THE CLASSIFICATION OF COMPACT SURFACES BY USING BERGER'S LEMMA

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Dedicated to Professor Tominosuke Otsuki on his 75th birthday

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0. Introduction.

Let M be a compact Riemannian manifold and $p, q \in M$ are the points such that

$$d(p, q) = d(M) \quad (:= \text{diameter of } M)$$

where d is the distance function of M . Then in [1], M. Berger gave the following interesting fact which is known as Berger's Lemma:

FACT (M. Berger). For any non-zero tangent vector $v \in T_p(M)$, there exists a shortest geodesic $\gamma: [0, d(p, q)] \rightarrow M$ from p to q satisfying

$$\angle(v, \dot{\gamma}(0)) \leq \frac{\pi}{2}.$$

Here $T_p(M)$ denotes the tangent space to M at p and $\angle(x, y)$ the angle between the vectors x and y . In this note, all geodesics are parametrized by arc-length. By $G(p, q)$, we will denote the set of all shortest geodesics from p to q .

Now the purpose of this note is to show that all compact surfaces which is homeomorphic to a 2-dimensional sphere S^2 can be classified into three types. To this classification, we will use the conditions considered by using Berger's Lemma.

1. Three conditions.

For a compact surface M , we will consider the following three conditions

[C0], [CI], [CII]:

[C0] For any pair of points $p, q \in M$ satisfying $d(p, q) = d(M)$ and any non-zero vector $v \in T_p(M)$, there exists a geodesic $\gamma \in G(p, q)$ such that

$$\angle(v, \dot{\gamma}(0)) \leq \frac{\pi}{2}$$

[CI] For any pair of points $p, q \in M$ satisfying $d(p, q) = d(M)$, any subset $G'(p, q) \subset G(p, q)$ which satisfies the following condition (*):

(*) For any non-zero vector $v \in T_p(M)$, there exists some

$$\gamma \in G'(p, q) \text{ such that } \angle(v, \dot{\gamma}(0)) \leq \frac{\pi}{2}$$

also satisfies the following condition (**):

(**) For any non-zero tangent vector $w \in T_q(M)$, there exists

$$\text{some } \gamma \in G'(p, q) \text{ such that } \angle(w, -\dot{\gamma}(d(p, q))) \leq \frac{\pi}{2}.$$

[CII] For any pair of points $p, q \in M$ satisfying $d(p, q) = d(M)$, following condition (***) is satisfied:

(***) For any two non-zero tangent vectors $v \in T_p(M)$ and $w \in T_q(M)$, there exists a geodesic $\gamma \in G(p, q)$ such that

$$\angle(v, \dot{\gamma}(0)) \leq \frac{\pi}{2} \text{ and } \angle(w, -\dot{\gamma}(d(p, q))) \leq \frac{\pi}{2}.$$

Of course by Berger's Lemma, condition [C0] is satisfied automorphically for all compact surfaces. Condition [CI] and [CII] are considered in connection with Berger's Lemma and are related mutually.

For later use, we will give a Proposition which is equivalent to Berger's Lemma.

PROPOSITION 1. Let $p, q \in M$ are the points satisfying $d(p, q) = d(M)$. Then there exists a subset $G'(p, q) \subset G(p, q)$ whose number of elements equals 2 or 3 and satisfying the following condition:

for any non-zero vector $v \in T_p(M)$, there exists some $\gamma \in G'(p, q)$ such that

$$\angle(v, \dot{\gamma}(0)) \leq \frac{\pi}{2}.$$

This can be seen easily but we will give a proof here because the method used in the proof of this Proposition is useful later.

PROOF OF PROPOSITION 1. Let $\gamma_1: [0, d(p, q)] \rightarrow M$ be a shortest geodesic from p to q . Then by Berger's Lemma, putting $v = -\dot{\gamma}_1(0)$, we get another geodesic $\gamma_2 \in G(p, q)$ such that

$$\star(v, \dot{\gamma}_2(0)) \leq \frac{\pi}{2}.$$

If $\star(-\dot{\gamma}_1(0), \dot{\gamma}_2(0)) = 0$ i.e. $-\dot{\gamma}_1(0) = \dot{\gamma}_2(0)$, then $G'(p, q) = \{\gamma_1, \gamma_2\}$ satisfies the required condition. In the case $\star(-\dot{\gamma}_1(0), \dot{\gamma}_2(0)) > 0$, putting $v = -(\dot{\gamma}_1(0) + \dot{\gamma}_2(0))$ we get a geodesic $\gamma_3 \in G(p, q)$ such that

$$\star(v, \dot{\gamma}_3(0)) \leq \frac{\pi}{2}.$$

If $\star(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) + \star(\dot{\gamma}_2(0), \dot{\gamma}_3(0)) \geq \pi$, $G'(p, q) = \{\gamma_1, \gamma_2, \gamma_3\}$ is the one which we want. If $\star(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) + \star(\dot{\gamma}_2(0), \dot{\gamma}_3(0)) < \pi$, then putting $v = -(\dot{\gamma}_1(0) + \dot{\gamma}_3(0))$ we also get a geodesic $\gamma_4 \in G(p, q)$ such that

$$\star(v, \dot{\gamma}_4(0)) \leq \frac{\pi}{2}.$$

If $\star(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) + \star(\dot{\gamma}_2(0), \dot{\gamma}_4(0)) \geq \pi$, then $G'(p, q) = \{\gamma_1, \gamma_2, \gamma_4\}$ is the one which we want. If $\star(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) + \star(\dot{\gamma}_2(0), \dot{\gamma}_4(0)) < \pi$, then putting $v = -(\dot{\gamma}_1(0) + \dot{\gamma}_4(0))$, we also get a geodesic $\gamma_5 \in G(p, q)$ such that

$$\star(v, \dot{\gamma}_5(0)) \leq \frac{\pi}{2}.$$

Repeating this process we can reduce to the following two cases:

Case 1. There exists a sequence of geodesics $\{\gamma_i\}_{i=1,2,\dots,m} \subset G(p, q)$

($m \geq 3$) such that $\sum_{i=1}^{m-2} \star(\dot{\gamma}_i(0), \dot{\gamma}_{i+1}(0)) < \pi$ and $\sum_{i=1}^{m-1} \star(\dot{\gamma}_i(0), \dot{\gamma}_{i+1}(0)) \geq \pi$

Case 2. There exists a sequence of geodesics $\{\gamma_i\}_{i=1,2,3,\dots} \subset G(p, q)$ such

that $\sum_{i=1}^k \star(\dot{\gamma}_i(0), \dot{\gamma}_{i+1}(0)) < \pi$ for all k and $\sum_{i=1}^{\infty} \star(\dot{\gamma}_i(0), \dot{\gamma}_{i+1}(0)) = \pi$.

If case 1 occur, then letting $G'(p, q) = \{\gamma_1, \gamma_2, \gamma_m\}$ and if case 2 occur, letting $G'(p, q) = \{\gamma_1, \gamma_2, \gamma_{\infty}\}$, we complete the proof of Proposition 1 where $\gamma_{\infty} \in G(p, q)$ is the limiting geodesic of $\{\gamma_i\}$. Q.E.D.

Proposition 1 is an improved version of Berger's Lemma.

In the following section, we will give an example of surface which does not satisfy condition [CI]. Thus condition [CI] is strictly stronger than condition [C0].

To investigate the difference between condition [CI] and [CII], we now further assume that M is homeomorphic to a 2-dimensional sphere S^2 .

Let $p, q \in M$ are points satisfying $d(p, q) = d(M)$. Then by two geodesics $\gamma_1, \gamma_2 \in G(p, q)$, M is divided into two disjoint domains D and D' such that

$$\partial D = \partial D' = \gamma_1([0, d(p, q)]) \cup \gamma_2([0, d(p, q)]).$$

In this situation, we will call D as an H -domain if it satisfy the following conditions:

$$\angle_D(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) \leq \pi \quad \text{and} \quad \angle_D(-\dot{\gamma}_1(d(p, q)), -\dot{\gamma}_2(d(p, q))) > \pi$$

or

$$\angle_D(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) > \pi \quad \text{and} \quad \angle_D(-\dot{\gamma}_1(d(p, q)), -\dot{\gamma}_2(d(p, q))) \leq \pi.$$

Here $\angle_D(,)$ denotes the inner angle measured on D .

LEMMA 1. A compact surface M which is homeomorphic to S^2 satisfies condition [CI] if and only if there exists no H -domain in M .

PROOF. Assume M satisfy condition [CI]. If there exist H -domain D such that for certain $\gamma_1, \gamma_2 \in G(p, q)$

$$\partial D = \gamma_1([0, d(p, q)]) \cup \gamma_2([0, d(p, q)])$$

and

$$\angle_D(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) > \pi, \quad \angle_D(-\dot{\gamma}_1(d(p, q)), -\dot{\gamma}_2(d(p, q))) \leq \pi.$$

Then by the argument in the proof of Proposition 1, we can get another geodesic $\gamma_3 \in G(p, q)$ such that $\gamma_3((0, d(p, q))) \subset D$ and

$$\angle_D(\dot{\gamma}_1(0), \dot{\gamma}_3(0)) \leq \pi, \quad \angle_D(\dot{\gamma}_2(0), \dot{\gamma}_3(0)) \leq \pi.$$

Thus the set $G'(p, q) = \{\gamma_1, \gamma_2, \gamma_3\} \subset G(p, q)$ satisfies the assumption (*) in [CI]. Hence by [CI], $G'(p, q) = \{\gamma_1, \gamma_2, \gamma_3\}$ must satisfy the condition (**). But this is impossible, because $\gamma_3((0, d(p, q))) \subset D$ and hence we have

$$\angle(-\dot{\gamma}_1(d(p, q)), -\dot{\gamma}_3(d(p, q))) \leq \angle_D(-\dot{\gamma}_1(d(p, q)), -\dot{\gamma}_3(d(p, q))) \leq \pi$$

and

$$\star(-\dot{\gamma}_2(d(p, q)), -\dot{\gamma}_3(d(p, q))) \leq \star_D(-\dot{\gamma}_2(d(p, q)), -\dot{\gamma}_3(d(p, q))) \leq \pi.$$

To show the converse, let $G'(p, q) \subset G(p, q)$ be any subset which satisfies the condition (*) in [CI]. Assuming that this $G'(p, q)$ does not satisfy the condition (**), we will derive a contradiction. Since $G'(p, q)$ does not satisfy (**), we can find a half circle $\tilde{S}^1$ in $T_q^1(M) := \{v \in T_p(M) : \|v\| := \text{norm of } v = 1\}$ such that

$$\{-\dot{\gamma}(d(p, q)) : \gamma \in G'(p, q)\} \subset \tilde{S}^1$$

and the number of elements of the set

$$\partial\tilde{S}^1 \cap \{-\dot{\gamma}(d(p, q)) : \gamma \in G'(p, q)\}$$

is at most one. Here we are identifying $T_q^1(M)$ with a circle S^1 .

Now take a geodesic $\gamma_1 \in G'(p, q)$ and put $v := -\dot{\gamma}_1(0)$. Then from the condition (*), we can find another geodesic $\gamma_2 \in G'(p, q)$ with $\star(-\dot{\gamma}_1(0), \dot{\gamma}_2(0)) \leq \pi/2$. If $\dot{\gamma}_2(0) = -\dot{\gamma}_1(0)$, then the subset $G''(p, q) = \{\gamma_1, \gamma_2\} \subset G'(p, q) \subset G(p, q)$ satisfies (*). If $\star(\dot{\gamma}_1(d(p, q)), \dot{\gamma}_2(d(p, q))) < \pi$, then we can easily find an H -domain. So it must be

$$\star(\dot{\gamma}_1(d(p, q)), \dot{\gamma}_2(d(p, q))) = \pi.$$

Thus $G''(p, q)$ satisfies the condition (**) and hence $G'(p, q)$ satisfies (**). This is a contradiction.

In the case $\star(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) < \pi$, by the argument in the proof of Proposition 1, we can find third geodesic $\gamma_3 \in G'(p, q)$ such that $G''(p, q) := \{\gamma_1, \gamma_2, \gamma_3\}$ satisfies the condition (*). Then since

$$\{\dot{\gamma}_i(d(p, q)) : i = 1, 2, 3\} \subset \{\dot{\gamma}(d(p, q)) : \gamma \in G'(p, q)\} \subset \tilde{S}^1$$

we can easily get an H -domain and this is a contradiction.

q.e.d.

Using this Lemma 1, we can show the following

PROPOSITION 2. Let M be a compact surface which is homeomorphic to S^2 and satisfies condition [CII]. Then M satisfies condition [CI] also.

In the following section, we will give an example of compact surfaces which satisfies condition [CI] but does not satisfy condition [CII]. Thus condition [CII] is strictly stronger than [CI].

PROOF OF PROPOSITION 2. Assume M does not satisfy condition [CI]. Then by Lemma 1, there exists a pair of points $p, q \in M$ satisfying $d(p, q) = d(M)$ and H -domain D such that

$$\partial D = \gamma_1([0, d(p, q)]) \cup \gamma_2([0, d(p, q)])$$

and

$$\star_D(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) \leq \pi,$$

$$\star_D(-\dot{\gamma}_1(d(p, q)), -\dot{\gamma}_2(d(p, q))) > \pi$$

where $\gamma_1, \gamma_2 \in G(p, q)$. Then putting $v := -(\dot{\gamma}_1(0) + \dot{\gamma}_2(0))$, $w := \dot{\gamma}_1(d(p, q)) + \dot{\gamma}_2(d(p, q))$, by [CII] we get a geodesic $\gamma_3 \in G(p, q)$ satisfying

$$\star(v, \dot{\gamma}_3(0)) \leq \pi/2 \quad \text{and} \quad \star(w, \dot{\gamma}_3(d(p, q))) \leq \pi/2.$$

Then since $\dot{\gamma}_3(d(p, q)) \neq \dot{\gamma}_1(d(p, q))$ and $\dot{\gamma}_3(d(p, q)) \neq \dot{\gamma}_2(d(p, q))$, we know that $\gamma_3 \neq \gamma_1, \gamma_2$ and $\gamma_3([d(p, q) - \varepsilon, d(p, q)]) \subset D$ for some $\varepsilon > 0$. Also we know that $\gamma_3((0, \varepsilon]) \subset \bar{D}^c$ for some $\varepsilon' > 0$. Thus $\gamma_3|[\varepsilon', d(p, q) - \varepsilon]$ is connecting a point $(= \gamma_3(\varepsilon'))$ in $\bar{D}^c$ to a point $(= \gamma_3(d(p, q) - \varepsilon))$ in D . Thus using the assumption that M is homeomorphic to S^2 , we see $\gamma_3|[\varepsilon', d(p, q) - \varepsilon]$ must intersect $\partial D = \gamma_1([0, d(p, q)]) \cup \gamma_2([0, d(p, q)])$ which is an image of simply closed curve $\gamma_2^{-1} \circ \gamma_1$. This contradicts the shortestness of γ_1 or γ_2 .

Q.E.D.

Now we will call a compact surface M homeomorphic to S^2 to be of type 0 if M does not satisfy condition [CI]. Note that condition [C0] is satisfied for all compact Riemannian manifold automorphically by Berger's Lemma. Similarly we will call M to be of type 1 (resp. type 2) if M satisfies condition [CI] but does not satisfy condition [CII] (resp. satisfies condition [CII]). So putting

$$\mathfrak{m} := \{M : M \text{ is a compact surface homeomorphic to } S^2\}$$

and

$$\mathfrak{m}_i := \{M : M \in \mathfrak{m} \text{ and is of type } i\} \quad i = 0, 1, 2,$$

we can summarize Proposition 1 and Proposition 2 as

$$\text{THEOREM 1. } \mathfrak{m} = \mathfrak{m}_0 \cup \mathfrak{m}_1 \cup \mathfrak{m}_2$$

REMARK. Each m_i is non-empty as is shown in the following section.

2. Examples.

In this section, we will give an example of compact surfaces of each type i , $i = 0, 1, 2$.

1°. Type 2 examples.

A standard sphere S^2 in a Euclidean 3-space E^3 is an example of surface which is of type 2. More generally, let M be a compact surface of revolution in E^3 which is homeomorphic to S^2 . If its two vertices $p, q \in M$ satisfy $d(p, q) = d(M)$ and

$$d(p, q) > d(p', q') \quad \text{for all } (p', q') (\neq (p, q)) \in M \times M,$$

then M is also of type 2. As such examples, we can take ellipsoids

$$x^2 + y^2 + \frac{z^2}{k^2} = 1 \quad (k > 1)$$

where (x, y, z) denotes standard orthogonal coordinates of E^3 .

2°. Type 1 examples.

As examples of this type, we also take ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (0 < a < b < c).$$

The diameter of these surfaces are attained for the pair of points (p, q) such that $p = (0, 0, c)$, $q = (0, 0, -c)$ and $G(p, q)$ consists of exactly two shortest geodesics γ_1, γ_2 with $\dot{\gamma}_1(0) = -\dot{\gamma}_2(0)$, $\dot{\gamma}_1(d(p, q)) = -\dot{\gamma}_2(d(p, q))$. So these surfaces satisfy the condition [CI] but not satisfy condition [CII] and hence of type 1.

3°. Type 0 examples.

In E^3 , we consider two rectangles R_1 and R_2 whose four vertices are

$$R_1: (0, 0, 0), (0, a, 0), (0, a, b), (0, 0, b)$$

$$R_2: (1, 0, 0), (1, a, 0), (1, a, b), (1, 0, b)$$

and four half cylinders C_1, C_2, C_3, C_4 which are defined by the following

equations:

$$C_1: (x - 1/2)^2 + (z - b)^2 = 1, z \geq b, y + z \geq b, -y + z \geq b - a$$

$$C_2: (x - 1/2)^2 + z^2 = 1, z \leq 0, -y + z \leq 0, y + z \leq a$$

$$C_3: (x - 1/2)^2 + y^2 = 1, y \leq 0, y + z \leq b, -y + z \geq 0$$

$$C_4: (x - 1/2)^2 + (y - a)^2 = 1, y \geq a, -y + z \leq b - a, y + z \geq a$$

where a, b are sufficiently large constants satisfying $0 < a < b$. Let S be a compact surface obtained by attaching four half cylinder $C_1 \sim C_4$ with R_1 and R_2 . Obviously this flat surface is homeomorphic to S^2 and has singularities. Furthermore S can be considered as a metric space whose metric d is induced from the flat metric. Then S has a center $s_o(1/2, a/2, b/2)$ i.e. there exists an isometry $\sigma: S \rightarrow S$ induced from the central symmetry $\Sigma: E^3 \rightarrow E^3$ with respect to s_o . Then we can check that

$$d(S) (= \text{diameter of } S) = d(p_o, q_o)$$

where $p_o = (1/2, -1, b+1)$, $q_o = (1/2, a+1, -1)$. Note that the points $p_o, q_o \in S$ satisfy $q_o = \sigma(p_o)$ ($p_o = \sigma(q_o)$). Furthermore we can see that there exist four shortest curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ connecting p_o to q_o . Let γ_1 and γ_2 (resp. γ_3 and γ_4) are those which are contained in the half space $\{x \geq 1/2\}$ (resp. $\{x \leq 1/2\}$) in E^3 . We can assume that $\sigma(\gamma_1) = \gamma_3$ and $\sigma(\gamma_2) = \gamma_4$. Since S is symmetric under the reflection t with respect to the plane $\{y = a/2\}$, we also have points $p_1 = t(p_o)$, $q_1 = t(q_o)$ satisfying $d(p_1, q_1) = d(S)$ and shortest curves $\sigma_i (= t(\gamma_i))$, $i = 1, 2, 3, 4$ connecting from p_1 to q_1 .

Now, we will regularize S so as to obtain a smooth convex surface $\tilde{S} \subset E^3$ satisfying the following conditions:

(i) For a fixed small $\varepsilon > 0$, $S - \{\varepsilon - \text{tubular neighborhood of four half circles } S_i, i = 1, 2, 3, 4 \text{ in } S\}$ is imbedded isometrically in $\tilde{S}$ as a flat surface. Here S_i is defined by the following equations:

$$S_1: (x - 1/2)^2 + (z - b)^2 = 1, y + z = b, z \geq b$$

$$S_2: (x - 1/2)^2 + (z - b)^2 = 1, -y + z = b - a, z \geq b$$

$$S_3: (x - 1/2)^2 + z^2 = 1, -y + z = 0, z \leq 0$$

$$S_4: (x - 1/2)^2 + z^2 = 1, y + z = a, z \leq 0$$

(in the following, we will add further restrictions).

(ii) In each ε -neighborhood of S_i , S is regularized symmetrically with respect to S_i .

(iii) $\tilde{S}$ has a centre $s_o = (1/2, a/2, b/2)$ i.e. there exists an isometry $\sigma: \tilde{S} \rightarrow \tilde{S}$ induced from the central symmetry $\Sigma: E^3 \rightarrow E^3$ with respect to s_o .

(iv) There exists a point $\tilde{p}_o$ (resp. $\tilde{q}_o$) $\in \tilde{S}$ near p_o (resp. q_o) $\in S$ such that $\tilde{p}_o, \tilde{q}_o$ are contained in the plane $\{x = 1/2\} \subset E^3$ and $d(\tilde{p}_o, \tilde{q}_o) = d(\tilde{S})$.

(v) There exists four shortest geodesics $\tilde{\gamma}_i$ near γ_i , $i = 1, 2, 3, 4$ connecting from $\tilde{p}_o$ to $\tilde{q}_o$ such that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ (resp. $\tilde{\gamma}_3$ and $\tilde{\gamma}_4$) are contained in the half space $\{x \geq 1/2\}$ (resp. $\{x \leq 1/2\}$) and

$$\sigma(\tilde{\gamma}_1) = \tilde{\gamma}_3, \quad \sigma(\tilde{\gamma}_2) = \tilde{\gamma}_4.$$

Let K denotes the Gaussian curvature of S . Then, by construction, K takes non-zero values only in a small neighborhood $V(\tilde{p}_o)$ (resp. $V(\tilde{q}_o)$, $V(\tilde{p}_1)$, $V(\tilde{q}_1)$) of point $\tilde{p}_o$ (resp. $\tilde{q}_o$, $\tilde{p}_1$, $\tilde{q}_1$). $\tilde{p}_1, \tilde{q}_1 \in \tilde{S}$ are the points near p_1 and q_1 respectively satisfying $d(\tilde{p}_1, \tilde{q}_1) = d(\tilde{S})$.

Now for a subset $U \subset \tilde{S}$, let denote the integral curvature of U by $c(U)$:

$$c(U) = \int_U K d\tilde{S}$$

where $d\tilde{S}$ is the area element of $\tilde{S}$. Since $c(U)$ equals the area of image $g(U)$ of U of the Gauss map $g: \tilde{S} \rightarrow S^2 \subset E^3$, we see

$$c(V(\tilde{p}_o)) = c(V(\tilde{q}_o)) = c(V(\tilde{p}_1)) = c(V(\tilde{q}_1)).$$

Thus from Gauss-Bonnet's Theorem, we have

$$c(V(\tilde{p}_i)) = c(V(\tilde{q}_i)) = \pi, \quad i = 1, 2.$$

Put $\alpha := \angle(\dot{\tilde{\gamma}}_1(0), \dot{\tilde{\gamma}}_2(0))$ and $\beta := \angle(\dot{\tilde{\gamma}}_1(d(p_o, q_o)), \dot{\tilde{\gamma}}_2(d(p_o, q_o)))$.

Here as was announced in (i), we will add a further restriction on $\tilde{S}$. Namely, we regularize S along S_1 and S_4 so as to have

$$\alpha \neq \beta.$$

Let D_1 be the domain bounded by $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ satisfying $\alpha = \angle_{D_1}(\dot{\tilde{\gamma}}_1(0), \dot{\tilde{\gamma}}_2(0))$. Then since $\beta \leq \pi$,

$$\beta = \angle_{D_1}(\dot{\tilde{\gamma}}_1(d(\tilde{p}_o, \tilde{q}_o)), \dot{\tilde{\gamma}}_2(d(\tilde{p}_o, \tilde{q}_o))).$$

By symmetry, we also have

$$\beta = \star (\dot{\gamma}_3(0), \dot{\gamma}_4(0)).$$

So if we put

$$\begin{aligned}\delta &:= \star (\dot{\gamma}_2(0), \dot{\gamma}_3(0)), \\ \mu &:= \star (\dot{\gamma}_2(d(\tilde{p}_o, \tilde{q}_o)), \dot{\gamma}_3(d(\tilde{p}_o, \tilde{q}_o))),\end{aligned}$$

then $\delta = \mu$ and

$$\begin{aligned}\mu &= \star (\dot{\gamma}_1(0), \dot{\gamma}_4(0)), \\ \delta &= \star (\dot{\gamma}_1(d(\tilde{p}_o, \tilde{q}_o)), \dot{\gamma}_4(d(\tilde{p}_o, \tilde{q}_o))).\end{aligned}$$

Thus we can find geodesics $\tilde{\gamma}_{i_o}$ and $\tilde{\gamma}_{j_o}$, $i_o \neq j_o$ bounding an H -domain.

3. Number of shortest geodesics.

In the previous sections, we showed that all compact surfaces which are homeomorphic to a 2-dimensional sphere S^2 are classified into three types and there exist examples of each type certainly. In its classification, the set $G(p, q)$ plays important roles. So in this section, we will study certain properties of $G(p, q)$.

Let M be a compact surface homeomorphic to a 2-dimensional sphere S^2 and $p, q \in M$ the points satisfying $d(p, q) = d(M)$.

(0). Type 0 case.

In proposition 1, we saw that

$$\# G(p, q) \geq 2$$

where $\#$ denotes the number of elements. If $\# G = 2$, then $G(p, q) = \{\gamma_1, \gamma_2\}$ and its composite $\gamma_2^{-1} \circ \gamma_1$ is a simple closed geodesic through p and q . Thus $G(p, q)$ satisfies condition [CI]. If $\# G(p, q) = 3$, let $G(p, q) = \{\gamma_1, \gamma_2, \gamma_3\}$. Then $G(q, p) = \{\gamma_1^{-1}, \gamma_2^{-1}, \gamma_3^{-1}\}$. Thus $G(p, q)$ satisfies condition [CI]. So if M is of type 0, then for some pair of points $p, q \in M$ satisfying $d(p, q) = d(M)$, it must hold that

$$\# G(p, q) \geq 4.$$

In section 2, we gave an example of surfaces of type 0 with $G(p, q)$ satisfying

$$\# G(p, q) = 4.$$

REMARK. It may be an interesting problem whether there exist an example of compact surface of type 0 with two pair of points $p, q \in M$ and $\tilde{p}, \tilde{q} \in M$ such that $d(p, q) = d(M)$, $d(\tilde{p}, \tilde{q}) = d(M)$ and satisfying

$$\# G(p, q) \geq 4, \quad \# G(\tilde{p}, \tilde{q}) \leq 3.$$

(2). Type 1 case.

In section 2, we gave an example of surfaces such that $\# G(p, q) = 2$ for all pair of points $p, q \in M$ with $d(p, q) = d(M)$ (In fact, such pair of points $p, q \in M$ was unique). Thus for type 1 surface, we can not obtain a further sharp estimate other than

$$\# G(p, q) \geq 2.$$

(3). Type 2 case.

As is stated in section 2, a standard 2-dimensional sphere S^2 is an example of surface of type 2. In this example, each geodesic $\gamma: [0, d(p, q)] \rightarrow S^2$ starting from p belongs to $G(p, q)$ and hence

$$\# G(p, q) = \infty$$

where $p, q \in S^2$ are the points satisfying $d(p, q) = d(S^2)$. We will show that this phenomenon is occurring for all compact surfaces of type 2.

Let M be a compact surface of type 2 and $p, q \in M$ are the points satisfying $d(p, q) = d(M)$. We identify $T_p^1 (= \{v \in T_p(M) : \|v\| = 1\})$ (resp. $T_q^1(M)$) with the standard unit circle $S^1 = [0, 2\pi]$ (2π is identified with 0). This identification is done so as to satisfy the following conditions:

(i) We fix a geodesic $\gamma_1 \in G(p, q)$ and identify $\dot{\gamma}_1(0)$ with $0 \in S^1$ (resp. $-\dot{\gamma}_1(d(p, q))$ with $0 \in S^1$).

(ii) For other geodesic $\gamma_2 \in G(p, q)$, if $\angle_+(\dot{\gamma}_1(0), \dot{\gamma}_2(0))$, the angle between $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ measured with respect to the positive orientation on $T_p(M)$, equals to α , then

$$\dot{\gamma}_2(0) = \alpha \in S^1$$

(resp. if $\angle_D(-\dot{\gamma}_1(d(p, q)), -\dot{\gamma}_2(d(p, q))) = \beta$, then $-\dot{\gamma}_2(d(p, q)) = \beta \in S^1$ where D is the domain in M bounded by $\gamma_2^{-1} \circ \gamma_1$ such that $\angle_D(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) = \alpha$). Here we note that two domains bounded by $\gamma_2^{-1} \circ \gamma_1$ are not H -domains. To avoid the confusion, we denote the circle $T_p^1(M)$ (resp. $T_q^1(M)$) by $S^1(p)$ (resp. $S^1(q)$). Then $T_p^1(M) \times T_q^1(M)$ can be considered as a flat torus $S^1(p) \times S^1(q)$. For a point $(s, t) \in S^1(p) \times S^1(q)$, we use the notation $(s, t) \in G(p, q)$. This

means that there exists a geodesic $\gamma \in G(p, q)$ satisfying $\dot{\gamma}(0) = s$ and $-\dot{\gamma}(d(p, q)) = t$.

We need some lemmas.

LEMMA 2 (Uniqueness). For any $s \in S^1(p)$, there exists at most one $t \in S^1(q)$ such that

$$(s, t) \in G(p, q).$$

This follows from the uniqueness of geodesic for the initial values.

LEMMA 3 (Continuity). If a sequence $\{(s_i, t_i)\}_{i=1,2,\dots} \subset G(p, q)$ converge to $(s_o, t_o) \in S^1(p) \times S^1(q)$, then $(s_o, t_o) \in G(p, q)$.

PROOF. Let $\gamma_i \in G(p, q)$ such that $\dot{\gamma}_i(0) = s_i$ and $-\dot{\gamma}_i(d(p, q)) = t_i$ for $i = 1, 2, 3, \dots$. Then the limit geodesic $\gamma := \lim_{i \rightarrow \infty} \gamma_i$ satisfies $\dot{\gamma}(0) = s_o$, $-\dot{\gamma}(d(p, q)) = t_o$ and $\gamma \in G(p, q)$. q.e.d.

LEMMA 4 (Monotonicity). For (s, t) and $(s', t') \in G(p, q)$, if $s < s'$ then $t < t'$.

PROOF. Let γ and $\sigma \in G(p, q)$ are the geodesics such that $\dot{\gamma}(0) = s$, $-\dot{\gamma}(d(p, q)) = t$ and $\dot{\sigma}(0) = s'$, $-\dot{\sigma}(d(p, q)) = t'$ respectively. Since γ and σ are shortest geodesics, γ and σ do not intersect except at p and q . Thus if D (resp. D') denote the domain bounded by $\gamma^{-1} \circ \gamma_1$ (resp. $\sigma^{-1} \circ \gamma_1$) such that $\star_D(\gamma(0), \gamma_1(0)) = s$ (resp. $\star_{D'}(\sigma(0), \gamma_1(0)) = s'$), then

$$D \subset D'.$$

Thus we know

$$\star_D(-\dot{\gamma}(d(p, q)), -\dot{\gamma}_1(d(p, q))) \leq \star_{D'}(-\dot{\sigma}(d(p, q)), -\dot{\gamma}_1(d(p, q)))$$

i.e. $t < t'$.

q.e.d.

Now putting $S_o^1(p) := \{s \in S^1(p) : (s, t) \in G(p, q) \text{ for some } t \in S^1(q)\} (\subset S^1(p))$, we will prove the following:

THEOREM 2. For type 2 surface M , $S_o^1(p) = S^1(p)$ holds i.e. all geodesics starting from p are shortest connections to q with length $d(M)$ where $p \in M$ and $q \in M$ are the points satisfying $d(p, q) = d(M)$.

PROOF. We will consider $S^1(p) \times S^1(q)$ as is obtained from the Euclidean plane E^2 by mod 2π identification for each variable. All argument in the

following are done under mod 2π identification if we do not state otherwise.

At first we note that for each $(s_o, t_o) \in G(p, q)$ and for any $(s, t) \in [s_o - \pi/2, s_o + \pi/2] \times [t_o - \pi/2, t_o + \pi/2]$, there exists some $(\tilde{s}, \tilde{t}) \in G(p, q)$ such that

$$\star(s, \tilde{s}) \leq \pi/2 \quad \text{and} \quad \star(t, \tilde{t}) \leq \pi/2$$

because we can take such $(\tilde{s}, \tilde{t})$ as $(\tilde{s}, \tilde{t}) = (s_o, t_o)$. Here $\star(s, \tilde{s}) \leq \pi/2$ (also $\star(t, \tilde{t}) \leq \pi/2$) means for the geodesics $\gamma, \tilde{\gamma} \in G(p, q)$ such that $\dot{\gamma}(0) = s, \dot{\tilde{\gamma}}(0) = \tilde{s}$, it holds $\star(\dot{\gamma}(0), \dot{\tilde{\gamma}}(0)) \leq \pi/2$.

Conversely, since $G(p, q)$ satisfies condition [CII], for any $(s, t) \in S^1(p) \times S^1(q)$, there exists $(\tilde{s}, \tilde{t}) \in G(p, q)$ satisfying

$$(s, t) \in [\tilde{s} - \pi/2, \tilde{s} + \pi/2] \times [\tilde{t} - \pi/2, \tilde{t} + \pi/2].$$

Thus to show the assertion, it suffices to show that if $S^1_\circ(p) \subsetneq S^1(p)$, then

$$S^1(p) \times S^1(q) \not\subset \bigcup_{s \in S^1_\circ(p)} [s - \pi/2, s + \pi/2] \times [t - \pi/2, t + \pi/2].$$

Here $t \in S^1(q)$ is the one such that $(s, t) \in G(p, q)$. So assuming $S^1_\circ(p) \subsetneq S^1(p)$ and $S^1(p) \times S^1(q) \subset \bigcup_{s \in S^1_\circ(p)} [s - \pi/2, s + \pi/2] \times [t - \pi/2, t + \pi/2]$, we will derive

a contradiction. Then since $S^1_\circ(p) \subsetneq S^1(p)$, there exist $s_1, s_2 \in S^1_\circ(p)$ such that $s \notin S^1_\circ(p)$ for any $s \in (s_1, s_2)$. This follows from Lemma 2 and Lemma 3. Let $t_1, t_2 \in S^1(q)$ satisfy $(s_1, t_1) \in G(p, q)$ and $(s_2, t_2) \in G(p, q)$. To simplify the argument, for $s \in S^1_\circ(p)$, put $I_s := [s - \pi/2, s + \pi/2] \times [t - \pi/2, t + \pi/2]$, where $t \in S^1(q)$ is the one satisfying $(s, t) \in G(p, q)$. Since the continuous curve c defined on $S^1_\circ(p)$ by $c(s) = (s, t)$, $s \in S^1_\circ(p)$ where $t \in S^1(q)$ is the one satisfying $(s, t) \in G(p, q)$ does not defined on the interval $(s_1, s_2) \subset S^1(p)$, so terminal point $c(s_1)$ of c jumps to terminal point $c(s_2)$ of c as s varies from s_1 to s_2 because of Lemma 3 and Lemma 4. So rectangle I_s jumps also from I_{s_1} to I_{s_2} as s varies from s_1 to s_2 . So as a point characterizing this jump from I_{s_1} to I_{s_2} , we choose the point $(\hat{s}, \hat{t}) := (s_1 + \pi/2, t_2 - \pi/2) \in S^1(p) \times S^1(q)$. Note that $\pi/2 \leq \hat{s} \leq 5\pi/2$ and $-\pi/2 \leq \hat{t} \leq 3\pi/2$.

Furthermore we have

$$\text{LEMMA 5.} \quad \bigcup_{s \in S^1_\circ(p)} I_s \subset [-\pi/2, 5\pi/2] \times [-\pi/2, 5\pi/2].$$

This follows from Lemma 4 and the fact that $(0, 0) = (2\pi, 2\pi) \in G(p, q)$ i.e. $0 = 2\pi \in S^1_\circ(p)$.

- LEMMA 6. (i). $(\hat{s}, \hat{t}) \notin (\pi/2, 5\pi/2) \times (3\pi/2, 5\pi/2)$,
(ii) $(\hat{s}, \hat{t}) \notin (-\pi/2, \pi/2) \times (-\pi/2, 5\pi/2)$.

(i) and (ii) follows from the fact that $\hat{t} \leq 3\pi/2$ and $\hat{s} \geq \pi/2$ respectively.

LEMMA 7. $(\hat{s}, \hat{t}) \notin (\pi/2, 3\pi/2) \times (\pi/2, 3\pi/2)$.

PROOF. If $(\hat{s}, \hat{t}) \in (\pi/2, 3\pi/2) \times (\pi/2, 3\pi/2)$, then for sufficiently small $\varepsilon > 0$, $(\hat{s} + \varepsilon, \hat{t} - \varepsilon) \in (\pi/2, 3\pi/2) \times (\pi/2, 3\pi/2)$ and $(\hat{s} + \varepsilon, \hat{t} - \varepsilon) \notin \bigcup_{s \in S^1_\circ(p)} I_s$ because I_{s_1} jumps to I_{s_2} as s varies from s_1 to s_2 and any point of $(\pi/2, 3\pi/2) \times (\pi/2, 3\pi/2)$ can not be covered twice by the rectangles in $\{I_s\}_{s \in S^1_\circ(p)}$ from Lemma 5. q.e.d.

LEMMA 8. $(\hat{s}, \hat{t}) \notin (3\pi/2, 5\pi/2) \times (-\pi/2, \pi/2)$.

PROOF. Consider another point $(\hat{s}', \hat{t}') := (s_2 - \pi/2, t_1 + \pi/2) \in S^1(p) \times S^1(q)$ characterizing the jump from I_{s_1} to I_{s_2} . If $(\hat{s}, \hat{t}) \in (3\pi/2, 5\pi/2) \times (-\pi/2, \pi/2)$, then we can check easily that $(\hat{s}', \hat{t}') \in (\pi/2, 3\pi/2) \times (\pi/2, 3\pi/2)$. Thus for sufficiently small $\varepsilon > 0$, it holds $(\hat{s}' - \varepsilon, \hat{t}' + \varepsilon) \in (\pi/2, 3\pi/2) \times (\pi/2, 3\pi/2)$ and $(\hat{s}' - \varepsilon, \hat{t}' + \varepsilon) \notin \bigcup_{s \in S^1_\circ(p)} I_s$ because I_{s_1} jumps to I_{s_2} also. Then by the same reason in the proof of Lemma 7, this leads to a contradiction. q.e.d.

LEMMA 9. $(\hat{s}, \hat{t}) \notin (\pi/2, 3\pi/2) \times (-\pi/2, \pi/2)$.

PROOF. If $(\hat{s}, \hat{t}) \in (\pi/2, 3\pi/2) \times (-\pi/2, \pi/2)$, then we can check that another point $(\hat{s}', \hat{t}')$ characterizing the jump from I_{s_1} to I_{s_2} satisfies $(\hat{s}', \hat{t}') \in (-\pi/2, 3\pi/2) \times (\pi/2, 3\pi/2)$. But just as the proof of Lemma 7, we can show that $(\hat{s}', \hat{t}') \notin [\pi/2, 3\pi/2) \times (\pi/2, 3\pi/2)$. Thus it suffices to show that $(\hat{s}', \hat{t}') \notin (-\pi/2, \pi/2) \times (\pi/2, 3\pi/2)$. So assuming that $(\hat{s}', \hat{t}') \in (-\pi/2, \pi/2) \times (\pi/2, 3\pi/2)$, we will derive a contradiction. Then since there is a jump from I_{s_1} to I_{s_2} , for sufficiently small $\varepsilon > 0$,

$$(\hat{s} + \varepsilon, \hat{t} - \varepsilon) \in (\pi/2, 3\pi/2) \times (-\pi/2, \pi/2)$$

$$(\hat{s}' - \varepsilon, \hat{t}' + \varepsilon) \in (-\pi/2, \pi/2) \times (\pi/2, 3\pi/2)$$

and

$$(\hat{s} + \varepsilon, \hat{t} - \varepsilon) \notin \bigcup_{s \leq s_2, s \in S^1_\circ(p)} I_s$$

$$(\hat{s}' - \varepsilon, \hat{t}' + \varepsilon) \notin \bigcup_{s \leq s_2, s \in S^1_\circ(p)} I_s.$$

Furthermore we add another condition on ε such that $\hat{s}' - \hat{s} + \pi > 2\varepsilon$. This is possible because $0 < \hat{s} - \hat{s}' < \pi$. Thus since $S^1(p) \times S^1(q) \subset \bigcup_{s \in S^1_\circ(p)} I_s$, there exists $\bar{s}, \bar{s}' \in S^1_\circ(p)$ satisfying $(\hat{s} + \varepsilon, \hat{t} - \varepsilon + 2\pi) \in I_{\bar{s}}$ and $(\hat{s}' - \varepsilon + 2\pi, \hat{t}' + \varepsilon) \in I_{\bar{s}'}$, because $\pi/2 < \hat{s} + \varepsilon < 3\pi/2$ and $\pi/2 < \hat{t}' + \varepsilon < 3\pi/2$. Let $\bar{t}, \bar{t}' \in S^1(q)$ are the elements satisfying $(\bar{s}, \bar{t}) \in G(p, q)$ and $(\bar{s}', \bar{t}') \in G(p, q)$. Thus

$$|(\hat{s} + \varepsilon) - \bar{s}| \leq \pi/2, |(\hat{t} - \varepsilon) + 2\pi - \bar{t}| \leq \pi/2$$

and

$$|(\hat{s}' - \varepsilon) + 2\pi - \bar{s}'| \leq \pi/2, |(\hat{t}' + \varepsilon) - \bar{t}| \leq \pi/2.$$

From this, we get $s < s'$ and $t' > t$. But this contradicts to Lemma 4.

q.e.d.

Similarly we can check the following:

LEMMA 10. $(\hat{s}, \hat{t}) \notin (3\pi/2, 5\pi/2) \times (\pi/2, 3\pi/2)$.

Finally by the proof of Lemma 9, we can see that $\hat{s} \neq \pi/2, 3\pi/2$ and $\hat{t} \neq \pi/2, 3\pi/2$.

Thus summarizing the above Lemmas, we know

$$(\hat{s}, \hat{t}) \notin S^1(p) \times S^1(q).$$

This is a contradiction and hence we proved $S^1_\circ(p) = S^1(p)$.

Q.E.D.

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