REAL ZEROS OF HOLOMORPHIC FUNCTIONS WITH PARAMETERS I

COMPLEX NUMERICAL INTEGRATION USING MATHEMATICA

Dedicated to Professor Toshio NISHINO on his sixtieth birthday

By

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Introduction

The aim of the present series is to discuss methods of identifying values of parameters $t = [t_1, t_2, \dots, t_m]$ so that a holomorphic function $f[z, t_1, t_2, \dots, t_m]$ of variables z and t has real valued zero points z for these identified values of parameter t .

Secular equations concerning determinants of type

$$\text{Det}[A] = 0, \quad A = \{a_{jk}; j, k = 1, 2, \dots, \ell\}$$

occur very often in theoretical discussions of several substantial sciences, for example, in the valence-theory of chemistry. Moreover, for instance, in the flutter-analysis of aeronautics, the entries a_{jk} 's of the matrix A are polynomials of several parameters $z, t = [t_1, t_2, \dots, t_n]$ and the mathematical procedure assigned is to seek the hypersurface consisting of $[z, t]$ such that z 's are real valued zero points of the polynomial $f[z, t] := \text{det}[A]$. In this case the polynomial $f[z, t]$ is expanded in the polynomial

$$f[z, t] = c_0[t]z^n + c_1[t]z^{n-1} + \dots + c_{n-1}[t]z + c_n[t] \quad (0.1)$$

of z where the coefficients $c_i[t]$'s are the usual ones.

Let $a_i[t]$ and $b_i[t]$ be, respectively, the real and imaginary part of coefficient $c_i[t]$. A real valued zero point x of the holomorphic function $f[z, t]$ satisfies the following two algebraic equations with real valued coefficients simultaneously:

$$a_0[t]x^n + a_1[t]x^{n-1} + \cdots + a_{n-1}[t]x + a_n[t] = 0 \quad (0.2),$$

$$b_0[t]x^n + b_1[t]x^{n-1} + \cdots + b_{n-1}[t]x + b_n[t] = 0 \quad (0.3).$$

Early, in 1683, Kowa SEKI [9] introduced the notion of determinants in order to give a condition that two algebraic equations with degrees not necessarily common have common solutions x . We learn this as Sylvester [1814–1897]’s elimination method which is vanishing of the Sylvester’s resultant, which we denote by Sylvester $[t]$. A pair of a solution t_{realzero} of the algebraic equation

$$\text{Sylvester } [t] = 0 \quad (0.4),$$

and a common solution z_{realzero} of the two algebraic equations (0.2) and (0.3), which coincide with a real solution of the algebraic equation $f[z, t_{\text{realzero}}] = 0$ forms the desired real valued solution $[z_{\text{realzero}}, t_{\text{realzero}}]$ of the original algebraic equation $f[z, t] = 0$. Making use of personal computers, we can obtain numerical solutions of the algebraic equations (0.4) and (0.1). For example, by the system “Mathematica” the solutions are defined by

$$t_{\text{realzero}} := \text{NSolve}[\text{Sylvester } [t] = 0, t] \quad (0.5)$$

and a real solution z_{realzero} is between solutions given by

$$\text{NSolve } [f[z, t_{\text{realzero}}] = 0, z] \quad (0.6).$$

For the sake of generality, we will deal with the above problem for a holomorphic function $f[z, t]$ which is not necessarily algebraic by numerical complex integration making use of the software Mathematica and the hardware Macintosh II, Macintosh IIfx of common use, which is placed in a computer room of our Department, abbreviated MacIIfx hereafter and Macintosh IIcx of individual use placed in a room of the professor abbreviated MacIIcx hereafter. Calculation of MacIIfx is 3 times as speedy as that of MacIIcx.

The main idea owes to the Weierstrass Division Theorem in the theory of functions of several complex variables which is proved by Residue Theorem (cf. [1], [2], [3] and [4]).

Let us seek a pair $[x, t]$ such that $[x, t]$ is a real valued zero point in a rectangle

$$R := \{[x, t] \in \mathbf{R}^2; \alpha < x < \beta, \gamma < t < \delta\} \quad (0.7),$$

of a function $f[z, t]$ holomorphic in a domain D in $\mathbf{C}^2$ containing the rectangle

R. We define for a suitable small positive number ε a cuboid

$$C := \{[z, t] \in \mathbb{C} \times \mathbb{R}; \alpha < \operatorname{Re}[z] < \beta, -\varepsilon < \operatorname{Im}[z] < \varepsilon, \gamma < t < \delta\} \quad (0.8)$$

and for each parameter t with $\alpha < t < \beta$ an integral contour

$$\Gamma := \partial K, \quad K := \{z \in \mathbb{C}: \alpha < \operatorname{Re}[z] < \beta, -\varepsilon < \operatorname{Im}[z] < \varepsilon\} \quad (0.9)$$

which is the boundary of the above rectangle K with counterclockwise orientation. In the present paper, we denote by I the imaginary unit instead of usual i , i.e.,

$$I := \sqrt{-1} \quad (0.10)$$

using the notation in the software Mathematica.

We define a function $n[t]$ of a real variable t with $\alpha < t < \beta$ by complex integral along Γ occurring in the argument principle of the theory of functions of a complex variable:

$$n[t_-] := \frac{1}{2\pi I} \int_{\Gamma} \frac{\frac{\partial f}{\partial z}}{f[z, t]} dz \quad (0.11)$$

that is,

$$n[t_-] := \text{NIntegrate}[D[f[z, t], z]/(2 \text{ Pi } I f[z, t]), \{z, \alpha - I \varepsilon, \beta - I \varepsilon, \beta + I \varepsilon, \alpha + I \varepsilon, \alpha - I \varepsilon\}] \quad (0.12).$$

Moreover, letting s be a positive integer, we define s functions $g[j, t]$ for $j = 1, 2, \dots, s$ by the following complex integral along Γ :

$$g[j_-, t_-] := \frac{1}{2\pi I} \int_{\Gamma} z^j \frac{\frac{\partial f}{\partial z}}{f[z, t]} dz \quad (j = 1, 2, \dots, s) \quad (0.13)$$

that is,

$$g[j_-, t_-] := \text{NIntegrate}[z^j D[f[z, t], z]/(2 \text{ Pi } I f[z, t]), \{z, \alpha - I \varepsilon, \beta - I \varepsilon, \beta + I \varepsilon, \alpha + I \varepsilon, \alpha - I \varepsilon\}] \quad (0.14).$$

For any pair $[x_{\text{realzero}}, t_{\text{realzero}}] \in \mathbf{R}^2$ satisfying $f[x_{\text{realzero}}, t_{\text{realzero}}] = 0$, the function $n[t]$ does not vanish at t_{realzero} and there holds

$$n[t_{\text{realzero}}] \geq 1 \quad (0.15).$$

More precisely, $n[t_{\text{realzero}}]$ represents the number of zero points of the holomorphic functions $f[z, t_{\text{realzero}}]$ of the complex variable z in the rectangle K counting up and overlapping corresponding the orders of the zero points. So, if we adapt a suitable small positive number h as mesh following demand on engeneering or knowledge on experience, we can seek parameter t_{realzero} 's exhaustly in the interval $\gamma < t < \delta$ which have real valued zero points z_{realzero} 's in the interval $\alpha < x < \beta$ of the holomorphic function $f[z, t_{\text{realzero}}]$ without omission by letting MacII load Mathematica and calculating the values of the function $n[t]$ at the points $t = \alpha + kh$ for $k = 0, 1, \dots, p$ which is the largest integer with $\alpha + ph < \beta$. For t with $n[t] > 0$, $n[t]$ is theoretically a positive integer. But, when there is a zero point of the function $f[z, t]$ close to the integration contour Γ , the value of the integral $n[t]$ is not numerically an integer and we feel a bite as in fishing. Then we choose the positive integer s nearest to $n[t]$:

$$s := n[t] \quad (0.16)$$

and we let MacII calculate s integrals of the functions $g[j, t]$ for $j = 1, 2, \dots, s$. Each $g[j, t]$ represents the j -th power sum of zero points of the function $f[z, t]$ in the rectangle K in the complex plane $\mathbf{C}$. As an elementary algebra teaches us, we can obtain s zero points of the function $f[z, t]$ as s solutions of an algebraic equation $a[z, t] = 0$ of order s by letting MacII calculate NSolve $[a[z, t] = 0, z]$, i.e.,

$$z_{\text{zero}}\text{'s} := \text{NSolve}[a[z, t] = 0, z] \quad (0.17)$$

the coefficients of z^k in the polynomial $a[z, t]$ are determined by $g[j, t]$ for $j = 1, 2, \dots, k$ ($0 \leq k \leq s$).

Carefully watching variation of values, and especially, of signatures of the imaginary parts of zero points thus obtained, let us seek a pair $[t01_{\text{zero}}, t02_{\text{zero}}]$ of the parameter t with $t01_{\text{zero}} < t02_{\text{zero}}$ and a pair $[z01_{\text{zero}}, z02_{\text{zero}}]$ such that $\text{Im}[z01_{\text{zero}}] \text{Im}[z02_{\text{zero}}] < 0$ for zero point $z01_{\text{zero}}$, one of zero points of the function $f[z, t01_{\text{zero}}]$ and zero point $z02_{\text{zero}}$, one of zero points of the function $f[z, t02_{\text{zero}}]$. Here we apply the inverse linear interpolation to the imaginary part of zero points of the function $f[z, t]$ and we put

$$t1 := (t01_{\text{zero}} \text{Abs} [\text{Im} [z02_{\text{zero}}]] + t02_{\text{zero}} \text{Abs} [\text{Im} [z01_{\text{zero}}]]) / (\text{Abs} [\text{Im} [z02_{\text{zero}}]] + \text{Abs} [\text{Im} [z01_{\text{zero}}]]) \quad (0.18)$$

and let MacII calculate $n[t1]$ by (0.12) and $g[j, t1]$ for $j = 1, 2, \dots, s$ by (0.14) and s zero points of the function $f[z, t1]$ in the rectangle K by (0.17). Let $z1_{\text{zero}}$ be the zero point closest to $z01_{\text{zero}}$ and $z02_{\text{zero}}$. We scan signature of $\text{Im}[z1_{\text{zero}}]$ and, when $\text{Im}[z01_{\text{zero}}] \text{Im}[z1_{\text{zero}}] < 0$, we choose $t1_{\text{zero}}$ instead of $t02_{\text{zero}}$, when $\text{Im}[z02_{\text{zero}}] \text{Im}[z1_{\text{zero}}] < 0$, we choose $t1$ instead of $t01_{\text{zero}}$. We continue this inverse linear interpolation with respect to the imaginary part of zero points of the function $f[z, tn]$ and we define successively a sequence $\{tn; n = 1, 2, 3, \dots\}$ of real numbers tn with $\gamma < tn < \delta$ and a sequence $\{zn_{\text{zero}}; n = 1, 2, 3, \dots\}$ of a zero point of the function $f[z, tn]$ for $n = 1, 2, 3, \dots$, then $\text{Im}[zn_{\text{zero}}]$ converges rapidly and numerically to zero, this numerical limit tn_{zero} is a parameter wanted and the corresponding zero point zn_{zero} of the function $f[z, tn_{\text{zero}}]$ is a desired real valued zero point.

In any numerical integration for engineering, computer must call the values of coefficients, for example aerodynamic force, from work station at each input of the values of the integrant. In order to load equivalent heavy conditions in this mathematical Gedankenexperiment on our Mac as well as in engineering, we will consider examples concerning Bessel functions in the present paper.

The merits of the methods presented in the present paper are non-existence of theoretical error because we use the residue theorem and the necessity of knowledge on neither Numerical Analysis nor Programming. Generally the numerical analysis using Mathematica on Mac is said to be slow. It seems to the authors that this excels the above demerit. They feel satisfaction at the results.

1. Real valued zero point of a holomorphic function $f[z, t]$

Let T be a compact Hausdorff space, let p be a non-negative integer and, for any k with $0 \leq k \leq p$, let $z_k[s, t]$ be a continuous mapping of the compact subset $\{[s, t] \in \mathbf{R} \times T; 0 \leq s \leq 1, t \in T\}$ into the complex plane $\mathbf{C}$ such that, for any $t \in T$, $z_k[s, t]$ forms a piecewise smooth closed Jordan curve $\Gamma_k[t]$ in the complex plane $\mathbf{C}$. Moreover, assume that, for any $t \in T$, the $p + 1$ curves $\Gamma_0[t]$ and $\Gamma_k[t]$ ($1 \leq k \leq p$) form a boundary $\Gamma[t]$ of a relatively compact domain $D[t]$ in the complex plane $\mathbf{C}$ of the one complex variable z

and that the interior D of the compact subset

$$K := \{[z, t] \in \mathbf{C} \times T; z \in D[t] \cup \Gamma[t], t \in T\} \quad (1.1)$$

of $\mathbf{C} \times \mathbf{R}$ is a domain in $\mathbf{C} \times \mathbf{R}$ and is represented by

$$D = \{[z, t] \in \mathbf{C} \times T; z \in D[t], t \in T\} \quad (1.2).$$

Under the above assumption, let $f[z, t]$ be a complex valued continuous function on the compact space K such that, for any $t \in T$, $f[z, t]$ is a holomorphic function of the one complex variable z in a neighborhood of the compact set

$$K[t] := D[t] \cup \Gamma[t] \quad (1.3)$$

in the complex plane $\mathbf{C}$ and let $T_{\text{boundaryzero}}$ be the set of points t of the compact space T such that the function $f[z, t]$ of the one complex variable z has zero points on the boundary $\Gamma[t]$ of the compact set $K[t]$ in the complex plane $\mathbf{C}$. The set

$$\delta K := \{[z, t] \in \mathbf{C} \times T; z \in \Gamma[t], t \in T\} \quad (1.4)$$

is a compact subset of the compact Hausdorff space K . We denote the canonical projections by $p_{\mathbf{C}}: \mathbf{C} \times T \rightarrow \mathbf{C}$ and $p_T: \mathbf{C} \times T \rightarrow T$.

LEMMA 1. $T_{\text{boundaryzero}}$ is a closed subset of the compact space T .

PROOF. Let t_0 be an adherent point of T , i.e., let t_0 belong to the closure of the set $T_{\text{boundaryzero}}$ in the space T . Let $\mathcal{U}$ be the set of neighborhoods of the point t_0 . By the definition the set

$$\mathcal{U} \cap T_{\text{boundaryzero}} := \{U \cap T_{\text{boundaryzero}}; U \in \mathcal{U}\} \quad (1.5)$$

generates a filter on the compact set T . For any $U \in \mathcal{U}$, we consider the subset

$$f^{-1}(\{0\})[U] := \{[z, t] \in f^{-1}(\{0\}) \cap \delta K; z \in \Gamma[t], t \in U\} \quad (1.6).$$

of δK and the set

$$f^{-1}(\{0\})[\mathcal{U}] := \{f^{-1}(\{0\})[U]; U \in \mathcal{U}\} \quad (1.7)$$

of subsets of δK . There exists an ultrafilter $\mathcal{F}$ on δK such that $f^{-1}(\{0\})[\mathcal{U}] \subset \mathcal{F}$. The ultrafilter $\mathcal{F}$ on the compact Hausdorff space δK converges to a point $[z_{\text{lim}}, t_{\text{lim}}]$ of δK . By the continuity of the mappings

p_C and p_T , the images of the convergent filter $\mathcal{F}$ converges to the points $z_{\lim}$ and $t_{\lim}$ respectively. Hence $p_T(\mathcal{F})$ converges to the point $t_{\lim}$. Since $p_T(\mathcal{F})$ is stronger than the trace of $\mathcal{U}$ on δK which equals $p_T(\mathcal{U})$ and which converges to t_0 on δK , the filter $p_T(\mathcal{F})$ converges to the point t_0 too. Since T is a Hausdorff space, the limit point $t_{\lim}$ coincides with the adherent point t_0 . Since $[z_{\lim}, t_0] \in f^{-1}(\{0\}) \cap \delta K$ and since

$$f[z, t] = 0 \quad ([z, t] \in f^{-1}(\{0\}) \cap \delta K) \quad (1.8),$$

we have

$$f[z_{\lim}, t_{\lim}] = 0, \quad z_{\lim} \in \Gamma[t_{\lim}] \quad (1.9).$$

Hence $t_0 = t_{\lim}$ belongs to $T_{\text{boundaryzero}}$. q.e.d.

We consider the subset

$$T_{\text{neverzero}} := \{t \in T; f[z, t] \neq 0 \text{ for any } z \in \Gamma[t]\} \quad (1.10)$$

of the Hausdorff space T . Because it is the complement of the closed subset $T_{\text{boundaryzero}}$ of the space T , the set $T_{\text{neverzero}}$ is open in the space T .

LEMMA 2. *The functions $n[t]$ and $g[j, t]$ ($j = 1, 2, \dots$) defined by*

$$n[t_-] := \frac{1}{2\pi i} \int_{\Gamma[t]} \frac{\frac{\partial f}{\partial z}}{f[z, t]} dz \quad (1.11)$$

and

$$g[j_-, t_-] := \frac{1}{2\pi i} \int_{\Gamma[t]} z^j \frac{\frac{\partial f}{\partial z}}{f[z, t]} dz \quad (1.12)$$

for $t \in T_{\text{neverzero}}$ are continuous on $T_{\text{neverzero}}$.

PROOF. Since the integrals are represented by those of $dz[s, t]/ds$ times the integrands, which are continuous on the product space of the compact set $\{s \in \mathbf{R}; 0 \leq s \leq 1\}$ and $T_{\text{neverzero}}$, over the compact space $\{s \in \mathbf{R}; 0 \leq s \leq 1\}$, they are continuous functions of one parameter t on $T_{\text{neverzero}}$.

THEOREM 1. *Let t_1 and t_2 be points of a same connected subset T_{12} of*

$T_{\text{neverzero}}$ such that $n[t1] = n[t2] = 1$, $\text{Im}[g[1, t1]] > 0 > \text{Im}[g[1, t2]]$. Then, there exists a point t_{realzero} of $T12$ and a point z_{realzero} of $D[t_{\text{realzero}}]$ such that $\text{Im}[z_{\text{realzero}}] = 0$ and $f[z_{\text{realzero}}, t_{\text{realzero}}] = 0$.

PROOF. For any non-negative integer p , we consider the subset N_p of points of $T12$ with $n[t] = p$. Since $n[t]$ is an integer valued continuous function, the set N_p 's are mutually disjoint open subsets of the connected space $T12$ and since $t1$ and $t2 \in N_1$, we have $N_1 = T12$. The image of the connected space $T12$ by the continuous mapping $\text{Im}[g[1, t]]$ is a connected subset of the real line $\mathbf{R}$ and it is an interval by the axiom concerning connectivity of the real line of Dedekind as is shown at page 75 in Kajiwara [5]. Because $\text{Im}[g[1, t1]] > 0$ and $\text{Im}[g[1, t2]] < 0$, there exists a point t_{realzero} of T_{12} such that $\text{Im}[g[1, t_{\text{realzero}}]] = 0$. As we have shown above, $n[t_{\text{realzero}}] = 1$. Hence there exists uniquely a point z_{realzero} , which coincides with $g[1, t_{\text{realzero}}]$, of $D[t_{\text{realzero}}]$ such that $f[z_{\text{realzero}}, t_{\text{realzero}}] = 0$ and $\text{Im}[g[1, t_{\text{realzero}}]] = 0$ what was to be proved.

THEOREM 2. We assume that T is an interval of the real line $\mathbf{R}$ and that $t1 < t2$ in addition to the assumptions of Theorem 1. Then the sequences $\{un; n = 1, 2, 3, \dots\}$ and $\{vn; n = 1, 2, 3, \dots\}$ defined by the method of successive bisection

$$u1 = t1, v1 = t2 \quad (1.13),$$

$$\text{Im}[g[1, un]] \geq 0 \quad (n > 1) \quad (1.14),$$

$$\text{Im}[g[1, vn]] < 0 \quad (n > 1) \quad (1.15),$$

either $un = un - 1$ and

$$vn = \frac{un - 1 + vn - 1}{2} \quad (n > 1) \quad (1.16),$$

or $vn = vn - 1$ and

$$un = \frac{un - 1 + vn - 1}{2} \quad (n > 1) \quad (1.17)$$

is well-defined and convergent to a point t_{realzero} for which the function $f[z, t_{\text{realzero}}]$ has a real zero point z_{realzero} in $D[t_{\text{realzero}}]$.

PROOF. Assume that uk 's and vk 's are defined for $k = 1, 2, \dots, n$. We

take the mean value τ_{n+1} of u_n and v_n . If $\text{Im}[g[1, \tau_{n+1}]] \geq 0$, we put $u_{n+1} = \tau_{n+1}$ and $v_{n+1} = v_n$. If $\text{Im}[g[1, \tau_{n+1}]] < 0$, we put $v_{n+1} = \tau_{n+1}$ and $u_{n+1} = u_n$. These sequences are monotone and converge by the axiom of Weierstrass concerning the continuity of the real line $\mathbf{R}$. Since

$$0 \leq u_n - u_{n-1}, v_n - v_{n-1} \leq \frac{t_2 - t_1}{2^{n-1}} \quad (n > 1) \quad (1.18)$$

and since the right hand side of (1.18) converges to 0 by the binomial theorem and the axiom of Archimedes, these limit points coincide with each other which we denote by t_{realzero} . By (1.14), (1.15) and the continuity of the function $\text{Im}[g[1, t]]$, we have $\text{Im}[g[1, t_{\text{realzero}}]] = 0$. This means that t_{realzero} is the desired parameter and $g[1, t_{\text{realzero}}]$ is the desired real zero point of $f[z, t_{\text{realzero}}]$ in the domain $D[t_{\text{realzero}}]$.

THEOREM 3. *We assume that T is an interval of the real line $\mathbf{R}$, that $t_1 < t_2$ and that $\text{Im}[g[1, t]]$ is a convex function of t on the interval T in addition to the assumptions of Theorem 2. Then the sequence $\{t_n; n = 3, 4, 5, \dots\}$ defined by*

$$t_n := \frac{\text{Im}[g[1, t_1]] * t_{n-1} - \text{Im}[g[1, t_{n-1}]] * t_1}{\text{Im}[g[1, t_1]] - \text{Im}[g[1, t_{n-1}]]} \quad (n > 2) \quad (1.19)$$

is decreasing and convergent to a point t_{realzero} for which the function $f[z, t_{\text{realzero}}]$ has a real zero point z_{realzero} in $D[t_{\text{realzero}}]$.

PROOF. We assume that t_k 's are defined so as to satisfy the above conditions for $k = 1, 2, \dots, n-1$. We define a linear function $\ell[t]$ by

$$\ell[t] := \frac{\text{Im}[g[1, t_{n-1}]] - \text{Im}[g[1, t_1]]}{t_{n-1} - t_1} (t - t_{n-1}) + \text{Im}[g[1, t_{n-1}]] \quad (1.20).$$

The solution of the linear equation

$$\ell[t] = 0 \quad (1.21)$$

coincides with the t_n of (1.19).

Since the point $[t_n, \text{Im}[g[1, t_n]]]$ is on a line segment which joins the two points $[t_1, \text{Im}[g[1, t_1]]]$ and $[t_{n-1}, \text{Im}[g[1, t_{n-1}]]]$, we have $t_1 < t_n < t_{n-1}$. By the convexity of the function $\text{Im}[g[1, t]]$, we have

$$\operatorname{Im}[g[1, tn]] < \ell[tn] = 0 \quad (1.22).$$

The bounded monotonically decreasing sequence $\{tn; n = 2, 3, 4, \dots\}$ converges to a point by the axiom of Weierstrass. We denote the limit point by t_{realzero} . Taking limits as $n \rightarrow \infty$ from the both sides of the equation (1.19), we have $\operatorname{Im}[g[1, t_{\text{realzero}}]] = 0$. Then

$$z_{\text{realzero}} := g[1, t_{\text{realzero}}] \quad (1.23)$$

is the desired real valued zero point of $f[z, t_{\text{realzero}}]$ in the domain $D[t_{\text{realzero}}]$. Thus, the sequence obtained by the above inverse linear interpolation with respect to the imaginary part, $\operatorname{Im}[g[1, t]]$, converges monotonically to the parameter t_{realzero} desired.

2. Example 1

In this chapter we consider the Bessel function $J_m[z]$ of order m , which is a positive integer, and denote it by

$$\text{BesselJ}[m, z] := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+m}}{k!(k+m)! 2^{2k+m}} \quad (2.1)$$

which is an entire function of one complex variable z .

2.1. Zeros of order one.

We consider the entire function

$$\begin{aligned} f[z, t] &:= \text{BesselJ}[1, z + i(t - \pi/10)] - \text{BesselJ}[2, z - i(t - \pi/10)^2] \\ &:= J_1\left[z + \sqrt{-1}\left(t - \frac{\pi}{10}\right)\right] - J_2\left[z - \sqrt{-1}\left(t - \frac{\pi}{10}\right)^2\right] \end{aligned} \quad (2.1.1)$$

of two complex variables z and t . Let

$$T := \{t \in \mathbf{R}; -1 \leq t \leq 1\} \quad (2.1.2)$$

be the compact Hausdorff space equipped with the topology reduced from that of the real line $\mathbf{R}$ and, for any $t \in T$,

$$D[t] := \{z \in \mathbf{C}; 1 < \operatorname{Re}[z] < 9, -0.1 < \operatorname{Im}[z] < 0.1\} \quad (2.1.3)$$

be the domain of the complex plane $\mathbf{C}$ and

$$\Gamma[t_-] := \partial D[t], \quad K[t_-] := D[t] \cup \Gamma[t] \quad (2.1.4)$$

be, respectively, its boundary with counterclockwise orientation and its closure. We put

$$D := \{[z, t] \in \mathbf{C} \times \mathbf{R}; 1 < \operatorname{Re}[z] < 9, -0.1 < \operatorname{Im}[z] < 0.1, -1 < t < 1\} \quad (2.1.5)$$

and

$$K := \{[z, t] \in \mathbf{C} \times \mathbf{R}; 1 \leq \operatorname{Re}[z] \leq 9, -0.1 \leq \operatorname{Im}[z] \leq 0.1, -1 \leq t \leq 1\} \quad (2.1.6).$$

We apply these datum to the preceding chapter and consider functions

$$\begin{aligned} h[z_-, t_-] &:= D[f[z, t], z] / (2 \pi i \int f[z, t]) \\ &:= \frac{\frac{\partial f[z, t]}{\partial z}}{2 \pi i \int f[z, t]} \end{aligned} \quad (2.1.7),$$

$$k[z_-, t_-] := h[z] z \quad (2.1.8)$$

and

$$s[z_-, t_-] := h[z] z^2 \quad (2.1.9).$$

We consider for $t \in T_{\text{neverzero}}$ the complex integral of the functions $h[z, t]$, $k[z, t]$ and $s[z, t]$ along the rectangle $\Gamma[t]$ getting the hardwares Macintosh IIx and IIfx to start with the software Mathematica:

$$\begin{aligned} n[t_-] &:= \text{NIntegrate}[h[z, t], \\ &\quad \{z, 1 - 0.1 i, 9 - 0.1 i, 9 + 0.1 i, 1 + 0.1 i, 1 - 0.1 i\}] \\ &:= \frac{1}{2 \pi i} \int_{\Gamma[t]} \frac{\frac{\partial f}{\partial z}}{f[z, t]} dz \end{aligned} \quad (2.1.10),$$

$$\begin{aligned} g[1, t_-] &:= \text{NIntegrate}[k[z, t], \\ &\quad \{z, 1 - 0.1 i, 9 - 0.1 i, 9 + 0.1 i, 1 + 0.1 i, 1 - 0.1 i\}] \end{aligned}$$

$$:= \frac{1}{2\pi I} \int_{\Gamma[t]} z \frac{\overline{\partial f}}{f[z, t]} dt \quad (2.1.11)$$

and

$$g[2, t_-] := \text{NIntegrate}[s[z, t], \{z, 1 - 0.1 I, 9 - 0.1 I, 9 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]$$

$$:= \frac{1}{2\pi I} \int_{\Gamma[t]} z^2 \frac{\overline{\partial f}}{f[z, t]} dt \quad (2.1.12).$$

We command Macintosh IIcx or IIx to do numerical calculation by typing NIntegrate [] or, when we ask for the output of calculation time too, typing Timing [NIntegrate[]]. Then Mac returns the numerical value of the complex integration, or the pair of calculation time and numerical value of complex integration. We show them in the following tables. Moreover, we let MacIIcx save the screen during the calculation using Pyro! 4.0.

t	Calculation time	The Value $n[t]$ With Accuracy Report * With Accuracy Report **	Number of Zero Points in $D[t]$
	By MacIIcx Second Minutes		
	By MacIIcx Second Minutes		
	Ratio IIcx/IIfx		
-1.0	IIfx 446.317 Sec. 7.4 Minutes.	0.	0
-0.9	IIfx 446.717 Sec. 7.5 Minutes	0.	0
-0.8	IIfx 446.717 Sec. 7.5 Minutes	0.	0
-0.7	IIfx 495.383 Sec. 8.3 Minutes	0.	0
-0.6	IIfx 587.55 Sec. 9.8 Minutes	0.	0

−0.5	IIfx 629.283 Sec. 10.5 Minutes	0.	0
−0.4	IIfx 627.6 Sec. 10.5 Minutes	0.	0
−0.3	IIfx 774.383 Sec. 12.9 Minutes	0.	0
−0.2	IIfx 822.517 Sec. 13.7 Minutes	0.	0
−0.1	IIfx 1113.12 Sec. 18.6 Minutes	0.	0
0.0	IIfx 1254.1 Sec. 20.9 Minutes	0.	0
0.1	IIfx 1451.55 Sec. 24.2 Minutes	0. *	0
0.2	IIfx 1498.73 Sec. 25.0 Minutes	2.00001−0.0000152872 I **	2
0.3	IIfx 1602.32 Sec. 26.7 Minutes	2.+2.44684 10 ^{−11} I	2
0.4	IIfx 1508.32 Sec. 25.1 Minutes	2.+2.54041 10 ^{−9} I	2
	IIfx 4546.83 Sec. 75.8 Minutes		
	Speed Ratio 3.0145		
0.5	IIfx 1500.1 Sec. 25.0 Minutes	1.0009−0.000540928 I **	1
	IIfx 4527.93 Sec. 75.5 Minutes		
	Speed Ratio 3.0184		
0.6	IIfx 1363.95 Sec. 22.7 Minutes	0.0000046528−0.0000155203 I *	0
0.7	IIfx 1364.12 Sec. 22.7 Minutes	0. *	0
0.8	IIfx 1221.13 Sec. 20.4 Minutes	0. *	0
0.9	IIfx 1215.17 Sec. 20.3 Minutes	0. *	0
1.0	IIfx 1313.62 Sec. 21.9 Minutes	0. *	0

The software Mathematica returns principal values even when there exists simple poles on the integral contour $\Gamma[t]$. A simple pole z on a side of $\Gamma[t]$ contributes $I\pi$ times residue at z and a simple pole z at a vertex contributes $I\pi/2$ times residue at z .

When zero points of $f[z, t]$ are close to the integral contour, Mathematica warns about accuracy and executes faithfully the commanded integral. For example, for the command

```
Timing[NIntegrate[h[z, 0.1],
  {z, 1 - 0.1 I, 9 - 0.1 I, 9 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I}]],
```

MacIfx reports in the mean time

```
NIntegrate::conv:
```

```
Numerical integration failed to reach specified accuracy
after 7 recursive subdivisions near 2.5905 + 0.1 I
for variable number 1.
```

, execute numerical complex integration and returns

```
{1451.55 Second, 0.}.
```

We denote such a report by single asterisk *. Moreover, when zero points of $f[z, t]$ are nearer to the integral contour $\Gamma[t]$ or even when there are zero points on $\Gamma[t]$, for example, for the command

```
Timing[NIntegrate[h[z, 0.2],
  {z, 1 - 0.1 I, 9 - 0.1 I, 9 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I}]],
```

MacIfx reports interim

```
NIntegrate::conv:
```

```
Numerical integration failed to reach specified accuracy
after 7 recursive subdivisions near 6.0649 + 0.1 I
for variable number 1.
..... omission of halves .....
```

```
General::stop:
```

```
Further output of NIntegrate::conv
will be suppressed during this calculation.
```

, execute faithfully the given command and returns

```
{1498.73 Second. 2.00001 - 0.0000152872 I}.
```

We denote such a report by double asterisk **.

Since the value of the function $n[t]$ is a quarter of non negative integer, Mac's return is enough to estimate numbers of zero points in the domain $D[t]$.

As the above table shows, we feel strikes when we obtain returns of Mac for $t = 0.2, 0.3, 0.4$ and 0.5 , we command Mac to execute the complex integrations (2.1.11) and (2.1.12), and obtain returns as in the following tables:

t	Number of Zeros $D[t]$	Value of $g[1, t]$ Calculation time By Ifx Second Minutes By Ifcx Second Minutes Speed Ratio	Value of $g[2, t]$ Calculation time By Ifx Second Minutes By Ifcx Second Minutes Speed Ratio	Zeros of $f[z, t]$ in $D[t]$
0.2	2	$8.71449 + 0.154908 I$	$43.9353 + 1.29214 I$	$2.62862 + 0.0858139 I$
		fx (12 Min.)	fx 2166.83 Sec. 36.1 Min.	and
		cx 5606.57 Sec. 93.4 Min.	cx 6483.18 Sec. 108.1 Min.	$6.08587 + 0.0690941 I$
		(fx : Cal. Back)	Speed Ratio 2.99	
0.3	2	$8.7162 + 0.0182089 I$	$43.9596 + 0.150667 I$	$2.62985 + 0.0102683 I$
		fx (37 Min.)	fx 2197.05 Sec. 36.6 Min.	and
		cx 6410.7 Sec. 106.9 Min.	cx 6706.15 Sec. 111.8 Min.	$6.08635 + 0.00794064 I$
		(fx : Cal. Back)	Speed Ratio 3.052	
0.4	2	$8.71473 - 0.104073 I$	$43.9437 - 0.85343 I$	$2.62879 - 0.0597796 I$
		fx (37 Min.)	fx 2206.28 Sec. 36.8 Min.	and
		cx 5721.72 Sec. 95.4 Min.	cx 6655.27 Sec. 110.9 Min.	$6.08594 - 0.0442934 I$
		(fx : Cal. Back)	Speed Ratio 3.0165	
0.5	1	$6.08957 - 0.0907361 I$		$6.08957 - 0.0907361 I$
		fx (30 Min.)		
		cx 5517.58 Sec. 92.0 Min.		

The above results shows that the signature of the imaginary parts of two zeros of $f[z, t]$ changes from positive to negative between 0.3 and 0.4 and suggests, by Theorem 1, the existence of parameters t between 0.3 and 0.4 at which the function $f[z, t]$ has a real valued zero points.

We perform the inverse linear interpolations and obtain parameters t for which the function $f[z, t]$ has real valued zeros.

Since in the 0-th approximation step, the value of the parameter t and the expected zeros are, respectively, $t = 0.3$, $z = 2.62985 + 0.0102683 I$, $6.08635 + 0.00794064 I$ and $t = 0.4$, $z = 2.62879 - 0.0597796 I$, $6.08594 - 0.0442934 I$, firstly we consider the case

$$D[t_-] := \{z \in \mathbb{C}; -0.1 < \operatorname{Re}[z] < 0.1, 1 < \operatorname{Im}[z] < 5\} \quad (2.1.13)$$

and let MacIfx execute

Timing[NIntegrate[h[z, 0.3],

$$\{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] \quad (2.1.14).$$

Then he returns

$$\{674.85 \text{ Second}, 1. + 2.9176 \cdot 10^{-11} I\}.$$

This means that there exists a unique zero point of the function $f[z, 0.3]$ in the domain $D[0.3]$. Next we let MacIfx execute

Timing[NIntegrate[k[z, 0.3],

$$\{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] \quad (2.1.15)$$

and obtain Mac's return

$$\{675.917 \text{ Second}, 2.62985 + 0.0102682 I\}$$

This means that

$$z01 := 2.62985 + 0.0102682 I \quad (2.1.16)$$

is the unique zero point and the absolute value of its imaginary part is given by

$$i01 := 0.0102682 \quad (2.1.17).$$

We understand that the zero point given by (2.1.16) is slight different from that in the above table obtained by a solution of the quadratic equation. We

command `MacIfx` to do the same thing for $t = 0.4$:

$$\begin{aligned} &\text{Timing}[\text{NIntegrate}[h[z, 0.4], \\ &\quad \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] \end{aligned} \quad (2.1.18),$$

his return

$$\{669.833 \text{ Second}, 1. + 1.55123 \cdot 10^{-9} I\}.$$

$$\begin{aligned} &\text{Timing}[\text{NIntegrate}[k[z, 0.4], \\ &\quad \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] \end{aligned} \quad (2.1.19),$$

his return

$$\{719.75 \text{ Second}, 2.62877 - 0.059788 I\}.$$

This means that

$$z02 := 2.62877 - 0.0597788 I \quad (2.1.20)$$

is the unique zero point of the function $f[z, 0.4]$ in the domain $D[0.4]$ and the absolute value of its imaginary part is given by

$$i02 := 0.0597788 \quad (2.1.21).$$

We regard these as 0-th approximation procedure and put

$$t01 := 0.3 \quad (2.1.22),$$

$$t02 := 0.4 \quad (2.1.23)$$

and define the value of the parameter t of the 1st approximation step as

$$\begin{aligned} t1 &:= (t01 * i02 + t02 * i01) / (i01 + i02) \\ &:= \frac{t01 i02 + t02 i01}{i01 + i02} \end{aligned} \quad (2.1.24),$$

and command him to show the numerical value by letting $N[t1]$ enter, then Mac returns

$$N[t1] = 0.314659.$$

Thus we command `MacIfx` to calculate the numerical values

$$\text{Timing}[\text{NIntegrate}[h[z, t1], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] \quad (2.1.25)$$

and

$$\text{Timing}[\text{NIntegrate}[k[z, t1], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] \quad (2.1.26).$$

His returns are

$$\{653.7 \text{ Second}, 1. + 3.97888 \cdot 10^{-13} I\},$$

$$\{654.9 \text{ Second}, 2.62987 - 0.000360381 I\}.$$

We can build and execute the 2nd step:

$$i1 := 0.000360381 \quad (2.1.27),$$

$$t2 := (t01 * i1 + t1 * i01) / (i01 + i1)$$

$$N[t2] = 0.314162$$

$$\text{Timing}[\text{NIntegrate}[h[z, t2], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] = \{655.467 \text{ Second}, 1. + 1.43377 \cdot 10^{-12} I\} \quad (2.1.29)$$

and

$$\text{Timing}[\text{NIntegrate}[k[z, t2], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] = \{657.283 \text{ Second}, 2.62987 - 0.00000195408 I\} \quad (2.1.30).$$

The commands and returns for the 3rd step are

$$i2 := 0.00000195408 \quad (2.1.31),$$

$$t3 := (t01 * i2 + t2 * i01) / (i01 + i2)$$

$$N[t3] = 0.314159 \quad (2.1.32),$$

$$\text{Timing}[\text{NIntegrate}[h[z, t3], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] = \{657.283 \text{ Second}, 1. + 1.43939 \cdot 10^{-12} I\} \quad (2.1.33)$$

and

$$\text{Timing}[\text{NIntegrate}[k[z, t3], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] = \{659.417 \text{ Second}, 2.62987 - 1.05922 \cdot 10^{-8} I\} \quad (2.1.34).$$

We prepare the 4th step as follows:

$$i3 := 1.05922 \cdot (10^{-8}) \quad (2.1.35),$$

$$t4 := (t01 \cdot i3 + t3 \cdot i01)/(i01 + i3)$$

$$N[t4] = 0.314159 \quad (2.1.36).$$

The above numerical value 0.314159 of the parameter t at the 3rd and 4th steps coincide with each other. So we ask MacIfx to show the more precise value and he returns

$$N[t3, 10] = 0.31415928, \quad N[t4, 10] = 0.3141592654 \quad (2.1.37)$$

which are larger than 0.314159. Hence we command MacIfx below and he returns as follows:

$$\text{Timing}[\text{NIntegrate}[h[z, 0.314159], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] = \{676. \text{ Second}, 1. + 1.43997 \cdot 10^{-12} I\} \quad (2.1.38)$$

and

$$\text{Timing}[\text{NIntegrate}[k[z, 0.314159], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] = \{677.067 \text{ Second}, 2.62987 + 1.91398 \cdot 10^{-7} I\} \quad (2.1.39)$$

To satisfy ourselves, we give the commands below and obtains the returns as follows:

$$\text{Timing}[\text{NIntegrate}[h[z, t4], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] = \{663.267 \text{ Second}, 1. + 1.43942 \cdot 10^{-12} I\} \quad (2.1.40)$$

and

$$\text{Timing}[\text{NIntegrate}[k[z, t4], \{z, 1 - 0.1 I, 5 - 0.1 I, 5 + 0.1 I, 1 + 0.1 I, 1 - 0.1 I\}]] = \{665.317 \text{ Second}, 2.62987 - 5.95019 \cdot 10^{-10} I\} \quad (2.1.41).$$

Since the imaginary part of the unique zero point of the function $f[z, t]$ in

$D[t]$ changes signature at points 0.314159 and $t_4 = 0.31415926\dots$, by Theorem 1, there exists a value t_{realzero} of the parameter t such that the function $f[z, t]$ has a real valued zero point z_{realzero} between 0.314159 and $t_4 = 0.31415926\dots$, so we reached at the target numerical value

$$t_{\text{realzero}} = 0.314159 \quad (2.1.42)$$

and the seeked real valued zero point of the function $f[z, t_{\text{realzero}}]$

$$z_{\text{realzero}} = 2.62987 \quad (2.1.43).$$

By Theorem 3, the sequence $\{tn; n = 1, 2, 3, \dots\}$ thus obtained by the inverse linear interpolation is monotonically decreasing and convergent. But the practical sequence converges very speedy in the numerical sense.

We summarize the above results in the table below:

number of step	about t left end right end interpolation procedure	about zero point real part left t 's right t 's	about zero point imaginary part left end's Calculation time right end's Calculation time
0	$t_{01} := 0.3$ $t_{02} := 0.4$ $t_1 := \frac{t_{01} i_{02} + t_{02} i_{01}}{i_{01} + i_{02}}$ $N[t_1] = 0.314659$	2.62985 2.62877	$i_{01} := 0.0102682$ 11.3 Minutes $-i_{02} := -0.059788$ 12.0 Minutes
1	$t_{01} := 0.3$ $t_1 := 0.314659$ $t_2 := \frac{t_{01} i_1 + t_1 i_{01}}{i_{01} + i_1}$ $N[t_2] = 0.314162$	2.62985 2.62987	$i_{01} := 0.0102682$ $-i_1 := -0.000360381$ 10.9 Minutes
2	$t_{01} := 0.3$ $t_2 := 0.314162$ $t_3 := \frac{t_{01} i_2 + t_2 i_{01}}{i_{01} + i_2}$ $N[t_3] = 0.314159$	2.62985 2.62987	$i_{01} := 0.0102682$ $-i_2 := -0.00000195408$ 11.0 Minutes
3	$t_{01} := 0.3$ $t_3 := 0.314159$ $t_4 := \frac{t_{01} i_3 + t_3 i_{01}}{i_{01} + i_3}$ $N[t_4] = 0.314159$ $N[t_4, 10] = 0.3141592654$	2.62985 2.62987	$i_{01} := 0.0102682$ $-i_3 := -1.05922 \cdot 10^{-8}$ 11.0 Minutes
Confirm.	0.314159	2.62987	$1.91398 \cdot 10^{-7}$

Next we consider the case

$$D[t_-] := \{z \in \mathbb{C}; -0.1 < \operatorname{Re}[z] < 0.1, 5 < \operatorname{Im}[z] < 9\} \quad (2.1.44)$$

similarly and summarize the results in the similar table below:

number of step	about t left end right end interpolation procedure	about zero point real part left t 's right t 's	about zero point imaginary part left end's Calculation time right end's Calculation time
0	$t01 := 0.3$ $t02 := 0.4$ $t1 := \frac{t01 \cdot i02 + t02 \cdot i01}{i01 + i02}$ $N[t1] = 0.315202$	6.08635 6.08595	$i01 := 0.00794068$ 16.5 Minutes $-i02 := -0.0442938$ 17.4 Minutes
1	$t01 := 0.3$ $t1 := 0.315202$ $t2 := \frac{t01 \cdot i1 + t1 \cdot i01}{i01 + i1}$ $N[t2] = 0.314171$	6.08635 6.08636	$i01 := 0.00794068$ $-i1 := -0.000577716$ 16.2 Minutes
2	$t01 := 0.3$ $t2 := 0.314171$ $t3 := \frac{t01 \cdot i2 + t2 \cdot i01}{i01 + i2}$ $N[t3] = 0.314159$	6.08635 6.08636	$i01 := 0.00794068$ $-i2 := -0.00000650311$ 16.3 Minutes
3	$t01 := 0.3$ $t3 := 0.314159$ $t4 := \frac{t01 \cdot i3 + t3 \cdot i01}{i01 + i3}$ $N[t4] = 0.314159$ $N[t4, 10] = 0.3141592668$	6.08635 6.08636	$i01 := 0.00794068$ $-i3 := -7.30838 \cdot 10^{-8}$ 16.3 Minutes
Confirm.	0.314159	6.08636	$1.47147 \cdot 10^{-7}$

Thus we have systematically obtained real valued zero points of the function $f[z, t]$ in the cube $\{[z, t] \in \mathbb{C} \times \mathbb{R}; 1 < \operatorname{Re}[z] < 9, -0.1 < \operatorname{Im}[z] < 0.1, -1 < t < 1\}$.

2.2. Value of parameter for which the zero point reaches at the boundary.

Firstly, we consider the case

$$D[t_-] := \{z \in \mathbb{C}; 0 < \operatorname{Im}[z] < 0.2, 6 < \operatorname{Re}[z] < 7\} \quad (2.2.1)$$

and let MacIfx execute

$$\text{Timing}[\text{NIntegrate}[k[z, 0.154], \{z, 6, 7, 7 + 0.2 I, 6 + 0.2 I, 6\}]] \quad (2.2.2).$$

Then he returns

$$\{508.8 \text{ Second}, 6.08551 + 0.100168 I\} \quad (2.2.3).$$

Next, we let MacIfx execute

$$\text{Timing}[\text{NIntegrate}[k[z, 0.155], \{z, 6, 7, 7 + 0.2 I, 6 + 0.2 I, 6\}]] \quad (2.2.4).$$

and obtain Mac's return

$$\{509.533 \text{ Second}, 6.08552 + 0.0994724 I\} \quad (2.2.5).$$

This means that, between $t = 0.154$ and $t = 0.155$, there exists a value of the parameter t such that the imaginary part of the zero point z of the function $f[z, t]$ in D is just 0.1. We will seek it by the inverse linear interpolation. We put

$$i01 := 0.000168 \quad (2.2.6).$$

$$t01 := 0.154 \quad (2.2.7),$$

$$i02 := 0.1 - 0.0994724 = 0.0005276 \quad (2.2.8),$$

$$t02 := 0.155 \quad (2.2.9).$$

We regard these as 0-th approximation procedure and define the value of the parameter t of the 1st approximation step as

$$\begin{aligned} t1 &:= (t01 * i02 + t02 * i01) / (i01 + i02) \\ &:= \frac{t01 i02 + t02 i01}{i01 + i02} \end{aligned} \quad (2.2.10)$$

and command Mac to show the numerical value by letting $N[t1, 10]$ enter, then Mac returns

$$N[t1, 10] = 0.1542415181 \quad (2.2.11).$$

We command MacIfx to calculate the numerical values

$$\text{Timing}[\text{NIntegrate}[k[z, t], \{z, 6, 7, 7 + 0.2 \text{ I}, 6 + 0.2 \text{ I}, 6\}]] \quad (2.2.12).$$

His return is

$$\{510.217 \text{ Second}, 6.08551 + 0.1 \text{ I}\} \quad (2.2.13).$$

Thus we can conclude that at

$$t := 0.15424 \quad (2.2.14)$$

the function $f[z, t]$ has a zero point

$$z := 6.0855 + 0.1 \text{ I} \quad (2.2.15)$$

at the boundary point of the domain $D[t]$ given by (2.1.3).

In this way, we can obtain parameters t for which the zero points z of the function $f[z, t]$ is at the boundary $\Gamma[t]$ of the domain $D[t]$. We give them in the following table:

value of t	right zero	left zero
0.15424	$6.0855 + 0.1 \text{ I}$	$2.6276 + 0.12223 \text{ I}$
0.18205	$6.08574 + 0.0809883 \text{ I}$	$2.62821 + 0.1 \text{ I}$
0.314159	$6.08636 + 1.47147 \cdot 10^{-7} \text{ I}$	$2.62987 + 1.91398 \cdot 10^{-7} \text{ I}$
0.461912	$6.08502 - 0.0720673 \text{ I}$	$2.62627 - 0.1 \text{ I}$
0.534453	$6.08305 - 0.1 \text{ I}$	

The above 0.15424, 0.18205, 0.461912 and 0.534453 approximate points of $T_{\text{boundaryzero}}$ of Lemma 1.

2.3. Repeated Integral.

In this paragraph, we obtained that the number of zero points of the holomorphic function $f[z, t]$ in the open rectangle $D[t]$ of (2.1.3) as the numerical value of single integral (2.1.10) along the circumference $\Gamma[t]$ of the rectangle $D[t]$ for $t = -1, -0.9, -0.8, -0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1$ and 0. According to more empirical knowledge, we convince ourself of non existence of zeros of the function $f[z, t]$ of (2.1.1) in the interior of $D[t]$ for $-1 \leq t \leq -0.1$. But, the above experimental results do not

immediately guarantee it theoretically. Since Mac returns a quarter of non-negative integers of $n[t]$ for any t and since $n[t]$ is continuous in the open subset $T_{\text{neverzero}}$ of the interval $\{t \in \mathbf{R}; -1 \leq t \leq 1\}$, the repeated integral

$$\begin{aligned} r[\alpha_-, \beta_-] &:= \text{NIntegrate}[h[z, t], \{t, \alpha, \beta\}, \\ &\quad \{z, 1 - 0.1 \text{ I}, 9 - 0.1 \text{ I}, 9 + 0.1 \text{ I}, 1 + 0.1 \text{ I}, 1 - 0.1 \text{ I}\}] \\ &:= \int_{\alpha}^{\beta} n[t] dt \end{aligned} \quad (2.3.1)$$

has a non negative real value and, hence if it is zero numerically, we can conclude that there is no zero point of the function $f[z, t]$ in the domain $D[t]$ for each t with $\alpha \leq t \leq \beta$. For example, we let the software Mathematica run on the Macintosh IIfx and command him to do

$$\begin{aligned} &\text{Timing}[\text{NIntegrate}[h[z, t], \{t, -1, 0\}, \\ &\quad \{z, 1 - 0.1 \text{ I}, 9 - 0.1 \text{ I}, 9 + 0.1 \text{ I}, 1 + 0.1 \text{ I}, 1 - 0.1 \text{ I}\}]] \end{aligned} \quad (2.3.2),$$

then he returns

$$\{47602.9 \text{ Second}, 0.\} \quad (2.3.3)$$

This means that there is no zero point of the function $f[z, t]$ in the domain $D[t]$ for any t with $-1 \leq t \leq 0$. The calculation time 47602.9 Second is about 13.223 hours and $13.223 \text{ hours}/3 = 4.4 \text{ hours}$ is conversion calculation time of Mac IIfx of common use. For example, we let the software Mathematica run on the Macintosh IIfx and command him to do

$$\begin{aligned} &\text{Timing}[\text{NIntegrate}[h[z, t], \{t, 0.16, 0.18\}, \\ &\quad \{z, 1 - 0.1 \text{ I}, 9 - 0.1 \text{ I}, 9 + 0.1 \text{ I}, 1 + 0.1 \text{ I}, 1 - 0.1 \text{ I}\}]] \end{aligned} \quad (2.3.4),$$

then he returns

$$\{44523.2 \text{ Second}, 0.0211981 - 0.00296181 \text{ I}\} \quad (2.3.5)$$

spending 12 hours and 22 minutes. By the results of the preceding paragraph, we can assume that

$$n[t] = 1 \quad (0.16 \leq t \leq 0.18) \quad (2.3.6)$$

and

$$\text{second component of (2.3.5)} := \int_{0.16}^{0.18} n[t] dt = 0.02 \quad (2.3.7).$$

Hence we can not satisfy in the sense of calculation speed and accuracy. There remains a task for the authors in another chance to identify parameters directly by repeated integrals.

We will give a table for the results obtained:

value of lower end α	value of upper end β	calculation time Ifx with 4MBRAM (converted) Ifcx	numerical value of the repeated integrals	value ought to be
-1	0	Ifx (4.4 hours) Ifcx 13.2 hours 47602.9 Second	0.	0.
0	0.16	Ifcx with 8MBRAM 583458. Second	0.00749909-0.00808293 I Pyro! unactivating	0.00576
0	0.15	Ifx (5.1 days) Ifcx 15.4 days 1.32958 10^6 Second	0.000840401-0.00512829 I	0.
0.16	0.18	Ifx 12.4 hours 44523.2 Second	0.0211981-0.00296181 I	0.02
0.18	0.19	Ifx 5.3 hours 19095.6 Second	0.0163725+0.000750783 I	0.01795
0.19	0.46	Ifx >3 days Ifcx 38 days 3.29034 10^6 Second	0.537657-0.00168507 I	0.54
0.46	0.47	Ifx (5.3 hours) Ifcx 15.8 hours 56883.1 Second	0.0135699-0.00137113 I	0.011912
0.47	0.5	Ifx 14.1 hours 50668.3 Second	0.0296248+0.000823652 I	0.03
0.5	0.54	Ifx (22.7 hours) Ifcx 68.2 hours 245596. Second	0.0291285+0.0109753 I	0.034453
0.54	0.7	Ifcx 6.5 days 558073. Second	0.00099732+0.0132291 I Pyro! unactivating	0.
0.7	1	Ifx 19.5 hours 70145.4 Second Ifcx 2.46 days 212453. Second Speed Ratio 3.029	-0.000218093+0.0000335967 I	0.

3. Example 2

In this chapter, we also consider Bessel functions $J_m[z]$ defined by (2.1).

3.1. Zero point of order two.

Instead of the function f defined by (2.1.1), the interval T defined by (2.1.2) and the rectangle $D[t]$ defined by (2.1.3), in this chapter we consider the function

$$\begin{aligned} f[z_-, t_-] &:= \text{I} (t - \text{Pi}/10) \text{ BesselJ}[1, z] - \text{BesselJ}[2, z - (t - \text{Pi}/10)^2] \\ &:= \left(t - \frac{\pi}{10}\right) J_1[z] \sqrt{-1} - J_2\left[z - \left(t - \frac{\pi}{10}\right)^2\right] \end{aligned} \quad (3.1.1)$$

of two complex variables z and t , the compact space equipped with the topology reduced from that of the real line $\mathbf{R}$

$$T := \{t \in \mathbf{R}; 0.3 \leq t \leq 0.4\} \quad (3.1.2)$$

and, for any $t \in T$, the closed rectangle

$$D[t_-] := \{z \in \mathbf{C}; -1 \leq \text{Re}[z] \leq 1, -0.1 \leq \text{Im}[z] \leq 0.1\} \quad (3.1.3)$$

as a closed domain of the complex plane $\mathbf{C}$ and

$$\Gamma[t_-] := \partial D[t] \quad (3.1.4).$$

For these datum, we use the notations $h[z, t]$, $k[z, t]$ and $s[z, t]$, defined respectively by (2.1.7), (2.1.8) and (2.1.9). Similarly as in the preceding chapter, we consider for $t \in T_{\text{neverzero}}$ the complex integral of the functions $h[z, t]$, $k[z, t]$ and $s[z, t]$ along the rectangle $\Gamma[t]$ letting the hardwares Macintosh IIx and IICx start and the software Mathematica run:

$$\begin{aligned} n[t_-] &:= \text{NIntegrate}[h[z, t], \\ &\quad \{z, -1 - 0.1 \text{ I}, 1 - 0.1 \text{ I}, 1 + 0.1 \text{ I}, -1 + 0.1 \text{ I}, -1 - 0.1 \text{ I}\}] \end{aligned} \quad (3.1.5),$$

$$\begin{aligned} g[1, t_-] &:= \text{NIntegrate}[k[z, t], \\ &\quad \{z, -1 - 0.1 \text{ I}, 1 - 0.1 \text{ I}, 1 + 0.1 \text{ I}, -1 + 0.1 \text{ I}, -1 - 0.1 \text{ I}\}] \end{aligned} \quad (3.1.6)$$

and

$$g[2, t_] := \text{NIntegrate}[s[z, t], \{z, -1 - 0.1 I, 1 - 0.1 I, 1 + 0.1 I, -1 + 0.1 I, -1 - 0.1 I\}] \quad (3.1.7)$$

We command to Macintosh IIfx to do numerical calculation by typing `Timing[NIntegrate[ ]]` asking to output the calculation time too. Then Mac returns the pair of calculation time and numerical value of complex integration. We show them in the following tables.

t	Calculation time By IIfx Second Minutes	Value of $n[t]$	Number of zero points in $D[t]$
0.3	446.05 Second 7.4 Minutes	$2. + 1.7578 \cdot 10^{-11} I$	2
0.4	354.267 Second 5.9 Minutes	$1. + 3.52845 \cdot 10^{-11} I$	1
0.35	446.083 Second 7.4 Minutes	$1. + 1.77219 \cdot 10^{-11} I$	1
0.325	400.517 Second 6.7 Minutes	$2. + 1.97671 \cdot 10^{-11} I$	2

By the method of bisection and the results given in the table below, we see that the zero point with positive real part of the function $f[z, t]$ changes the signature of the imaginary part between 0.3 and 0.325.

t	Number of Zeros $D[t]$	Value of $g[1, t]$ Calculation time By IIfx Second Minutes	Value of $g[2, t]$ Calculation time By IIfx Second Minutes	Zeros of $f[z, t]$ in $D[t]$
0.3	2	0.000401184 -0.0566446 I 319.183 Second 5.3 Minutes	-0.00320853 -0.0000454499 I 258.667 Second 4.3 Minutes	0.0285562 -0.0297248 I -0.028155 -0.0269198 I
0.325	2	0.000235117 +0.0433663 I 355.15 Second 5.9 Minutes	-0.00188061 +0.000020392 I 255.483 Second 4.3 Minutes	0.000240065 +0.0424899 I, -0.00000494811 +0.000876424 I

The above results shows that the signature of the imaginary parts of the zero points with positive real part of $f[z, t]$ changes its signature from negative to positive between 0.3 and 0.325 and suggests, by Theorem 1, the existence of

parameters t between 0.3 and 0.325 at which the function $f[z, t]$ has a real valued zero points.

We continue the method of bisection and obtain the following sequence $\{t_n; n = 0, 1, 2, \dots\}$ of parameters t_n so that the product of imaginary parts of zero points of $f[z, t_{n-1}]$ and $f[z, t_n]$ with positive real parts is negative:

number of step n	about t left end right end interpolation procedure	about two zero points that with positive real part that with negative real part Calculation time, $n[ti]$ Calculation time, $g[1, ti]$ Calculation time, $g[2, ti]$
0	0.3	0.0285562−0.0297248 I −0.028155−0.0269198 I {446.05 Second 2. +1.7578 10^{-11} I} {319.183 Second 0.000401184−0.0566446 I} {255.8 Second −0.00320853−0.0000454499 I}
	0.325	0.000240065+0.0424899 I −0.00000494811+0.000876424 I {400.517 Second 2. +1.97671 10^{-11} I} {355.15 Second 0.000235117+0.0433663 I} {255.483 Second −0.00188061+0.000020392 I}
1	0.3125	0.00000550645−0.00663722 I −2.41943 10^{-10} +1.53086 10^{-7} I {448.133 Second 2. −5.99702 10^{-15} I} {262.283 Second 0.00000550621−0.00663707 I} {153.75 Second −0.0000440527−7.30951 10^{-8} I}
	0.325	the same as the above
2	0.3125	the same as the above
	0.31875	0.0000421869+0.0183633 I −3.47621 10^{-8} −9.98048 10^{-8} I {428.067 Second 2. −6.75316 10^{-11} I} {311.883 Second 0.0000421521+0.0183632 I} {155.333 Second −0.000337209+0.00000154938 I}

.....omission of halves.....		
14	0.314158630371094	0.00000126796-0.00000126802 I -0.00000126802-0.00000126796 I {457.6 Second 2.-3.8093 10 ⁻¹⁵ I} {105.05 Second -5.91869 10 ⁻¹¹ -0.00000253598 I} {105.567 Second 0.}
	0.31416015625	the same as the above, i.e., 0.00000177904+0.00000177896 I -0.00000177896+0.00000177904 I {451.167 Second 2.-3.80933 10 ⁻¹⁵ I} {99.8833 Second 8.5754 10 ⁻¹¹ +0.0000035558 I} {100.4 Second 0.}
15	0.314159393310547	0. 0. {462.1 Second 2.-3.80936 10 ⁻¹⁵ I} {110.3 Second 0.} {110.55 Second 0.}

For the function defined by (3.1.1), we command

$$d[z_-, t_-] := D[f[z, t], z] \quad (3.1.8).$$

Then, Mac returns immediately an explicit formula remembering the recurrence formula. We seek the values of f and d by commanding Mac to do below. Then Mac returns as follows.

$$N[f[0, 0.314158]] = 3.20453 \cdot 10^{-25} \quad (3.1.9),$$

$$N[d[0, 0.314158]] = -6.32679 \cdot 10^{-7} I \quad (3.1.10),$$

$$N[f[0, 0.314159]] = -6.19791 \cdot 10^{-28} \quad (3.1.11),$$

$$N[d[0, 0.314159]] = -1.32679 \cdot 10^{-7} I \quad (3.1.12),$$

$$N[f[0, 0.314160]] = -3.64092 \cdot 10^{-26} \quad (3.1.13),$$

$$N[d[0, 0.314160]] = 3.67321 \cdot 10^{-7} I \quad (3.1.14).$$

The derivative of f with respect to z changes the signatures between (3.1.12) and (3.1.14). Thus we conclude that at

$$t = 0.314159 \quad (3.1.15)$$

the function $f[z, t]$ has numerically a zero point in the interior of $D[t]$

$$z = 0 \quad (3.1.16)$$

of order 2.

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