3)}{12} \equiv \frac{1}{4} \equiv 1 \pmod{3}$. Hence, $a_1 = t_0 u_1 + t_1 u_0 \equiv \frac{1}{3} - \frac{1}{3} \equiv 0 \pmod{3^0}$ and $a_2 \equiv t_0 u_2 \equiv \frac{1}{3} \pmod{3^0}$.

(ii) If $v = 1$, then we similarly obtain $t_0 \equiv -h^r \pmod{3}$, $t_1 \equiv 3h^r \pmod{3^2}$ and $t_2 \equiv 0 \pmod{3^2}$. Since $u_0 \equiv \frac{1}{3} \pmod{3^0}$ and $u_1 \equiv u_2 \equiv 1 \pmod{3}$, we have $a_1 \equiv 0 \pmod{3}$ and $a_2 \equiv -h^r \pmod{3}$. If $v \geq 2$, then we have $t_0 \equiv -3^{v-1}h^r \pmod{3^v}$, $t_1 \equiv 3^v r h^r \pmod{3^{v+1}}$ and $t_2 \equiv -3^v h^r \pmod{3^{v+1}}$. Since $u_0 \equiv \frac{1}{3^v} \pmod{3^{-v+1}}$, $u_1 \equiv 1 \pmod{3}$ and $u_2 \equiv 3^{v-1} \pmod{3^v}$, we get $a_1 \equiv r h^r \pmod{3}$ and $a_2 \equiv -h^r \pmod{3}$. This completes the proof of Theorem 5.

We put $\theta = -1$ if $(k, 3) = 1$ and r is even, and $\theta = 0$ if $k \equiv 0 \pmod{3}$. Let s, t be in $\mathbf{Z}_3$. Then, it follows from Theorem 5 that if $v = 0, 1$ and $s + t \in 1 - r + 3\mathbf{Z}_3$, or $v \geq 2$ and $s + t \in 1 + 3\mathbf{Z}_3$, then

$$|S_3(s; r, h, k) - S_3(t; r, h, k)| \leq |3^{1+\theta}| |s - t|,$$

and otherwise,

$$|S_3(s; r, h, k) - S_3(t; r, h, k)| = |3^\theta| |s - t|.$$

If $(k, 3) = 1$ and r is odd, we have $S_3(s; r, h, k) = \bar{S}_{3,2}(s; r, h, k)$. Hence, we have already known by (16) that

$$a_0 = 0, \quad |a_n| \leq \left| \frac{r 3^{n-1}}{(n-1)!} \right|, \quad n \geq 1.$$

We can see, in the manner similar to the proof of (44), that

$$(51) \quad \frac{a_1}{r} \equiv \frac{1}{(m+1-r)r} k^m S_{m+1}^{(r)}(h, k) \pmod{3}$$

for all odd integers $m > r$, and obtain the result corresponding to Theorem 4(i).

The argument of the first half of 4. for $p \geq 5$ is valid also for $p = 3$. Hence, the results a°) and b°) hold also for the function $S_3(s; r, h, k)$ with $(k, 3) = 1$ and $(r, 2) = 1$.

Applying Theorem 5(i) to $k = 1$, we obtain

$$(52) \quad B_n \equiv 1 - \frac{1}{3} - n^2 \pmod{3^{1+\text{ord}_3(n)}}$$

for all even integers $n \geq 4$.

§8. $S_2(s; r, h, k)$

In this section, we consider the case $p = 2$. In this case, $\alpha = e = 2$, and the function $S_{2,2}(s; r, h, k)$ will simply be denoted by $S_2(s; r, h, k)$. Let $k = k_0 2^v$, $(k_0, 2) = 1$, $v \geq 0$.

We first consider the case that $S_2(s; r, h, k)$ is not identical with $\bar{S}_{2,2}(s; r, h, k)$.

THEOREM 6. *Let $p = 2$ and let $S_2(s; r, h, k) = \sum_{n=0}^{\infty} a_n(s+1-r)^n$.*

(i) *If $k \not\equiv 0 \pmod{2}$ and r is even, then*

$$|a_0| = \left| \frac{1}{4} \right|, \quad |a_1| = \left| \frac{1}{2} \right|, \quad |a_n| \leq \left| \frac{2q^{n-2}}{n!} \right| \quad (n \geq 2).$$

In particular,

$$|S_2(s; r, h, k) - S_2(t; r, h, k)| = \left| \frac{1}{2} \right| |s - t|, \quad s, t \in \mathbf{Z}_2.$$

(ii) *If $k = 2k_0$ and r is even, or $k \equiv 0 \pmod{4}$, then*

$$|a_0| = \left| \frac{1}{2} \right|, \quad |a_1| = 1, \quad |a_n| \leq \left| \frac{q^{n-1}}{n!} \right| \quad (n \geq 2).$$

In particular,

$$|S_2(s; r, h, k) - S_2(t; r, h, k)| = |s - t|, \quad s, t \in \mathbf{Z}_2.$$

PROOF. It is sufficient to show the assertions for $|a_n|$. Since $|\bar{a}_n| \leq \left| \frac{q^n}{n!} \right|$, $n \geq 1$, by (17), we may assume for proofs that $k = 2^v$, $v \geq 0$.

(i) If $v = 0$ and r is even, then by Propositions 4 and 3, we have $|t_0| = 1$, $|t_1| \leq |q|$ and $|t_n| \leq \left| \frac{q^n}{n!} \right|$, $n \geq 2$. On the other hand, $|u_0| = \left| \frac{1}{q} \right|$, $|u_1| = \left| \frac{1}{2} \right|$ and $|u_n| \leq \left| \frac{2q^{n-2}}{n!} \right|$, $n \geq 2$. Hence we see $|a_0| = |t_0 u_0| = \left| \frac{1}{q} \right|$ and $|a_1| = |t_0 u_1| = \left| \frac{1}{2} \right|$. For $n \geq 2$, we have $|t_{n-i} u_i| \leq \left| \frac{q^{n-i}}{(n-i)!} \frac{2q^{i-2}}{i!} \right| \leq \left| \frac{2q^{n-2}}{n!} \right|$, $0 \leq i \leq n$, so $|a_n| \leq \left| \frac{2q^{n-2}}{n!} \right|$, $n \geq 2$. This proves (i).

(ii) The proofs for $1 \leq v \leq 2$ are quite similar to the above. For $v \geq 3$, we see from Propositions 4, 3 that $|t_0| = |2^{v-1}|$ and $|t_1| = |2^v|$. Since $|u_0| = \left| \frac{1}{2^v} \right|$ and $|u_1| = \left| \frac{1}{2} \right|$, we obtain $|a_0| = \left| \frac{1}{2} \right|$ and $|a_1| = 1$. For $n \geq 2$, we have $|t_n| \leq \left| \frac{2^v q^{n-1}}{n!} \right|$ and $|u_n| \leq \left| \frac{2^{v(n-1)-1}}{n!} \right|$. Hence, we have $|t_n u_0| \leq \left| \frac{q^{n-1}}{n!} \right|$, $|t_0 u_n| \leq \left| \frac{2^{vn-2}}{n!} \right| < \left| \frac{q^{n-1}}{n!} \right|$, and $|t_{n-i} u_i| \leq \left| \frac{2^v q^{n-i-1}}{(n-i)!} \frac{2^{v(i-1)-1}}{i!} \right| \leq \left| \frac{2^{vi-1} q^{n-i-1}}{n!} \right| \leq \left| \frac{q^{n-1}}{n!} \right|$ for $1 \leq i \leq n-1$. Therefore we get $|a_n| \leq \left| \frac{q^{n-1}}{n!} \right|$, $n \geq 2$. This completes the proof of Theorem 6.

REMARK. Let $1 \leq r \leq m$ with odd integer m . (I) Let k be odd. Then, by (4), we have (i) $|S_{m+1}^{(r)}(h, k)| = \left| \frac{1}{4} \right|$ for even r , (ii) $|S_{m+1}^{(r)}(h, k)| \leq 1$, $m \geq 3$, for odd r , and (iii) $S_2^{(1)}(h, k) \equiv \frac{k-1}{4} \pmod{2^0}$.

(II) Let $k = 2k_0$. Then, by Theorem 1 and the formula (5) we see

$$S_2(m; r, h, k) = k^m S_{m+1}^{(r)}(h, k) - 2^m k_0^m S_{m+1}^{(r)}(h, k_0).$$

Now, if r is even then $S_2(m; r, h, k) \equiv \frac{1}{2} \pmod{2^0}$ by Theorem 6(ii), and if r is odd then $S_2(m; r, h, k) \equiv 0 \pmod{4}$ by (17). Hence, by the properties of

$S_{m+1}^{(r)}(h, k_0)$ described above, we have (i) $k^m S_{m+1}^{(r)}(h, k) \equiv \frac{1}{2} \pmod{2^0}$ for even r , (ii) $k^m S_{m+1}^{(r)}(h, k) \equiv 0 \pmod{4}$, $m \geq 3$, for odd r , and (iii) $S_2^{(1)}(h, k) \equiv \frac{k_0 - 1}{4} \pmod{2^0}$.

(III) Finally, let $k \equiv 0 \pmod{4}$. Then, by Theorems 1, 6 (ii) and induction on $\text{ord}_2(k) \geq 2$, we have

$$\begin{aligned} S_2(m; r, h, k) &= k^m S_{m+1}^{(r)}(h, k) - 2^m \left(\frac{k}{2}\right)^m S_{m+1}^{(r)}\left(h, \frac{k}{2}\right) \\ &\equiv k^m S_{m+1}^{(r)}(h, k) \pmod{2^0} \end{aligned}$$

and

$$k^m S_{m+1}^{(r)}(h, k) \equiv \frac{1}{2} \pmod{2^0}.$$

By the way, we see from Theorem 6 (i), in the same manner as showing (42), that

$$(53) \quad B_n \equiv \frac{1}{2} - n \pmod{2^{1+\text{ord}_2(n)}},$$

for all even integers $n \geq 6$.

We next consider the case that $k \not\equiv 0 \pmod{4}$ and r is odd. In this case, we have $S_2(s; r, h, k) = \bar{S}_{2,2}(s; r, h, k)$.

First, let $k \not\equiv 0 \pmod{2}$. Then, for odd $r \geq 1$, we have

$$a_1 = \bar{a}_1 = r \sum_{\zeta^k=1, \zeta \neq 1} c_{0,r}(\zeta^h) E_{r-1}(\zeta^{-1}).$$

Further we have

$$\begin{aligned} (54) \quad c_{0,r}(\zeta) &= \int_{\mathbf{Z}_2^*} \omega(x) x^{-1} d\mu_\zeta(x) \equiv \int_{\mathbf{Z}_2^*} d\mu_\zeta(x) \pmod{4} \\ &= E_0(\zeta) - E_0(\zeta^2). \end{aligned}$$

Hence, for $r = 1$,

$$\begin{aligned} a_1 &\equiv \sum_{\zeta^k=1, \zeta \neq 1} (E_0(\zeta^h) - E_0(\zeta^{2h})) E_0(\zeta^{-1}) \pmod{4} \\ &= k S_2^{(1)}(h, k) - k S_2^{(1)}(2h, k). \end{aligned}$$

We shall observe this number on the right. It is easy to see from (1) that

$$(55) \quad kS_2^{(1)}(h, k) - kS_2^{(1)}(2h, k) = \sum_{j=0}^{k-1} j \left\{ \frac{jh}{k} \right\} - \sum_{j=0}^{k-1} j \left\{ \frac{2jh}{k} \right\}.$$

Now we have

$$\begin{aligned} \sum_{j=0}^{4k-1} j \left\{ \frac{jh}{k} \right\} &= \sum_{j=0}^{k-1} \sum_{i=0}^3 (j + ki) \left\{ \frac{jh}{k} \right\} = 4 \sum_{j=0}^{k-1} j \left\{ \frac{jh}{k} \right\} + 3k(k-1), \\ \sum_{j=0}^{k-1} (4j+1) \left\{ \frac{(4j+1)h}{k} \right\} &= 4 \sum_{j=0}^{k-1} j \left\{ \frac{(4j+1)h}{k} \right\} + \frac{k-1}{2}, \\ \sum_{j=0}^{2k-1} 2j \left\{ \frac{2jh}{k} \right\} &= 2 \sum_{j=0}^{k-1} \sum_{i=0}^1 (j + ki) \left\{ \frac{2jh}{k} \right\} = 4 \sum_{j=0}^{k-1} j \left\{ \frac{2jh}{k} \right\} + k(k-1), \end{aligned}$$

and, changing j to $k-1-j$,

$$\begin{aligned} \sum_{j=0}^{k-1} (4j+3) \left\{ \frac{(4j+3)h}{k} \right\} &= \sum_{j=0}^{k-1} (4k-4j-1) \left\{ \frac{(-4j-1)h}{k} \right\} \\ &= \frac{(4k-1)(k-1)}{2} - 4 \sum_{\substack{j=0 \\ j \neq \bar{j}}}^{k-1} j \left(1 - \left\{ \frac{(4j+1)h}{k} \right\} \right) \\ &= -\frac{k-1}{2} + 4\bar{j} + 4 \sum_{j=0}^{k-1} j \left\{ \frac{(4j+1)h}{k} \right\}, \end{aligned}$$

where $\bar{j}$ denotes the integer such that $0 < \bar{j} < k$ and $4\bar{j} + 1 \equiv 0 \pmod{k}$. Substituting the above equalities into the both sides of

$$\begin{aligned} \sum_{j=0}^{4k-1} j \left\{ \frac{jh}{k} \right\} &= \sum_{j=0}^{k-1} (4j+1) \left\{ \frac{(4j+1)h}{k} \right\} + \sum_{j=0}^{2k-1} 2j \left\{ \frac{2jh}{k} \right\} \\ &\quad + \sum_{j=0}^{k-1} (4j+3) \left\{ \frac{(4j+3)h}{k} \right\}, \end{aligned}$$

we obtain

$$\sum_{j=0}^{k-1} j \left\{ \frac{jh}{k} \right\} - \sum_{j=0}^{k-1} j \left\{ \frac{2jh}{k} \right\} = 2 \sum_{j=0}^{k-1} j \left\{ \frac{(4j+1)h}{k} \right\} - \frac{k(k-1)}{2} + \bar{j}.$$

If $k = 4l + 1$ then $\bar{j} = l = \frac{k-1}{4}$, and if $k = 4l - 1$ then $\bar{j} = k - l = \frac{3k-1}{4}$.

Hence, (55) becomes

$$(56) \quad kS_2^{(1)}(h, k) - kS_2^{(1)}(2h, k) = 2 \sum_{j=0}^{k-1} j \left\{ \frac{(4j+1)h}{k} \right\} - \frac{(k-1)^2}{2} + \frac{1 \mp k}{4},$$

according as $k \equiv \pm 1 \pmod{4}$. Therefore, we obtain

THEOREM 7. *Let $(k, 2) = 1$, $k > 1$ and $r = 1$. Then,*

$$a_0 = 0, \quad |a_n| \leq \left| \frac{q^{n-1}}{(n-1)!} \right|, \quad n \geq 1,$$

and we have

$$a_1 \equiv 2 \sum_{j=0}^{k-1} j \left\{ \frac{(4j+1)h}{k} \right\} - \frac{(k-1)^2}{2} + \frac{1 \mp k}{4} \pmod{4},$$

according as $k \equiv \pm 1 \pmod{4}$. In particular, $a_1 \equiv 0 \pmod{2}$ if and only if $k \equiv \pm 1 \pmod{8}$.

REMARK. Let k be odd and $m \geq 3$. Then, by Theorem 1 and the Remark for Theorem 6, we have

$$\begin{aligned} S_2(m; 1, h, k) &= k^m S_{m+1}^{(1)}(h, k) - 2^{m-1} k^m S_{m+1}^{(1)}(2h, k) \\ &\equiv k^m S_{m+1}^{(1)}(h, k) \pmod{4}. \end{aligned}$$

Hence by Theorem 7 and (5)' we obtain for $m \geq 3$

$$(57) \quad \frac{1}{m} k^m S_{m+1}^{(1)}(h, k) \equiv 2 \sum_{j=0}^{k-1} j \left\{ \frac{(4j+1)h}{k} \right\} - \frac{(k-1)^2}{2} + \frac{1 \mp k}{4} \pmod{4},$$

$$(58) \quad \frac{1}{m} k^m S_{m+1}^{(m)}(h, k) \equiv 2 \sum_{j=0}^{k-1} j \left\{ \frac{(4j+1)h^*}{k} \right\} - \frac{(k-1)^2}{2} + \frac{1 \mp k}{4} \pmod{4},$$

according as $k \equiv \pm 1 \pmod{4}$, where h^* is an integer such that $hh^* \equiv 1 \pmod{k}$. In particular,

$$(59) \quad S_{m+1}^{(1)}(h, k) \equiv S_{m+1}^{(m)}(h, k) \equiv \frac{k^2 - 1}{8} \pmod{2} \quad \text{for } m \geq 3.$$

Finally, we consider the case that $k = k_0$, $r > 1$ or $k = 2k_0$, $r \geq 1$, where r and k_0 are odd, and $k_0 > 1$.

Let W_{k_0} be the group of k_0 -th roots of unity, and let $\mathcal{A}$ be a complete set of

representatives for the equivalence relation $' \sim '$ on $W_{k_0} - \{1\}$, defined by $\zeta \sim \zeta' \Leftrightarrow \zeta = \zeta'$ or $\zeta^{-1} = \zeta'$. We see by a little calculation that

$$(60) \quad c_{n,r}(\zeta) = (-1)^r c_{n,r}(\zeta^{-1}), \quad n \geq 0, \quad r \geq 1,$$

$$(61) \quad c_{n,r}(-\zeta) = -c_{n,r}(\zeta), \quad n \geq 0, \quad r \geq 1,$$

for $\zeta \in W_{k_0} - \{1\}$.

Let $k = k_0$ and r be odd > 1 . Then, by (26) and (60), we have

$$a_1 = r \sum_{\zeta^k=1, \zeta \neq 1} c_{0,r}(\zeta^h) E_{r-1}(\zeta^{-1}) = 2r \sum_{\zeta \in A} c_{0,r}(\zeta^h) E_{r-1}(\zeta^{-1}).$$

For odd $r > 1$, we obtain

$$(62) \quad E_{r-1}(\zeta) \equiv \int_{\mathbf{Z}_2^*} d\mu_\zeta(x) = E_0(\zeta) - E_0(\zeta^2) \pmod{4}.$$

Hence, we have

$$(63) \quad \begin{aligned} a_1 &\equiv 2r \sum_{\zeta \in A} (E_0(\zeta^h) - E_0(\zeta^{2h}))(E_0(\zeta^{-1}) - E_0(\zeta^{-2})) \pmod{8} \\ &= r \sum_{\zeta^k=1, \zeta \neq 1} (E_0(\zeta^h) - E_0(\zeta^{2h}))(E_0(\zeta^{-1}) - E_0(\zeta^{-2})) \\ &= r(kS_2^{(1)}(h, k) - kS_2^{(1)}(2h, k) + kS_2^{(1)}(h^*, k) - kS_2^{(1)}(2h^*, k)), \end{aligned}$$

where $hh^* \equiv 1 \pmod{k}$.

Let $k = 2k_0$ and r be odd ≥ 1 . Then,

$$(64) \quad \begin{aligned} a_1 &= \sum_{\zeta^k=1, \zeta^2 \neq 1} c_{0,r}(\zeta^h) E_{r-1}(\zeta^{-1}) \\ &= r \sum_{\zeta^{k_0}=1, \zeta \neq 1} c_{0,r}(\zeta^h) (E_{r-1}(\zeta^{-1}) - E_{r-1}(-\zeta^{-1})) && \text{by (61)} \\ &= 2r \sum_{\zeta \in A} c_{0,r}(\zeta^h) (E_{r-1}(\zeta^{-1}) - E_{r-1}(-\zeta^{-1})) && \text{by (26) and (60)} \\ &= 4r \sum_{\zeta \in A} c_{0,r}(\zeta^h) (E_{r-1}(\zeta^{-1}) - 2^{r-1} E_{r-1}(\zeta^{-2})) && \text{by (2)} \\ &\equiv 4r \sum_{\zeta \in A} (E_0(\zeta^h) - E_0(\zeta^{2h}))(E_0(\zeta^{-1}) - E_0(\zeta^{-2})) \pmod{16} \end{aligned}$$

by (54) and (62)

$$\begin{aligned}
&= 2r \sum_{\zeta^{k_0}=1, \zeta \neq 1} (E_0(\zeta^h) - E_0(\zeta^{2h}))(E_0(\zeta^{-1}) - E_0(\zeta^{-2})) \quad \text{by (26)} \\
&= 2r(k_0 S_2^{(1)}(h, k_0) - k_0 S_2^{(1)}(2h, k_0) + k_0 S_2^{(1)}(h^*, k_0) - k_0 S_2^{(1)}(2h^*, k_0)).
\end{aligned}$$

For relatively prime integers h and k with k odd > 1 , let $hh^* \equiv 1 \pmod{k}$, and put

$$R(h, k) = \sum_{j=0}^{k-1} j \left\{ \frac{(4j+1)h}{k} \right\} + \sum_{j=0}^{k-1} j \left\{ \frac{(4j+1)h^*}{k} \right\} - \frac{(k-1)^2}{2} + \frac{1 \mp k}{4},$$

according as $k \equiv \pm 1 \pmod{4}$. Then we have obtained the following

THEOREM 8. *Let k_0 be an odd integer > 1 .*

(i) *If $k = k_0$ and r is odd > 1 , then*

$$a_0 = 0, \quad |a_n| \leq \left| \frac{2q^{n-1}}{(n-1)!} \right|, \quad n \geq 1,$$

and we have $a_1 \equiv 2rR(h, k) \pmod{8}$.

(ii) *If $k = 2k_0$ and r is odd ≥ 1 , then*

$$a_0 = 0, \quad |a_n| \leq \left| \frac{q^n}{(n-1)!} \right|, \quad n \geq 1,$$

and we have $a_1 \equiv 4rR(h, k_0) \pmod{16}$.

REMARK. Let k and r be odd. Then, we have by Theorem 1

$$S_2(m; r, h, k) = k^m S_{m+1}^{(r)}(h, k) - 2^{m-r} k^m S_{m+1}^{(r)}(2h, k), \quad 1 \leq r \leq m.$$

If $3 \leq r \leq m-2$, then it follows from Theorem 8(i) that

$$(65) \quad S_{m+1}^{(r)}(h, k) \equiv 0 \pmod{2} \quad \text{and}$$

$$(66) \quad \frac{1}{(m+1-r)r} k^m S_{m+1}^{(r)}(h, k) \equiv 2R(h, k) \pmod{8}.$$

H. Lang [6] has proved that $S_{m+1}^{(r)}(h, k) \equiv 0 \pmod{2}$ for odd r , $3 \leq r \leq m-2$, and $S_{m+1}^{(1)}(h, k) \equiv S_{m+1}^{(m)}(h, k) \equiv \frac{k^2-1}{8} \pmod{2}$ for $m \geq 3$, and applied them to his theory of real quadratic field.

For $k = 2k_0$ and r odd, we have in the same manner as above that

$$S_2(s; r, h, k) = 2S_2(s; r, h, k_0) - 2^*S_2(s; r, 2^*h, k_0),$$

where 2^* denotes an integer x such that $2x \equiv 1 \pmod{k_0}$.

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