BIHARMONIC SURFACES IN PSEUDO-EUCLIDEAN SPACES

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1. Introduction.

Let M be a connected n -dimensional Riemannian manifold and E_s^m the m -dimensional pseudo-Euclidean space with the canonical pseudo-Euclidean metric of signature $(s, m-s)$ given by

$$(1.1) \quad g_0 = - \sum_{i=1}^s dy_i^2 + \sum_{j=s+1}^m dy_j^2,$$

where $(y_1, \dots, y_m)$ is a rectangular coordinate system of E_s^m . Denote by Δ the Laplacian of M . Assume $x: M \rightarrow E_s^m$ is an isometric immersion of M into E_s^m . Then the position vector of M in E_s^m satisfies

$$(1.2) \quad \Delta x = -nH,$$

where H is the mean curvature vector of M in E_s^m . From (1.2) it is clear that the submanifold M is minimal in E_s^m if and only if the immersion x is harmonic, i.e., $\Delta x = 0$. In the following an isometric immersion $x: M \rightarrow E_s^m$ is said to be *biharmonic* if

$$(1.3) \quad \Delta^2 x = 0.$$

It is obvious that every minimal immersion is biharmonic. So the first-named author asked in [3] the following simple geometric question:

“Other than minimal submanifolds, which submanifolds are biharmonic?”

It follows from (1.2) and (1.3) that there exist no compact (space-like)

biharmonic submanifolds of positive dimension in E_s^m . Moreover, the first-named author proved in 1985 that *every biharmonic surface in a Euclidean 3-space E^3 is a minimal surface* [3]. This non-existence result was later generalized to some more general submanifolds in E^m by I. Dimitric in his 1989 doctoral thesis [4] at Michigan State University. So far, there exist no known examples of biharmonic submanifolds in a Euclidean space other than the harmonic ones. On the other hand, one may construct many nontrivial examples of biharmonic submanifolds in pseudo-Euclidean spaces (see sections 3, 5 and 6). Hence, it is natural to ask the following [3]:

“PROBLEM: *Classify biharmonic submanifolds.*”

In this paper we investigate this problem. As a consequence, several classification theorems in this respect are obtained.

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2. Preliminaries.

Throughout this paper, *submanifolds are assumed to be connected, smooth, and of positive dimension unless mentioned otherwise.*

Let $x: M \rightarrow E_s^m$ be an isometric immersion of an n -dimensional Riemannian manifold M into E_s^m . Let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of M and E_s^m , respectively. For any vector fields X, Y tangent to M , the Gauss formula is given by

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where h denotes the second fundamental form of M in E_s^m . The mean curvature vector H is given by $H = \text{trace } h/n$, $n = \dim M$. Let A denote the Weingarten map. Then h and A are related by

$$(2.2) \quad \langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$$

for X, Y tangent to M and ξ normal to M , where $\langle, \rangle$ is the inner product in

E_s^m . Let D denote the normal connection of M in E_s^m . Then the Weingarten formula is given by

$$(2.3) \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi.$$

Let R denote the Riemannian curvature tensor of M . The equation of Gauss is then given by

$$(2.4) \quad \langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle.$$

For the second fundamental form h , we define the covariant derivative of h by

$$(2.5) \quad (\tilde{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The equation of Codazzi is given by

$$(2.6) \quad (\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Y h)(X, Z).$$

If we denote by R^D the curvature tensor associated with the normal connection D of M in E_s^m , then the equation of Ricci is given by

$$(2.7) \quad \langle R^D(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle$$

for X, Y tangent to M and ξ, η normal to M . Since M is Riemannian, one may choose orthonormal local frame fields $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$ on M such that $e_1, \dots, e_n$ are tangent to M , $e_{n+1}, \dots, e_m$ are normal to M with $\langle e_A, e_A \rangle = \varepsilon_A = \pm 1$. Let $\omega^1, \dots, \omega^m$ be the fields of dual 1-forms of $e_1, \dots, e_m$. The connection forms (ω_B^A) is then given by

$$(2.8) \quad \tilde{\nabla} e_A = \sum_{B=1}^m \omega_A^B e_B, \quad \omega_A^B = -\varepsilon_A \varepsilon_B \omega_B^A, \quad A, B, C = 1, \dots, m.$$

From (2.8) we find

$$(2.9) \quad d\omega_A^B = -\sum_{C=1}^m \omega_A^C \wedge \omega_C^B.$$

Put

$$(2.10) \quad h = \sum_{i,j,r} h_{ij}^r \omega^i \omega^j e_r, \quad i, j = 1, \dots, n, \quad r = n+1, \dots, m.$$

Then we have

$$(2.11) \quad \omega_i^r = \sum_j h_{ij}^r \omega^j, \quad \langle A_r e_i, e_j \rangle = \varepsilon_r h_{ij}^r,$$

where $A_r = A_{e_r}$.

A submanifold M in E_s^m is said to be of *finite type* (cf. [1, 2] for details) if each component of its position vector field x can be written as a finite sum of eigenfunctions of the Laplacian Δ of M , i.e.,

$$(2.12) \quad x = x_0 + \sum_{i=1}^k x_{t_i},$$

where x_0 is a constant vector and $\Delta x_{t_i} = \lambda_{t_i} x_{t_i}$, $i = 1, \dots, k$. If in particular all eigenvalues $\{\lambda_{t_1}, \dots, \lambda_{t_k}\}$ are mutually different, then M is said to be of k -type. A k -type submanifold M is said to be of *null k -type* if one of eigenvalues $\lambda_{t_1}, \dots, \lambda_{t_k}$ is zero. A submanifold is said to be of *infinite type* if it is not of finite type.

A biharmonic submanifold M of a pseudo-Euclidean space E_s^m is either of null 1-type (i.e., it is minimal) or of infinite type. This fact can be seen as follows: Since $\Delta^2 x = 0$ and $x_{t_1}, \dots, x_{t_k}$ are linearly independent, we have $\lambda_{t_1}^2 = \dots = \lambda_{t_k}^2 = 0$.

For the isometric immersion $x: M \rightarrow E_s^m$, we have the following useful formula obtained in [1, 2]:

$$(2.13) \quad \Delta H = \Delta^D H + \sum_{i=1}^n h(A_H e_i, e_i) + (\Delta H)^T,$$

where Δ^D is the Laplacian operator associated with the normal connection D and $(\Delta H)^T$ is the tangential component of ΔH satisfying [2]

$$(2.14) \quad (\Delta H)^T = \frac{n}{2} \text{grad} \langle H, H \rangle + 2 \text{trace } A_{DH}.$$

From (2.13) and (2.14) we obtain the following

LEMMA 2.1. *The isometric immersion $x: M \rightarrow E_s^m$ is biharmonic if and only if the following two differential equations hold:*

$$(2.15) \quad \Delta^D H + \sum_{i=1}^n h(A_H e_i, e_i) = 0,$$

$$(2.16) \quad \frac{n}{2} \text{grad} \langle H, H \rangle + 2 \text{trace } A_{DH} = 0.$$

3. Classification of Biharmonic Curves.

Let $x: C \rightarrow E_s^m$ be an isometric immersion of a 1-dimensional Riemannian manifold C into E_s^m parametrized by arclength s . Then the Laplacian Δ of C is given by $-\frac{d^2}{ds^2}$ and the mean curvature vector H is given by $H = x''$, where the derivative is taking with respect to the arclength s . The condition that x being biharmonic is equivalent to $x^{iv} = 0$. Thus if the immersion x is biharmonic, then there exist constant vectors c_1, c_2, c_3 , and c_4 in E_s^m such that

$$(3.1) \quad x(s) = c_1 s^3 + c_2 s^2 + c_3 s + c_4.$$

Therefore, we have the following

LEMMA 3.1. *Let $x: C \rightarrow E_s^m$ be an isometric immersion of a 1-dimensional Riemannian manifold C into E_s^m . If x is biharmonic, then the image of x is contained in a 3-dimensional linear subspace of E_s^m .*

The main purpose of this section is to prove the following classification theorem for biharmonic curves.

THEOREM 3.2. *Let $x: C \rightarrow E_s^3$ be an isometric immersion of a 1-dimensional Riemannian manifold C into E_s^3 . Then the immersion x is biharmonic if and only if one of the following three statements holds:*

- (i) $s = 0, 1$, or 2 and x is linear.
- (ii) $s = 1$ and, up to rigid motions of E_1^3 , x is given by

$$(3.2) \quad x(s) = (\delta s^3 + \bar{\delta} s^2, \delta s^3 + \bar{\delta} s^2, s)$$

for some constant δ and $\bar{\delta}$ satisfying $\delta^2 + \bar{\delta}^2 \neq 0$ or given by

$$(3.2)' \quad x(s) = \left(\frac{\delta^2}{6} s^3, \frac{\delta}{2} s^2, -\frac{\delta^2}{6} s^3 + s \right)$$

for some nonzero constant δ .

- (iii) $s = 2$ and, up to rigid motions of E_2^3 , x is given by

$$(3.3) \quad x(s) = \left(\frac{\delta^2}{6} s^3, \frac{\delta}{2} s^2, \frac{\delta^2}{6} s^3 + s \right)$$

for some nonzero constant δ .

PROOF. If x is given by (i), (ii) or (iii), then it is easy to verify that x is biharmonic. Conversely, suppose that $x: C \rightarrow E_s^3$ is biharmonic. Then (3.1) holds. So, by using the condition that s is the arclength of x we may obtain

$$(3.4) \quad \langle c_1, c_1 \rangle = \langle c_1, c_2 \rangle = \langle c_2, c_3 \rangle = 0,$$

$$(3.5) \quad 3 \langle c_1, c_3 \rangle + 2 \langle c_2, c_2 \rangle = 0, \quad \langle c_3, c_3 \rangle = 1.$$

Without loss of generality, we may assume

$$(3.6) \quad c_3 = (0, 0, 1).$$

It is clear that the mean curvature vector $H = 0$ if and only if x is linear. So, in the remaining part of the proof we assume that $H \neq 0$.

Case 1: $\langle H, H \rangle = 0$. In this case we have

$$(3.7) \quad \langle c_2, c_2 \rangle = \langle c_1, c_3 \rangle = 0.$$

From (3.4)–(3.7), we have $s = 1$ and, up to rigid motions of E_1^3 , c_1 and c_2 are given by $c_1 = (\delta, \delta, 0)$, $c_2 = (\bar{\delta}, \bar{\delta}, 0)$ for some constants δ and $\bar{\delta}$ satisfying $\delta^2 + \bar{\delta}^2 \neq 0$. Thus, x is a curve given by (3.2).

Case 2: $\langle H, H \rangle \neq 0$. In this case, (3.1) implies $H = 6c_1s + 2c_2$. Thus, by (3.4), $\langle H, H \rangle = 4 \langle c_2, c_2 \rangle$ which is a nonzero constant. Let N denote the unit vector in the direction of H so that $H = \kappa N$ with $\kappa > 0$. We treat the cases $\langle N, N \rangle = 1$ and $\langle N, N \rangle = -1$, separately.

Case 2.1: $\langle N, N \rangle = 1$. In this case, $\kappa^2 = 4 \langle c_2, c_2 \rangle = \text{constant} \neq 0$ and $T(=x')$, N are orthonormal space-like vectors in E_s^3 . Let $Y = N' + \kappa T$. Then Y is perpendicular to both T and N . Since

$$Y = (6\kappa^{-1} + 3\kappa s^2)c_1 + 2\kappa s c_2 + \kappa c_3,$$

(3.4) and (3.5) imply

$$\langle Y, Y \rangle = -8 \langle c_2, c_2 \rangle + \kappa^2 = -\kappa^2 < 0.$$

Thus, $s = 1$, Y is time-like and T , N , $\kappa^{-1}Y$ are orthonormal. Thus we have

$$\langle c_2, c_2 \rangle = \frac{\kappa^2}{4}, \quad \langle c_1, c_3 \rangle = -\frac{\kappa^2}{6}.$$

For this and (3.4)–(3.6), we may choose a coordinate system on E_1^3 such that c_1

$= \left(\frac{\delta^2}{6}, 0, -\frac{\delta^2}{6} \right), c_2 = \left(0, \frac{\delta}{2}, 0 \right)$. Therefore, up to rigid motions of E_1^3 , x is a curve given by (3.2)'.

Case 2.2: $\langle N, N \rangle = -1$. In this case, we have $\kappa^2 = -4 \langle c_2, c_2 \rangle$. Thus, from (3.4) and (3.5), we find

$$(3.8) \quad \langle c_2, c_2 \rangle = -\frac{\kappa^2}{4}, \quad \langle c_1, c_3 \rangle = \frac{\kappa^2}{6}.$$

Since $\kappa \neq 0$, c_2 is time-like. Because $T(=x')$ and N are orthonormal, we may obtain from direct computation that the vector $Y = N' - \kappa T$ satisfies

$$\langle Y, Y \rangle = 8 \langle c_2, c_2 \rangle + \kappa^2 = -\kappa^2.$$

Thus, Y is time-like. Since $T, N, \kappa^{-1}Y$ are orthonormal, $s = 2$. Moreover, from (3.4), (3.5), and $s = 2$, we may obtain

$$x(s) = \left(as^3 + bcs^2, cs^3 - abs^2, \frac{3}{2}b^{-1}s^3 + s \right)$$

for some constants a, b, c satisfying $4(a^2 + c^2)b^4 = 9$. We then obtain (3.3) by applying the motion on E_2^3 given by

$$\begin{pmatrix} 6\delta^{-2}a & 6\delta^{-2}c & 0 \\ 6\delta^{-2}c & -6\delta^{-2}a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \left(\delta = \frac{3}{b} \right).$$

This completes the proof.

REMARK 3.1. It is easy to verify that the mean curvature vector H of a curve defined by (3.2) is a null vector and the mean curvature vector of a curve defined by (3.2)' and (3.3) satisfies $\langle H, H \rangle = \delta^2$ and $\langle H, H \rangle = -\delta^2 < 0$, respectively.

4. Biharmonic Surfaces in E_s^3 .

In this section we prove the following non-existence theorem.

THEOREM 4.1. *Let $x: M \rightarrow E_s^3$ ($s = 0, 1$) be a biharmonic isometric*

immersion of a Riemannian surface M into E_s^3 . Then x is harmonic.

PROOF. Let $x: M \rightarrow E_s^3$ be a biharmonic isometric immersion. We put $H = \alpha e_3$. Then $\langle H, H \rangle = \alpha^2 \varepsilon_3$ and from (2.15) and (2.16) we have

$$(4.1) \quad \Delta \alpha + \varepsilon_3 \alpha \|A_3\|^2 = 0,$$

$$(4.2) \quad A_3(\text{grad } \alpha) = -\varepsilon_3 \alpha (\text{grad } \alpha).$$

Thus we have

$$(4.3) \quad h(e_1, e_1) = \varepsilon_3 \beta e_3, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \varepsilon_3 \gamma e_3$$

for some functions β, γ on the open subset $U = \{p \in M \mid \text{grad } \alpha^2 \neq 0 \text{ at } p\}$. From equation (2.4) of Codazzi and (4.3) we have

$$(4.4) \quad e_2 \beta = (\beta - \gamma) \omega_1^2(e_1), \quad e_1 \gamma = (\beta - \gamma) \omega_1^2(e_2).$$

Assume $U \neq \emptyset$. Then, by (4.2), $\text{grad } \alpha$ is an eigenvector of A_3 with eigenvalue $-\varepsilon_3 \alpha$ on U . We choose e_1 in the direction of $\text{grad } \alpha$ on U . Then we have $e_2 \alpha = 0$ on U . Moreover, the Weingarten map A_3 satisfies

$$(4.5) \quad \beta = -\varepsilon_3 \alpha, \quad \gamma = 3\varepsilon_3 \alpha, \quad \|A_3\|^2 = 10\alpha^2.$$

Since $e_2 \alpha = 0$, (4.4) and (4.5) imply

$$(4.6) \quad \omega_1^2(e_1) = 0, \quad d\omega^1 = 0.$$

Thus, locally, $\omega^1 = du$ for some function u . Since $d\alpha \wedge \omega^1 = d\alpha \wedge du = 0$, α is a function of u . We denote by α' and α'' the first two derivatives of α with respect to u . From (4.4), (4.5), and (4.6), we have

$$(4.7) \quad 4\alpha \omega_1^2 = -3\alpha' \omega^2.$$

Also from $e_2 \alpha = 0$ and (4.7), we have

$$(4.8) \quad 4\alpha \Delta \alpha = 3(\alpha')^2 - 4\alpha \alpha''.$$

By using (4.1), (4.5), (4.6), and (4.8), we get

$$(4.9) \quad 4\alpha \alpha'' - 3(\alpha')^2 - 40\varepsilon_3 \alpha^4 = 0.$$

From (4.9) we may obtain

$$(4.10) \quad (\alpha')^2 = 8\varepsilon_3 \alpha^4 + C\alpha^{3/2}$$

for some constant C . On the other hand, equation (2.4) of Gauss, (4.3) and (4.7) imply

$$(4.11) \quad \alpha\alpha'' - \frac{7}{4}(\alpha')^2 + 4\epsilon_3\alpha^4 = 0.$$

From (4.9) and (4.11) we get

$$(4.12) \quad (\alpha')^2 = 16\epsilon_3\alpha^4.$$

Combining (4.10) and (4.12) we conclude that α is constant on U which is a contradiction. Therefore, U is empty. So, M has constant mean curvature. Thus, by applying (4.1), we obtain $\alpha = 0$. Hence the immersion x is harmonic. This completes the proof of the theorem.

REMARK 4.1. If $s = 0$, Theorem 4.1 was the results of the first-named author mentioned in Introduction.

5. Biharmonic Surfaces with Nonzero Constat Mean Curvature.

First we give some nontrivial examples of biharmonic surfaces in E_s^4 , $s = 1$ or 2 with nonzero constant mean curvature.

EXAMPLE 5.1. Let M be a surface in E_2^4 defined by

$$(5.1) \quad x(u, v) = \left(\frac{a^2}{6}u^3, \frac{a}{2}u^2, \frac{a^2}{6}u^3 + u, v \right)$$

for some constant a . Then we have

$$H = \frac{1}{2}(a^2u, a, a^2u, 0).$$

It is easy to verify that $\Delta^2 x = 0$. Thus, x is a biharmonic isometric immersion with respect to the induced metric. And $\langle H, H \rangle = -\frac{1}{4}a^2$ is a nonzero constant.

EXAMPLE 5.2. Let M be a surface in E_1^4 defined by

$$(5.2) \quad x(u, v) = \left(\frac{a^2}{6}u^3, \frac{a}{2}u^2, -\frac{a^2}{6}u^3 + u, v \right)$$

for some nonzero constant a . Then we have

$$H = \frac{1}{2}(a^2u, a, -a^2u, 0).$$

It is easy to verify that x is a biharmonic isometric immersion with respect to the induced metric and $\langle H, H \rangle = \frac{1}{4}a^2$ is nonzero constant.

The main purpose of this section is to prove the following classification theorem.

THEOREM 5.1. *Let $x: M \rightarrow E_s^4$ be an isometric immersion of a Riemannian surface M into E_s^4 . Then x is a biharmonic immersion with nonzero constant mean curvature if and only if $s = 2$ and, up to rigid motions of E_2^4 , M is a surface given by Example 5.1 or $s = 1$ and, up to rigid motions of E_1^4 , M is a surface given by Example 5.2.*

We derive some general formulas concerning biharmonic surfaces in E_s^4 for later use.

Assume M is a biharmonic surface in E_s^4 with $\langle H, H \rangle \neq 0$. We choose e_3 in the direction of H so that $H = \alpha e_3$, $\alpha > 0$. Then we have

$$(5.3) \quad \text{trace } A_4 = 0,$$

$$(5.4) \quad \Delta^D H = (\Delta \alpha + \alpha \varepsilon_3 \varepsilon_4 \|\omega_4^3\|^2) e_3 - (2\omega_3^4(\text{grad } \alpha) + \alpha \text{trace } (\nabla \omega_3^4)) e_4,$$

$$(5.5) \quad \sum_{i=1}^2 h(A_H e_i, e_i) = (\varepsilon_3 \alpha \|A_3\|^2) e_3 + (\varepsilon_4 \text{trace } A_H A_4) e_4,$$

$$(5.6) \quad \text{trace } A_{DH} = A_3(\text{grad } \alpha) + \alpha \omega_3^4(e_1) A_4 e_1 + \alpha \omega_3^4(e_2) A_4 e_2,$$

where

$$\text{trace } (\nabla \omega_3^4) = \sum_{i=1}^2 (\nabla_{e_i} \omega_3^4)(e_i) \quad \text{and} \quad \text{trace } A_{DH} = \sum_{i=1}^2 (A_{D_{e_i} H})(e_i).$$

From Lemma 2.1, (2.15), (2.16), (5.4), (5.5), and (5.6), we may obtain the following

LEMMA 5.1. *Let M be a biharmonic surface in E_s^4 . If $\langle H, H \rangle \neq 0$ and $H = \alpha e_3$ with $\alpha > 0$, then we have*

$$(5.7) \quad \Delta \alpha + \alpha \varepsilon_3 \varepsilon_4 \|\omega_4^3\|^2 + \alpha \varepsilon_3 \|A_3\|^2 = 0,$$

$$(5.8) \quad \varepsilon_4 \operatorname{trace} A_H A_4 = 2\omega_3^4(\operatorname{grad} \alpha) + \alpha \operatorname{trace} (\nabla \omega_3^4),$$

$$(5.9) \quad \varepsilon_3 \alpha \operatorname{grad} \alpha + A_3(\operatorname{grad} \alpha) + \alpha \omega_3^4(e_1) A_4 e_1 + \alpha \omega_3^4(e_2) A_4 e_2 = 0.$$

PROOF OF THEOREM 5.1. Assume x is a biharmonic isometric immersion with nonzero constant mean curvature. Since $H = \alpha e_3$, $\langle H, H \rangle = \varepsilon_3 \alpha^2$ is a nonzero constant. Thus (5.9) implies

$$(5.10) \quad \omega_3^4(e_1) A_4 e_1 + \omega_3^4(e_2) A_4 e_2 = 0.$$

Put

$$(5.11) \quad \omega_2^1 = -\omega_1^2 = f_1 \omega^1 + f_2 \omega^2, \quad \omega_3^4 = -\varepsilon_3 \varepsilon_4 \omega_4^3 = g_1 \omega^1 + g_2 \omega^2.$$

Equations (5.3) and (5.10) yield

$$(5.12) \quad \|A_4\| \|\omega_3^4\| = 0.$$

If $\omega_3^4 = 0$, then (5.7) and $\operatorname{grad} \alpha = 0$ give $A_3 = 0$ which implies $\alpha = 0$. This contradicts to the assumption $\alpha \neq 0$. Thus we have

$$(5.13) \quad \omega_3^4 \neq 0 \text{ and } A_4 \equiv 0.$$

From (5.8) and (5.13) we find

$$(5.14) \quad \operatorname{trace} (\nabla \omega_3^4) = 0.$$

From (5.11) and (5.14) we get

$$(5.15) \quad e_1 g_1 + e_2 g_2 + f_1 g_2 - f_2 g_1 = 0.$$

From equation (2.7) of Ricci, (5.11), and (5.13) we have

$$(5.16) \quad e_1 g_2 - e_2 g_1 = f_1 g_1 + f_2 g_2.$$

Since $A_4 = 0$, we may choose e_1, e_2 on the open subset $V = \{p \in M \mid A_3 \neq \alpha I \text{ at } p\}$ such that

$$(5.17) \quad h(e_1, e_1) = \varepsilon_3 \beta e_3, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \varepsilon_3 \gamma e_3$$

for some functions β and γ . From equation (2.6) of Codazzi, (5.11), and (5.17), we obtain

$$(5.18) \quad \beta g_2 = 0, \quad e_2 \beta = f_1(\gamma - \beta),$$

$$(5.19) \quad \gamma g_1 = 0, \quad e_1 \gamma = f_2(\gamma - \beta).$$

Since $A_3 \neq 0$ and $\omega_3^4 \neq 0$, (5.18), (5.19), and the assumption imply that both β and γ are constant on V , exactly one of β, γ is zero, and also exactly one of g_1, g_2 is zero on V . Since M is biharmonic, V is nonempty. Thus, by continuity, we get $V = M$. Without loss of generality, we may assume

$$(5.20) \quad g_1 \neq 0, g_2 = 0, \beta \neq 0, \gamma = 0.$$

In particular, this implies $\beta - \gamma \neq 0$ and β is constant. Thus, from (5.18), (5.19) and (5.20), we get

$$(5.21) \quad f_1 = f_2 = 0, \text{ i.e., } \omega_1^2 = \omega_2^1 = 0,$$

from which we conclude that M is flat. Moreover, from (5.15), (5.16), (5.20), and (5.21) we get

$$(5.22) \quad \omega_1^2 = \omega_2^3 = \omega_1^4 = \omega_2^4 = 0,$$

$$(5.23) \quad \omega_1^3 = \varepsilon_3 \beta \omega^1, \omega_3^4 = \delta \omega^1,$$

where β and δ are nonzero constants.

On the other hand, from (5.7), (5.23), and the constancy of α ($\neq 0$), we have

$$(5.24) \quad \varepsilon_4 \delta^2 + \beta^2 = 0.$$

Thus we get

$$(5.25) \quad \varepsilon_4 = -1, \delta^2 = \beta^2.$$

Thus, $s > 0$. Without loss of generality, we may choose e_1, e_2, e_3, e_4 in such way that $\delta = \beta$. Thus we obtain

$$(5.26) \quad \omega_1^2 = \omega_2^3 = \omega_1^4 = \omega_2^4 = 0, \omega_1^3 = \varepsilon_3 \beta \omega^1, \omega_3^4 = \beta \omega^1.$$

From (5.26) we have $\tilde{\nabla} e_2 = 0, d\omega^1 = d\omega^2 = 0$. Thus, e_2 is a constant space-like unit vector in E_s^4 . Let C_1 and C_2 be the integral curves of e_1 and e_2 through a fixed point in M , respectively. Then C_2 is an open portion of a straight line and C_1 is a curve lying in a E_s^3 perpendicular to the line containing C_2 . Since M is locally the Riemannian product of C_1 and C_2 , the biharmonicity of M implies the biharmonicity of C_1 in E_s^3 . Furthermore, the mean curvature vector H_1 of C_1 is given by $\varepsilon_3 \beta e_3$ which satisfies $\langle H_1, H_1 \rangle = \varepsilon_3 \beta^2 \neq 0$. Thus, by applying Theorem 3.2 and Remark 3.1, we conclude either $s = 1$ and, up to rigid motions, C_1 is given by (3.2)' or $s = 2$ and, up to rigid motions, C_1 is given by (3.3).

The converse of this was mentioned in Example 5.1 and Example 5.2. This completes the proof of the theorem.

6. Biharmonic Surfaces with Light-Like Mean Curvature Vector.

First, we give some nontrivial examples of biharmonic surfaces in E_1^4 with light-like mean curvature vector $H \neq 0$.

EXAMPLE 6.1. Let M be an open portion of the Euclidean plane E^2 with Euclidean coordinates (u, v) . Let $\varphi = \varphi(u, v)$ be a smooth function on M such that $\Delta\varphi \neq 0$ and $\Delta^2\varphi = 0$. Let $x: M \rightarrow E_1^4$ be defined by

$$(6.1) \quad x(u, v) = (\varphi(u, v), \varphi(u, v), u, v).$$

Then x is a biharmonic isometric immersion whose mean curvature vector is given by

$$(6.2) \quad H = -\frac{1}{2}(\Delta\varphi, \Delta\varphi, 0, 0) \neq 0.$$

From (6.2) it is easy to see that the mean curvature vector H of the immersion x is light-like, i.e., $\langle H, H \rangle = 0$.

A submanifold M is said to be *quasi-minimal* if the mean curvature vector of M is nonzero and light-like.

The main purpose of this section is to prove the following classification theorem.

THEOREM 6.1. Let $x: M \rightarrow E_s^4$ be an isometric immersion of a Riemannian surface M into E_s^4 . Then x is quasi-minimal and biharmonic if and only if $s = 1$ and, up to rigid motions of E_1^4 , x is a surface given by Example 6.1.

PROOF. Let $x: M \rightarrow E_s^4$ be an isometric immersion of M into E_s^4 . If x is quasi-minimal, then $s = 1$. Thus, we may choose e_3 and e_4 so that e_3 is space-like and e_4 is time-like, i.e., $\varepsilon_3 = 1$, $\varepsilon_4 = -1$. Because H is light-like, we have

$$(6.3) \quad H = \delta(e_3 + e_4)$$

for some function $\delta \neq 0$. By straight-forward computation we have

$$(6.4) \quad \sum_{i=1}^2 h(A_H e_i, e_i) = \delta(\|A_3\|^2 + \text{trace } A_3 A_4)e_3 - \delta(\|A_4\|^2 + \text{trace } A_3 A_4)e_4,$$

$$(6.5) \quad \Delta^p H = \{\Delta \delta - \delta \|\omega_3^4\|^2 - \delta \operatorname{trace}(\nabla \omega_3^4) - 2\omega_3^4(\operatorname{grad} \delta)\}(e_3 + e_4),$$

$$(6.6) \quad \operatorname{trace} A_{DH} = \sum_{i=1}^2 \{e_i \delta + \delta \omega_3^4(e_i)\}(A_3 e_i + A_4 e_i).$$

From Lemma 2.1, (6.4), (6.5), and (6.6) we get

$$(6.7) \quad \delta\{\|A_3\|^2 + \operatorname{trace} A_3 A_4 - \operatorname{trace}(\nabla \omega_3^4) - \|\omega_3^4\|^2\} + \Delta \delta = 2\omega_3^4(\operatorname{grad} \delta),$$

$$(6.8) \quad \delta\{\|A_4\|^2 + \operatorname{trace} A_3 A_4 + \operatorname{trace}(\nabla \omega_3^4) + \|\omega_3^4\|^2\} - \Delta \delta = -2\omega_3^4(\operatorname{grad} \delta),$$

$$(6.9) \quad \sum_{i=1}^2 \{e_i \delta + \delta \omega_3^4(e_i)\}(A_3 e_i + A_4 e_i) = 0.$$

From (6.7) and (6.8) we obtain

$$(6.10) \quad A_3 = -A_4, \quad A_H = 0.$$

Thus we have

$$(6.11) \quad h(e_1, e_1) = \beta(e_3 + e_4), \quad h(e_1, e_2) = \mu(e_3 + e_4), \quad h(e_2, e_2) = \gamma(e_3 + e_4)$$

for some functions β, γ and μ . Equation (2.4) of Gauss and (6.10) imply $R = 0$, i.e., M is flat. Thus, we may choose e_1, e_2 such that

$$(6.12) \quad \omega_1^2 = 0.$$

Moreover, from equation (2.7) of Ricci and (6.11) we get $R^p = 0$, i.e., x has flat normal connection. Thus, we may also choose e_3, e_4 so that

$$(6.13) \quad \omega_3^4 = 0.$$

Equations (6.10) and (6.13) imply

$$(6.14) \quad \tilde{\nabla}(e_3 + e_4) = 0.$$

Hence, $e_3 + e_4$ is a constant light-like vector in E_1^4 . Therefore, we may choose a pseudo-Euclidean coordinate system on E_1^4 such that $e_3 + e_4$ is given by

$$(6.15) \quad e_3 + e_4 = (\varepsilon, \varepsilon, 0, 0)$$

for some constant $\varepsilon \neq 0$. Since M is flat, M is locally an open portion of a Euclidean plane. Assume x is given by

$$(6.16) \quad x(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v), x_4(u, v)),$$

where (u, v) is a Euclidean coordinate system of the Euclidean plane E^2 . Let $e_1 = \frac{\partial}{\partial u}$, $e_2 = \frac{\partial}{\partial v}$. Since $\langle e_1, e_3 + e_4 \rangle = \langle e_2, e_3 + e_4 \rangle = 0$, (6.15) and (6.16) imply

$$(6.17) \quad \frac{\partial(x_1 - x_2)}{\partial u} = \frac{\partial(x_1 - x_2)}{\partial v} = 0.$$

Thus, by applying a translation on E_1^4 if necessary, we may assume

$$(6.18) \quad x_1(u, v) = x_2(u, v) = \varphi(u, v)$$

for some function $\varphi = \varphi(u, v)$. Let $z = x_3(u, v)$, $w = x_4(u, v)$. Then

$$e_1 = (\varphi_u, \varphi_u, z_u, w_u), \quad e_2 = (\varphi_v, \varphi_v, z_v, w_v).$$

Since e_1 and e_2 are orthonormal, we find

$$(6.19) \quad z_u^2 + w_u^2 = z_v^2 + w_v^2 = 1, \quad z_u z_v + w_u w_v = 0.$$

From (1.2), (6.3), (6.15), and (6.16) we have

$$(6.20) \quad z_{uu} + z_{vv} = w_{uu} + w_{vv} = 0.$$

From (6.19) and (6.20) we obtain

$$(6.21) \quad (z_u w_v - z_v w_u) z_{vv} = (z_v w_u - z_u w_v) w_{uu} = 0,$$

$$(6.22) \quad (z_u w_v - z_v w_u) w_{vv} = (z_v w_u - z_u w_v) z_{uu} = 0.$$

Case 1: $z_u = 0$. In this case, (6.19) implies $w_u^2 = 1$, $w_v = 0$, and $z_v^2 = 1$. Thus, up to rigid motions, (6.16) can be expressed by

$$(6.23) \quad x(u, v) = (\varphi(u, v), \varphi(u, v), u, v).$$

Case 2: $z_v = 0$, or $w_u = 0$, or $w_v = 0$. In this case, by an argument similar to Case 1, we obtain the same result as Case 1.

Case 3: $z_u z_v w_u w_v \neq 0$. In this case, since $z_u \neq 0$, the last equation of (6.19) implies

$$(6.24) \quad z_v = -\frac{w_u w_v}{z_u}.$$

From (6.21), (6.22) and (6.24) we obtain

$$(6.25) \quad z_{uu} = z_{vv} = w_{uu} = w_{vv} = 0.$$

By taking the derivatives of the last equation of (6.19) with respect to u and v , respectively, we obtain

$$(6.26) \quad z_u z_{uv} + w_u w_{uv} = 0, \quad z_v z_{uv} + w_v w_{uv} = 0,$$

from which we get

$$(6.27) \quad (z_u w_v - z_v w_u) z_{uv} = 0, \quad (z_u w_v - z_v w_u) w_{uv} = 0.$$

From (6.24) and (6.27) we obtain

$$(6.28) \quad z_{uv} = 0, \quad w_{uv} = 0.$$

Equations (6.25) and (6.28) imply that $z(u, v)$ and $w(u, v)$ are both linear functions in u and v . This we have

$$z = a_{11}u + a_{12}v + c_1, \quad w = a_{21}u + a_{22}v + c_2$$

for some constants $a_{11}, a_{12}, a_{21}, a_{22}, c_1$ and c_2 . Without loss of generality, we may assume $c_1 = c_2 = 0$. From these we have

$$e_1 = (\varphi_u, \varphi_v, a_{11}, a_{12}), \quad e_2 = (\varphi_u, \varphi_v, a_{21}, a_{22}).$$

Since e_1, e_2 are orthonormal, $(a_{ij}) \in O(2)$. Thus, by applying a rigid motion of E_1^4 if necessary, we have $a_{11} = a_{22} = 1, a_{12} = a_{21} = 0$. Therefore, up to rigid motions of E_1^4 , (6.16) can be expressed in the form of (6.23) too. Since $\Delta x \neq 0$ and $\Delta^2 x = 0$, the function φ satisfies $\Delta \varphi \neq 0$ and $\Delta^2 \varphi = 0$.

The converse of this is easy to verify. This completes the proof of the theorem.

7. Biharmonic Surfaces with Constant Gauss Curvature.

The purpose of this section is to prove the following

THEOREM 7.1. *Let $x: M \rightarrow E_s^4$ be a non-minimal isometric immersion of a Riemannian surface M of constant Gauss curvature into E_s^4 . Then x is biharmonic immersion with flat normal connection, if and only if, up to rigid motions of E_s^4 , M is a surface given by Example 5.1, or a surface given by Example 5.2, or a surface given by Example 6.1.*

PROOF. Assume that M is a Riemannian surface of constant Gauss

curvature and $x: M \rightarrow E_s^4$ is a biharmonic isometric immersion of M into E_s^4 with flat normal connection. According to Theorems 5.1 and 6.1, we may assume that the mean curvature vector H satisfies $\langle H, H \rangle \neq 0$ and $\langle H, H \rangle \neq \text{constant}$. Thus, there exists a nonempty connected open subset U of M such that both $\langle H, H \rangle$ and $\text{grad } \langle H, H \rangle$ are nowhere zero on U . In the remaining part of the proof we shall only work on U . We choose e_3 such that $H = \alpha e_3$, $\alpha > 0$ where α is nowhere constant, i.e., $\text{grad } \alpha \neq 0$ everywhere. As in section 5, we put

$$(7.1) \quad \omega_2^1 = -\omega_1^2 = f_1\omega^1 + f_2\omega^2, \quad \omega_3^4 = -\varepsilon_3\varepsilon_4\omega_4^3 = g_1\omega^1 + g_2\omega^2.$$

Since the normal connection is flat, i.e., $R^D = 0$, equation (2.7) of Ricci implies $[A_3, A_4] = 0$. Hence, A_3 and A_4 are diagonalizable simultaneously. We consider the open subset W of M on which there exists local orthonormal frame e_1, e_2 so that A_3 and A_4 are given by

$$(7.2) \quad A_3 = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \quad A_4 = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \quad \alpha = \varepsilon_3 \left(\frac{\beta + \gamma}{2} \right)$$

for some functions β, γ and μ . It is clear that the open subset W is dense in M . We will show that each connected component of W is an open part of one of the surfaces mentioned in the theorem. Thus, by continuity, it is sufficient to work only on this open dense subset W . From (7.2) we have

$$(7.3) \quad h(e_1, e_1) = \varepsilon_3\beta e_3 + \varepsilon_4\mu e_4, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \varepsilon_3\gamma e_3 - \varepsilon_4\mu e_4.$$

Equation (2.6) of Codazzi, (7.1) and (7.3) yield

$$(7.4) \quad e_2\beta = f_1(\gamma - \beta) + \mu g_2, \quad e_2\mu = -2f_1\mu - \varepsilon_3\varepsilon_4\beta g_2,$$

$$(7.5) \quad e_1\gamma = f_2(\gamma - \beta) - \mu g_1, \quad e_1\mu = 2f_2\mu + \varepsilon_3\varepsilon_4\gamma g_1.$$

From equation (2.4) of Gauss, (7.1) and (7.3) we get

$$(7.6) \quad e_1f_2 - e_2f_1 = f_1^2 + f_2^2 + \varepsilon_3\beta\gamma - \varepsilon_4\mu^2.$$

From (5.7), (7.1) and (7.2) we have

$$(7.7) \quad \Delta\alpha = -\alpha\varepsilon_3\varepsilon_4(g_1^2 + g_2^2) - \alpha\varepsilon_3(\beta^2 + \gamma^2).$$

From (5.9), (7.1) and (7.2) we obtain

$$(7.8) \quad (3\beta + \gamma)(e_1\alpha) = -2\alpha g_1\mu,$$

$$(7.9) \quad (\beta + 3\gamma)(e_2\alpha) = 2\alpha g_2\mu.$$

From (5.8), (7.1) and (7.2) we have

$$(7.10) \quad \alpha(e_1g_1 + e_2g_2) = \alpha\{\varepsilon_4(\beta - \gamma)\mu + f_2g_1 - f_1g_2\} - 2\{(e_1\alpha)g_1 + (e_2\alpha)g_2\}.$$

Since $R^D = 0$, equation (2.7) of Ricci, (7.1) and (7.2) yield

$$(7.11) \quad e_1g_2 - e_2g_1 = f_1g_1 + f_2g_2.$$

If $g_1 = g_2 = 0$, then (7.8), (7.9) and (7.10) imply $(3\beta + \gamma)(e_1\alpha) = 0$, $(\beta + 3\gamma)(e_2\alpha) = 0$, $\alpha(\beta - \gamma)\mu = 0$. Since $\alpha \text{ grad } \alpha^2 \neq 0$, we get $\mu = 0$. From $g_1 = g_2 = \mu = 0$, we may conclude that U is contained in a hyperplane E_t^3 ($t = 0$ or 1) of E_s^4 which contradicts to Theorem 4.1. Consequently, we have $g_1^2 + g_2^2 \neq 0$. Since $\beta = 2\varepsilon_3\alpha - \gamma$, by taking derivative of this equation with respect to e_1 and using (7.5) and (7.8), we obtain

$$(7.12) \quad 2\alpha(e_1\beta) = (2\alpha f_2 - (e_1\alpha))(\beta - \gamma).$$

From equation (2.4) of Gauss and (7.3), we see that the Gauss curvature of M is given by $G = \varepsilon_3\beta\gamma - \varepsilon_4\mu^2$ which is constant by our assumption. Taking the derivative of this equation with respect to e_1 and by applying (7.5), (7.8), and (7.12), we may obtain

$$(7.13) \quad 3(e_1\alpha^2) = f_2\{(\beta - \gamma)^2 + 4\varepsilon_3\varepsilon_4\mu^2\}.$$

Similarly, we may also obtain

$$(7.14) \quad 2\alpha(e_2\gamma) = (2\alpha f_1 + (e_2\alpha))(\beta - \gamma),$$

$$(7.15) \quad 3(e_2\alpha^2) = -f_1\{(\beta - \gamma)^2 + 4\varepsilon_3\varepsilon_4\mu^2\}.$$

For any function φ on M , we have

$$(7.16) \quad e_1e_2\varphi - e_2e_1\varphi = [e_1, e_2]\varphi = f_1(e_1\varphi) + f_2(e_2\varphi).$$

By using (7.13) and (7.15), we get $e_1e_2\alpha^2 - e_2e_1\alpha^2 = 0$. Therefore, by applying (7.13), (7.15) and (7.16), we may obtain $(e_1f_1 + e_2f_2)\{(\beta - \gamma)^2 + 4\varepsilon_3\varepsilon_4\mu^2\} = 0$. Since G is constant and α is nowhere constant on U , this equation implies

$$(7.17) \quad e_1f_1 + e_2f_2 = 0.$$

By using (7.4), (7.5), (7.12), (7.14) and (7.17) we may obtain

$$\begin{aligned} e_1 e_2 \gamma - e_2 e_1 \gamma &= (f_2 g_2 - f_1 g_1) \mu + \mu(e_2 g_1) + \left(\frac{\beta - \gamma}{2\alpha} \right) (e_1 e_2 \alpha - f_1 e_1 \alpha + f_2 e_2 \alpha) \\ &\quad - \frac{3(\beta - \gamma)}{4\alpha^2} (e_1 \alpha)(e_2 \alpha) + \frac{1}{2\alpha} g_1 \mu(e_2 \alpha) - \varepsilon_3 \varepsilon_4 \beta g_1 g_2. \end{aligned}$$

On the other hand, by (7.5), (7.14) and (7.16), we may obtain

$$e_1 e_2 \gamma - e_2 e_1 \gamma = -f_1 g_1 \mu + \left(\frac{\beta - \gamma}{2\alpha} \right) f_2 e_2 \alpha.$$

Therefore we have

$$\begin{aligned} (7.18) \quad 0 &= 4\alpha^2 \{f_2 g_2 \mu - \varepsilon_3 \varepsilon_4 \beta g_1 g_2 + \mu(e_2 g_1)\} \\ &\quad + (\beta - \gamma) \{2\alpha(e_1 e_2 \alpha) - 2\alpha f_1(e_1 \alpha) - 3(e_1 \alpha)(e_2 \alpha)\} + 2\alpha g_1 \mu(e_2 \alpha). \end{aligned}$$

Similarly, by using (7.4), (7.5), (7.12), (7.14), (7.16), and (7.17), we may also obtain

$$\begin{aligned} (7.19) \quad 0 &= 4\alpha^2 \{-f_1 g_1 \mu + \varepsilon_3 \varepsilon_4 \gamma g_1 g_2 + \mu(e_1 g_2)\} + 2\alpha \mu g_2(e_1 \alpha) \\ &\quad + (\beta - \gamma) \{2\alpha(e_2 e_1 \alpha) + 2\alpha f_2(e_2 \alpha) - 3(e_1 \alpha)(e_2 \alpha)\}. \end{aligned}$$

From (7.11), (7.16), (7.18) and (7.19) we find

$$(7.20) \quad 4\varepsilon_4 \alpha^2 g_1 g_2 + \mu(g_2 e_1 \alpha - g_1 e_2 \alpha) = 0.$$

From (7.8), (7.9) and (7.20) we may obtain

$$(7.21) \quad g_1 g_2 (3\alpha^2 + \varepsilon_3 G) = 0.$$

Since G is constant and α is nowhere constant on U , (7.21) implies $g_1 g_2 = 0$. Because $g_1^2 + g_2^2 \neq 0$, without loss of generality we may assume

$$(7.22) \quad g_1 = 0, \quad g_2 \neq 0.$$

Thus, by (7.8), (7.20) and (7.22), we find

$$(7.23) \quad (3\beta + \gamma)(e_1 \alpha) = 0, \quad \mu(e_1 \alpha) = 0.$$

Since α is nowhere constant and G is constant, (7.23) implies

$$(7.24) \quad e_1 \alpha = 0.$$

Therefore, from (7.13), we find $f_2 \{(\beta - \gamma)^2 + 4\varepsilon_3 \varepsilon_4 \mu^2\} = 0$. Since $4(\alpha^2 - \varepsilon_3 G)$

$= (\beta - \gamma)^2 + 4\varepsilon_3\varepsilon_4\mu^2$ and α is nowhere constant, we get

$$(7.25) \quad f_2 = 0.$$

From (7.5), (7.11), (7.12), (7.17), (7.22), (7.24) and (7.25), we find

$$(7.26) \quad e_1\gamma = e_1\mu = e_1g_2 = e_1\beta = e_1f_1 = 0.$$

(7.25) implies $d\omega^2 = 0$. Thus, there is a local function u such that $\omega^2 = du$. From (7.24) and (7.26), we conclude that the functions α , β , γ , f_1 and g_2 are functions of u . In the following, we denote by $'$ the derivative with respect to u . From (7.4), (7.6), (7.14) and (7.15) we have

$$(7.27) \quad 6\alpha\alpha' = -f_1\{(\beta - \gamma)^2 + 4\varepsilon_3\varepsilon_4\mu^2\},$$

$$(7.28) \quad \beta' = f_1(\gamma - \beta) + \mu g_2, \quad \gamma' = f_1(\beta - \gamma) + \frac{\alpha'}{2\alpha}(\beta - \gamma),$$

$$(7.29) \quad \mu' = -2f_1\mu - \varepsilon_3\varepsilon_4\beta g_2,$$

$$(7.30) \quad f_1' = -f_1^2 - \varepsilon_3\beta\gamma + \varepsilon_4\mu^2.$$

We put $H_1 = \tilde{V}_{e_1}e_1$. Then by (7.1) and (7.3) we find

$$(7.31) \quad H_1 = -f_1e_2 + \varepsilon_3\beta e_3 + \varepsilon_4\mu e_4.$$

From (7.1), (7.2), (7.3), (7.22), and (7.26) we obtain

$$(7.32) \quad \tilde{V}_{e_1}H_1 = -(f_1^2 + \varepsilon_3\beta^2 + \varepsilon_4\mu^2)e_1.$$

Also, by using (7.1), (7.2), (7.3), (7.25), (7.28), (7.29), (7.30), and (7.31), we may also obtain

$$(7.33) \quad \tilde{V}_{e_2}H_1 = -f_1H_1.$$

Case (i): $\langle H_1, H_1 \rangle \neq 0$. In this case we put

$$(7.34) \quad H_1 = \delta\bar{e}_2, \quad \delta^2 = \bar{e}_2(f_1^2 + \varepsilon_3\beta^2 + \varepsilon_4\mu^2), \quad \delta > 0, \quad \bar{e}_2 = \langle \bar{e}_2, \bar{e}_2 \rangle = \pm 1.$$

Thus, we have

$$\tilde{V}_{e_2}H_1 = (e_2\delta)\bar{e}_2 + \delta\tilde{V}_{e_2}\bar{e}_2.$$

Combining this with (7.26), (7.33) and (7.34) we obtain

$$(7.35) \quad e_1\delta = 0, \quad e_2\delta = -f_1\delta, \quad \tilde{V}_{e_2}\bar{e}_2 = 0.$$

The third equation of (7.35) shows that $\bar{e}_2$ is a parallel vector field in E_s^4 along an integral curve $\sigma(u)$ of e_2 . Also, from (7.3) and (7.25), we also have

$$(7.36) \quad \tilde{\nabla}_{e_2} e_1 = 0$$

which means that the vector field e_1 is also a parallel vector field in E_s^4 along $\sigma(u)$. Let $\tau = \tau(s)$ be an integral curve of e_1 parametrized by arclength s . Then we have $\tau'(s) = e_1$, $\tau''(s) = H_1$ and $\tau'''(s) = \tilde{\nabla}_{e_1} H_1$ along the integral curve τ . Combining these with (7.32) and (7.34) we find $\tau'''(s) + \bar{e}_2 \delta^2 \tau'(s) = 0$. This implies that an integral curve $\tau(s)$ of e_1 takes the following form:

$$(7.37) \quad \tau(s) = c_1 + c_2 \cos \delta s + c_3 \sin \delta s, \text{ if } \bar{e}_2 = 1, \\ \text{(respectively, } \tau(s) = c_1 + c_2 \cosh \delta s + c_3 \sinh \delta s \text{ if } \bar{e}_2 = -1),$$

for some vectors c_1, c_2, c_3 in E_s^4 . Since $\langle \tau'(s), \tau'(s) \rangle = 1$, we obtain

$$(7.38) \quad \langle c_2, c_2 \rangle = \langle c_3, c_3 \rangle = \delta^{-2}, \langle c_2, c_3 \rangle = 0, \text{ if } \bar{e}_2 = 1. \\ \text{(respectively, } \langle c_2, c_2 \rangle = -\langle c_3, c_3 \rangle = -\delta^{-2}, \langle c_2, c_3 \rangle = 0, \text{ if } \bar{e}_2 = -1.)$$

Let $\sigma = \sigma(u)$ be a fixed integral curve of e_2 . Throught each point $\sigma(u)$ on the integral curve σ , there is an integral curve $\tau_u(s)$ of e_1 perpendicular to $\sigma(u)$. Thus, locally, the surface takes the form:

$$(7.39) \quad x(u, v) = \rho(u) + r(u)(C_1(u) \sin v + C_2(u) \cos v), \text{ if } \bar{e}_2 = 1, \\ \text{(respectively, } x(u, v) = \rho(u) + r(u)(C_1(u) \sinh v + C_2(u) \cosh v), \text{ if } \bar{e}_2 = -1),$$

where $r = \delta^{-1}$, $C_1(u), C_2(u)$ are vector functions of u satisfying $\langle C_1, C_1 \rangle = \langle C_2, C_2 \rangle = 1$, $\langle C_1, C_2 \rangle = 0$ if $\bar{e}_2 = 1$ (respectively, $\langle C_1, C_1 \rangle = -\langle C_2, C_2 \rangle = 1$, $\langle C_1, C_2 \rangle = 0$ if $\bar{e}_2 = -1$). If we assume that $x(u, 0) = \sigma(u)$, then we get

$$(7.40) \quad x(u, v) = \sigma(u) + r(u)(C_1(u) \sin v + C_2(u)(\cos v - 1)), \text{ if } \bar{e}_2 = 1. \\ \text{(respectively, } x(u, v) = \sigma(u) + r(u)(C_1(u) \sinh v + C_2(u)(\cosh v - 1)), \text{ if } \bar{e}_2 = -1.)$$

From (7.34) and (7.40) we obtain $C_1(u) = e_1(\sigma(u))$, $C_2(u) = \bar{e}_2(\sigma(u))$. Since e_1 and $\bar{e}_2$ are parallel vector fields along the integral curve $\sigma(u)$, both C_1 and C_2 are constant vectors in E_s^4 . Since the v -curves of (7.40) are integral curves of e_1 and $x_v(u, v) = r(u)(C_1 \cos v - C_2 \sin v)$ if $\bar{e}_2 = 1$ (respectively, $x_v(u, v) = r(u)(C_1 \cosh v + C_2 \sinh v)$ if $\bar{e}_2 = -1$), we find

$$(7.41) \quad e_1 = C_1 \cos v - C_2 \sin v \text{ if } \bar{e}_2 = 1.$$

(respectively, $e_1 = C_1 \cosh v + C_2 \sinh v$ if $\bar{e}_2 = -1$.)

From (7.36) and (7.41) we get $e_2 v = 0$. This shows that v is constant along integral curves of e_2 . Thus $x_u(u, v)$ is parallel to e_2 . Therefore $\langle x_u(u, v), x_v(u, v) \rangle = 0$. Hence, by (7.40), we may get

$$(7.42) \quad \langle \sigma'(u), C_2 \rangle \sin v - \langle \sigma'(u), C_1 \rangle \cos v = r'(u) \sin v, \text{ if } \bar{e}_2 = 1.$$

(respectively, $\langle \sigma'(u), C_1 \rangle \cosh v + \langle \sigma'(u), C_2 \rangle \sinh v = r'(u) \sinh v$, if $\bar{e}_2 = -1$.)

From this we have

$$(7.43) \quad \langle e_2, \bar{e}_2 \rangle|_{\sigma(u)} \sin v - \langle e_2, e_1 \rangle|_{\sigma(u)} \cos v = r'(u) \sin v, \text{ if } \bar{e}_2 = 1.$$

(respectively, $\langle e_2, e_1 \rangle|_{\sigma(u)} \cosh v + \langle e_2, \bar{e}_2 \rangle|_{\sigma(u)} \sinh v = r'(u) \sinh v$, if $\bar{e}_2 = -1$.)

Since $\langle e_1, e_2 \rangle = 0$, $\langle e_2, \bar{e}_2 \rangle = -f_1 r$ by virtue of (7.31) and (7.36), equation (7.43) implies

$$(7.44) \quad (\delta^{-1})' = -\frac{f_1}{\delta}.$$

Combining (7.35) and (7.44) we obtain $f_1 = 0$ which implies $\alpha' = 0$ by virtue of (7.27). This contradicts to the assumption.

Case (ii): $\langle H_1, H_1 \rangle = 0$. In this case we have

$$(7.45) \quad f_1^2 + \varepsilon_3 \beta^2 + \varepsilon_4 \mu^2 = 0.$$

Since the Gaussian curvature G of M is constant, by applying a homothetic transformation if necessary, we may assume that $G = 0$, $G = 1$ or $G = -1$.

Case (ii-1): $G = 0$. In this case (7.30) implies $f_1 = (u + a)^{-1}$, for some constant a . Thus we have

$$(7.46) \quad f_1 = w^{-1}, \quad w = u + a.$$

From (7.27) and (7.46) we get

$$(7.47) \quad \alpha = bw^{-2/3},$$

where b is a non-zero constant. Since $\gamma = 2\varepsilon_3\alpha - \beta$,

$$(7.48) \quad \gamma = 2\varepsilon_3 b w^{-2/3} - \beta.$$

Because $0 = G = \varepsilon_3 \beta \gamma - \varepsilon_4 \mu^2$, (7.45), (7.46) and (7.48) yield

$$(7.49) \quad \beta = \frac{-1}{2b w^{4/3}}, \quad \gamma = \frac{2\varepsilon_3 b}{w^{2/3}} + \frac{1}{2b w^{4/3}}.$$

From (7.45), (7.46) and (7.49) we get

$$(7.50) \quad \mu^2 = \frac{-\varepsilon_4}{w^2} - \frac{\varepsilon_3 \varepsilon_4}{4b^2 w^{8/3}}.$$

From (7.28), (7.46) and (7.49) we find

$$(7.51) \quad \mu g_2 = \frac{-1}{3b w^{7/3}} - \frac{2\varepsilon_3 b}{w^{5/3}}.$$

From (7.1), (7.24) and (7.25) we get $\Delta \alpha = -f_1 \alpha' - \alpha''$. Combining this with (7.7) and using (7.22), (7.46), (7.47) and (7.49) we may obtain

$$(7.52) \quad g_2^2 = -\frac{14\varepsilon_3 \varepsilon_4}{9w^2} - \frac{\varepsilon_4}{2b^2 w^{8/3}} - \frac{4\varepsilon_4 b^2}{w^{4/3}}.$$

From (7.50), (7.51) and (7.52) we obtain $88b^4 w^{4/3} + 56\varepsilon_3 b^2 w^{2/3} + 9 = 0$ which is impossible.

Case (ii-2): $G = 1$. In this case (7.30) implies

$$(7.53) \quad f_1 = \tan w, \quad w = a - u$$

for some constant a . Then, by (7.27), we get

$$(7.54) \quad (\beta + \gamma)^2 = 4\alpha^2 = 4\varepsilon_3 + 4b \sec^{4/3} w,$$

where b is a nonzero constant. Since $1 = G = \varepsilon_3 \beta \gamma - \varepsilon_4 \mu^2$, (7.45), (7.53) and (7.54) give

$$(7.55) \quad \beta^2 = \frac{(1 - \tan^2 w)^2}{4(\varepsilon_3 + b \sec^{4/3} w)}.$$

From (7.45), (7.53) and (7.55) we get

$$(7.56) \quad \mu^2 = -\varepsilon_4 \tan^2 w - \frac{\varepsilon_3 \varepsilon_4 (1 - \tan^2 w)^2}{4(\varepsilon_3 + b \sec^{4/3} w)}.$$

Since $\beta^2 + \gamma^2 = 4\alpha^2 - 2\varepsilon_3 - 2\varepsilon_3\varepsilon_4\mu^2$ by virtue of $G = \varepsilon_3\beta\gamma - \varepsilon_4\mu^2$, (7.54), (7.55) and (7.56) imply

$$(7.57) \quad \gamma^2 = 4b \sec^{4/3} w + 2\varepsilon_3 \sec^2 w + \frac{(2 - \sec^2 w)^2}{4(\varepsilon_3 + b \sec^{4/3} w)}.$$

From (7.1), (7.7), (7.22), (7.24), and (7.25) we have $-\Delta\alpha = f_1\alpha' + \alpha'' = \varepsilon_3\varepsilon_4\alpha g_2^2 + \alpha\varepsilon_3(\beta^2 + \gamma^2)$. Thus, by applying (7.53), (7.54), (7.55) and (7.57) we may find

$$\begin{aligned} 18g_2^2 = & -\varepsilon_3\varepsilon_4 \sec^{4/3} w (\varepsilon_3 + b \sec^{4/3} w)^{-2} \{112\varepsilon_3 b \\ & + 140b^2 \sec^{4/3} w + 28b^2 \sec^{10/3} w + 20\varepsilon_3 b \sec^2 w \\ & + 72\varepsilon_3 b^3 \sec^{8/3} w + 36 \sec^{-4/3} w + 9 \sec^{8/3} w + 9\varepsilon_3 b \sec^4 w\}. \end{aligned}$$

From (7.28) we get

$$(7.59) \quad 2\mu\beta g_2 = (\beta^2)' - 2\varepsilon_3 f_1 \{G - \varepsilon_3\beta^2 + \varepsilon_4\mu^2\}.$$

Therefore, by (7.53), (7.55), and (7.56), we find

$$(7.60) \quad \begin{aligned} 6\mu\beta g_2 = & b \tan w \sec^{4/3} w (\varepsilon_3 + b \sec^{4/3} w)^{-2} \{2 \sec^2 w \\ & + \sec^4 w - 8 - 6\varepsilon_3 b \sec^{4/3} w (2 - \sec^2 w)\}. \end{aligned}$$

From (7.55), (7.56), (5.58), and (7.60) we obtain a contradiction since $w = a - u$ is a non-constant smooth function. Consequently, this case is impossible.

Case (ii-3): $G = -1$. In this case (7.30) implies $f'_1 = 1 - f_1^2$. If f_1 is a constat, then $f_1^2 = 1$. Then, $f_1 = 1$ or -1 . Replacing e_2 by $-e_2$, if necessary, we have $f_1 = 1$. From (7.45), we find

$$(7.61) \quad \beta^2 = -\varepsilon_3 - \varepsilon_3\varepsilon_4\mu^2.$$

Since $G = \varepsilon_3\beta\gamma - \varepsilon_4\mu^2 = -1$ and $4\alpha^2 = (\beta + \gamma)^2$, (7.61) yields

$$(7.62) \quad \beta^2 = \alpha^{-2}, \quad \gamma^2 = 4\alpha^2 + \alpha^{-2} + 4\varepsilon_3.$$

(7.61) and (7.62) give

$$(7.63) \quad \mu^2 = -\varepsilon_3\varepsilon_4\alpha^{-2} - \varepsilon_4.$$

Since $f_1 = 1$, (7.27) implies

$$(7.64) \quad \alpha^2 = e^{-\frac{4}{3}w} - \varepsilon_3, \quad w = u + a,$$

for some constant a . From (7.1), (7.7), (7.22), (7.24), (7.25) and $f_1 = 1$, we find $\alpha\alpha' + \alpha\alpha'' = \varepsilon_3\varepsilon_4\alpha^2g_2^2 + \varepsilon_3\alpha^2(\beta^2 + \gamma^2)$. Therefore, by (7.62), (7.63) and (7.64), we find

$$(7.65) \quad 9\varepsilon_3\varepsilon_4\alpha^4g_2^2 = 2\alpha^2e^{\frac{-4}{3}w} - 4e^{\frac{-8}{3}w} - 18\varepsilon_3\alpha^2 - 36(\varepsilon_3\alpha^2 + 1)\alpha^4.$$

From the first equation of (7.28), (7.62), (7.63), (7.64) and (7.65), we may conclude that this cannot occur. Consequently, f_1 is non-constant. Therefore, at least one of the following two open subsets $U_1 = \{p \in U \mid |f_1| < 1\}$ and $U_2 = \{p \in U \mid |f_1| > 1\}$ is non-empty. By integration of $f_1' = 1 - f_1^2$ we have

$$f_1 = \tanh w \text{ on } U_1 \text{ (respectively, } f_1 = \coth w \text{ on } U_2).$$

Therefore, by applying arguments similar to Case (ii-2), we may conclude that Case (ii-3) is also impossible.

The remaining part of the proof is easy to verify. This completes the proof of the theorem.

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