GLOBAL EXISTENCE OF HOLOMORPHIC
SOLUTIONS OF DIFFERENTIAL
EQUATIONS WITH COMPLEX
PARAMETERS -I

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Dedicated to Professor Katsumi Shiratani on his sixtieth birthday
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Introduction

As early as in 1956 L. Ehrenpreis [1] discoursed on an application of the sheaf theory to differential equations and gave a criterion for the existence of global solutions of differential equations \( Tf = g \) in a domain \( D \) when the local existence of local solutions are assured. J. Kajiwara [2] applied Ehrenpreis' method to linear ordinary differential equations with meromorphic coefficients and gave a necessary and sufficient condition for the global existence in the meromorphic category. In the holomorphic category the condition is that \( D \) is either simply connected or doubly connected without non trivial global single-valued holomorphic homogeneous solutions.


Concerning partial differential equation, J. Kajiwara [3]-[6] discussed several concrete cases. In the previous paper K. H. Shon [9] reported promptly the main theorem and showed sheaf-theoretically a route of its proof. The aim of this series is to give complete analytical foundations to it. In the present paper, we give a necessary the sufficient condition for the

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global existence stated in the last paragraph in case of polycylindrical domains.

1. Global existence and local existence

In the present paper, a connected and open set in a topological space is called a domain. Let $m$ be a positive integer and, for each integer $j$ with $1 \leq j \leq m$, $D_j$ be a domain in the complex plane $\mathbb{C}$. We put

$$D = D_1 \times D_2 \times \ldots \times D_m$$  \hspace{1cm} (1.1).

The set $D$ is a domain of holomorphy in the complex $m$-space $\mathbb{C}^m$. Let $M$ be a Stein manifold. We use the manifold $M$ as a parameter space, denote by $r$ a point of $M$ and regard it as a complex parameter. We consider the product manifold $D \times M$. For each $i$ with $1 \leq i \leq m$, let $a^i = (a^i_{jk}(z, r))$ be a square matrix of degree $m$ whose each $(j, k)$ element $a^i_{jk}(z, r)$ is a holomorphic function in $D \times M$. For each $i$ with $1 \leq i \leq m$, let $T_i$ be a differential operator defined by

$$T_i = \frac{\partial}{\partial z_i} + a^i$$  \hspace{1cm} (1.2).

Let $\mathcal{O}_{D \times M}$ be the sheaf of germs of all holomorphic functions on $D \times M$. Then each differential operator $T_i$ defines a sheaf homomorphism

$$T_i : (\mathcal{O}_{D \times M})^m \rightarrow (\mathcal{O}_{D \times M})^m$$  \hspace{1cm} (1.3)

as

$$T_i u = \left( \begin{array}{c}
\frac{\partial u_1}{\partial z_i} + \sum_{k=1}^{m} a^1_{ik}(z, r) u_k \\
\frac{\partial u_2}{\partial z_i} + \sum_{k=1}^{m} a^2_{ik}(z, r) u_k \\
\vdots \\
\frac{\partial u_m}{\partial z_i} + \sum_{k=1}^{m} a^m_{ik}(z, r) u_k
\end{array} \right)$$  \hspace{1cm} (1.4)

for any germ.
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\[
\begin{pmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_m
\end{pmatrix}
\]

of \((\mathcal{O}_{D \times M})^m\). We put

\[S = T_1 T_2 \cdots T_m\]  \hspace{1cm} (1.5).

For the accuracy, we redefine

\[Su = T_1(T_2(\cdots T_m(u)))\]  \hspace{1cm} (1.6)

for any element \(u\) of \((\mathcal{O}_{D \times M})^m\) and remark that \(T_m\) is the first and \(T_1\) is the last operator. Hereafter, we denote ecologically the column vector \(u\) above as a row vector \(u = (u_1, u_2, \ldots, u_m)\) for the sake of conservation of materials.

For any positive integer \(i\) with \(1 \leq i \leq m\), we also define halfway

\[S_i = T_1 T_2 \cdots T_i\]  \hspace{1cm} (1.7).

Let

\[H(D \times M) = H^0(D \times M, (\mathcal{O}_{D \times M})^m)\]  \hspace{1cm} (1.8)

be the set of global sections of \((\mathcal{O}_{D \times M})^m\) over \(D \times M\), etc. \(H^0(D \times M, \mathcal{O}_{D \times M})\) is the \(C\)-module of all holomorphic functions on \(D \times M\). \(H(D \times M)\) is the \(H^0(D \times M, \mathcal{O}_{D \times M})\)-module of all \(m\)-column vector valued holomorphic functions on \(D \times M\).

Let \(\text{Ker} S\) and \(\text{Im} S\) be, respectively, the kernel and image of the homomorphism

\[S: H(D \times M) \rightarrow H(D \times M)\]  \hspace{1cm} (1.9).

For any positive integer \(i\) with \(m \geq i \geq 1\), let \(\text{Ker} S_i\) and \(\text{Im} S_i\) be, respectively, the kernel and image of the homomorphism

\[S_i: H(D \times M) \rightarrow H(D \times M)\]  \hspace{1cm} (1.10).

Their elements are \(m\)-column vector valued global holomorphic functions on \(D \times M\).

We prove the following lemma by induction on \(i(1 \leq i \leq m)\):
PROPOSITION 1. If \( D_1, D_2, \ldots \) and \( D_i \) are simply connected, then for the above \( S_i \), we have

\[
H(D \times M) = S_i(H(D \times M)) \tag{1.11}
\]

Proof. By Riemann's mapping theorem, we may assume that the first domain \( D_1 \) is either the unit disc or the complex plane \( \mathbb{C} \) and that \( D \) contains the origin \( 0 = (0, 0, \ldots, 0) \) without losing the generality. In case \( i = 1 \), \( S_1 \) coincides with \( T_1 \).

Let \( g(z, r) = (g_j(z, r)) \) be any element of \( H(D \times M) \). We separate the first component

\[
z'_1 = z_1 \tag{1.12}
\]

and the others

\[
z''_1 = (z_2, z_3, \ldots, z_m) \tag{1.13}
\]
of \( z = (z_1, z_2, z_3, \ldots, z_m) \). Let \( f(z''_1, r) = f(z_2, z_3, \ldots, z_m, r) \) be any holomorphic functions in

\[
D''_1 \times M = D_2 \times \cdots \times D_m \times M \tag{1.14}
\]

We consider an initial value problem

\[
\begin{align*}
S_1 u &= g \\
u(0, z''_1, r) &= f(z''_1, r)
\end{align*} \tag{1.15}
\]

which is equivalent to a system of integral equations

\[
\begin{align*}
u_j(z'_1, z''_1, r) &= f_j(z''_1, r) \\
&\quad + \int_0^{z'_{1}} (g_j(s_1, z''_1, r) - \sum_{k=1}^{m} a^k_k(s_1, z''_1, r)u_k(s_1, z''_1, r)) \, ds_1 \\
&\quad \quad (1 \leq j \leq m)
\end{align*} \tag{1.16}
\]

where the complex integral is done along any piecewise smooth simple curve in \( D_1 \) joining 0 and \( z'_1 = z_1 \). By a theorem of Cauchy the righthand side of (1.16) is independent of the special choice of a curve and is well-defined.

We solve the above (1.16) by the method of successive approximation:

\[
u_j(1, z'_1, z''_1, r) = f_j(z''_1, r) \tag{1.17},
\]
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\[ u_j(p, z'_1, z''_1, r) = f_j(z''_1, r) + \int_0^{z_1} \left( g_j(s_1, z''_1, r) - \sum_{k=1}^{m} a^j_k(s_1, z''_1, r) u_k(p - 1, s_1, z''_1, r) \right) ds_1 \]

\[ (1 \leq j \leq m, p = 2, 3, 4,...) \quad (1.18) \]

We introduce

\[ w_j(p, z'_1, z''_1, r) = u_j(p + 1, z'_1, z''_1, r) - u_j(p, z'_1, z''_1, r) \]

\[ (1 \leq j \leq m, p = 2, 3, 4,...) \quad (1.19) \]

Then each \( w_j(p, z'_1, z''_1, r) \) satisfies a homogeneous recurrence relation

\[ w_j(p, z'_1, z''_1, r) = -\int_0^{z_1} a^1(s_1, z''_1, r) w_j(p - 1, s_1, z''_1, r) ds_1 \]

\[ (p = 2, 3,...) \quad (1.20) \]

For the holomorphic function \( w(p, z'_1, z''_1, r) = (w_j(p, z'_1, z''_1, r)) \), we define semi-norms

\[ w(p, |z'_1|, z''_1, r) = \max \{|w_j(p, x, z''_1, r)|; |x| \leq |z'_1|, 1 \leq j \leq m\} \]

\[ (p = 1, 2, 3,...) \quad (1.21) \]

We also put

\[ |a^1(z'_1, z''_1, r)| = \max \left\{ \sum_{k=1}^{m} |a^j_k(z'_1, z''_1, r)|; 1 \leq j \leq m \right\} \quad (1.22) \]

and

\[ a^1(|z'_1|, z''_1, r) = \max \{|a^1(x, z''_1, r)|; |x| \leq |z'_1|, 1 \leq j \leq m\} \quad (1.23) \]

By induction with respect to \( p \), we have

\[ w(p, |z'_1|, z''_1, r) \leq w(1, |z'_1|, z''_1, r) \frac{(a^1(|z'_1|, z''_1, r)|z'_1|)^{p-1}}{(p - 1)!} \]

\[ (p = 1, 2, 3,...) \quad (1.24) \]

Hence each sequences \( \{u_j(p, z'_1, z''_1, r); p = 1, 2, 3,...\} \) of holomorphic functions in \( D \times M \) converges uniformly on any compact subset of \( D \times M \) to a holomorphic function \( u_j(z'_1, u''_1, r) \) in \( D \times M \) with majoration
\[ |u_j(z'_1, z''_1, r) - f_j(z''_1, r)| \leq w(1, |z'_1|, z''_1, r) \exp(a^1(|z'_1|, z''_1, r)|z'_1|) \]

\[ (1 \leq j \leq m) \quad (1.25) \]

where

\[ w_j(1, z'_1, z''_1, r) = u_j(2, z'_1, z''_1, r) - u_j(1, z'_1, z''_1, r) \]

\[ = \int_0^{z'_1} (g_j(s_1, z''_1, r) - \sum_{k=1}^{m} a^1_k(s_1, z''_1, r)f_k(z''_1, r))ds_1 \]

\[ (1 \leq j \leq m) \quad (1.26) \]

Let \( q \) be an integer with \( 1 \leq q \leq m \), we separate the \( q \)-th variable \( z_q \) from other \( m-1 \) variables and put

\[ z'_q = z_q, \quad z''_q = (z_1, z_2, \ldots, z_{q-1}, z_{q+1}, \ldots, z_m) \quad (1.27), \]

\[ D''_q = D_1 \times D_2 \times \cdots \times D_{q-1} \times D_{q+1} \times \cdots \times D_m \quad (1.28) \]

and we regard the domain \( D \) as

\[ D = D_q \times D''_q \quad (1.29). \]

We assume that, for an integer \( q \), Proposition 1\(_q\) holds in the following more precise sense: Let \( g(z, r) = (g_j(z, r)) \) be any holomorphic function belonging to \( H(D \times M) \) and, for any \( q \) with \( 1 \leq q < m \), \( f(q; z'_q, z''_q, r) = (f_j(q; z'_q, z''_q, r)) \) be a holomorphic function belonging to \( H(D''_q \times M) \). We put

\[ g(0; z, r) = g(z, r) \quad (1.30) \]

in \( D \times M \).

Our claim is as follows: For any integer \( q \) with \( 1 \leq q < m \), there exists uniquely a holomorphic function \( g(q; z, r) = (g_j(q; z, r)) \) belonging to \( H(D \times M) \) which coincides with the unique solution \( u \) of an initial value problem

\[ S_q u = g(q - 1; z, r) \quad u_j(z'_q, z''_q, r) = f_j(q; z''_q, r) \quad \text{for } z'_q = 0 \]

\[ (1.31) \]

and \( u = g(q; z, r) = (g_j(q; z, r)) \) satisfies

\[ |g_j(q; z'_q, z''_q, r) - f_j(q; z''_q, r)| \leq w(q, 1, |z'_q|, z''_q, r) \exp(a^q(|z'_q|, z''_q, r)|z'_q|) \]

\[ (1 \leq j \leq m) \quad (1.32) \]

under similar notations to (1.21), (1.22) and (1.23), where
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\[ w_j(q; 1, z_q', z_q'', r) = u_j(q; 2, z_q', z_q'', r) - u_j(q; 1, z_q', z_q'', r) \]
\[ = \int_0^{z_q} (g_j(q - 1; s_q, z_q'', r) - \sum_{k=1}^{m} a_{jk}(s_q, z_q'', r)f_k(q; z_q'', r)) \, ds_q \]
\[ (1 \leq j \leq m) \quad (1.33) \]

For an integer \( q + 1 \), as we have done in case \( p = 1 \), we can construct \( g(q + 1; z, r) \), which is a solution to the initial value problem

\[
\begin{align*}
S_{q+1} u &= g(q; z, r) \\
u_j(z_{q+1}', z_{q+1}'', r) &= f_j(q + 1; z_{q+1}'', r) \quad \text{for} \quad z_{q+1}' = 0
\end{align*}
\]

solving equivalent integral equation

\[
\begin{align*}
u_j(z_{q+1}', z_{q+1}'', r) &= f_j(q + 1; z_{q+1}'', r) \\
&+ \int_0^{z_{q+1}'} (g_j(q; s, z_{q+1}'', r) - \sum_{k=1}^{m} a_{jk}^{q+1}(s, z_{q+1}'', r)u_k(s, z_{q+1}'', r)) \, ds_{q+1} \\
&\quad (1 \leq j \leq m) \quad (1.35)
\end{align*}
\]

by the method of successive approximation. Since the difference of two solutions satisfies the inequality corresponding to (1.24), the solution is unique.

q.e.d.

For any \( p \) with \( 1 \leq p \leq m \), we consider a short exact sequence of sheaves

\[ 0 \longrightarrow \text{Ker} S_p \longrightarrow (\mathcal{O}_{D \times M})^m \overset{S_p}{\longrightarrow} \text{Im} S_p \longrightarrow 0 \quad (1.36) \]

where \( \text{Ker} S_p \rightarrow (\mathcal{O}_{D \times M})^m \) is the canonical injection. The above short exact sequence of sheaves induces the following long exact sequence of cohomologies over \( D \times M \):

\[ 0 \longrightarrow H^0(D \times M, \text{Ker} S_p) \longrightarrow H(D \times M) \overset{S_p}{\longrightarrow} H^0(D \times M, \text{Im} S_p) \longrightarrow \]

\[ H^1(D \times M, \text{Ker} S_p) \longrightarrow H^1(D \times M, (\mathcal{O}_{D \times M})^m) \longrightarrow \cdots \quad (1.37) \]

Since \( D \times M \) is a domain of holomorphy from the theorem of Weierstrass, we have

\[ H^1(D \times M, (\mathcal{O}_{D \times M})^m) = 0 \quad (1.38) \]
from the theorem of Oka-Cartan-Serre. From Proposition 1, we have $\text{Im} S_p = (O_{D \times M})^m$. Hence we have the following Proposition:

**Proposition 2.** For the above $D$, $M$ and $S_p$, there holds

$$H^1(D \times M, \text{Ker } S_p) = H(D \times M)/S_p(H(D \times M))$$  \hspace{1cm} (1.39)

i.e.,

$$H^1(D \times M, \text{Ker } S_p) = 0$$  \hspace{1cm} (1.40)

if and only if

$$H(D \times M) = S_p(H(D \times M))$$  \hspace{1cm} (1.41).

Especially, in case that $p = m$, we have the following proposition:

**Proposition 3.** For the domain $D \times M$ and the differential operator $S$ given in (1.1)--(1.7), there holds

$$H^1(D \times M, \text{Ker } S) = H(D \times M)/S(H(D \times M))$$  \hspace{1cm} (1.42)

i.e.,

$$H^1(D \times M, \text{Ker } S) = 0$$  \hspace{1cm} (1.43)

if and only if

$$H(D \times M) = S(H(D \times M))$$  \hspace{1cm} (1.44).

**Proposition 4.** If, for an integer $p$ with $1 \leq p \leq m$, there holds

$$H^1(D \times M, \text{Ker } S_p) = 0$$  \hspace{1cm} (1.45)

then we have

$$H^1(D \times M, \text{Ker } T_1) = 0$$  \hspace{1cm} (1.46)

**Proof.** We prove the proposition by induction with respect to $p$. For any $g(z, r)$ of $H(D \times M)$, by (1.45) and Proposition 2, there exists $f(p; z, r)$ of $H(D \times M)$ such that

$$S_p f(p; z, r) = T_1 T_2 \cdots T_p f(p; z, r) = g(z, r)$$  \hspace{1cm} (1.47).

Then

$$u(z, r) = T_2 T_3 \cdots T_p f(p; z, r)$$  \hspace{1cm} (1.48)
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\begin{equation}
T_i u(z, r) = g(z, r)
\end{equation}

This means

\begin{equation}
H^1(D \times M, \text{Ker } T_i) = H(D \times M)/T_i(H(D \times M))
\end{equation}

By Proposition 3, we have (1.46).

**Proposition 5.** Let \( p \) be a positive integer with \( 1 \leq p \leq m \). For the domain \( D \), Stein manifold \( M \) and operator \( S_p \), if \( D_1, D_2, \ldots \) and \( D_p \) are simply connected, then we have

\begin{equation}
H^1(D \times M, \text{Ker } S_p) = 0
\end{equation}

**Proof.** By Proposition 1, we have

\begin{equation}
H(D \times M) = S_p(H(D \times M))
\end{equation}

and by Proposition 2, we have (1.51).

\section{Analytic prolongation around a closed curve}

Let \( i \) be a positive integer with \( 1 \leq i \leq m \). We separate the \( i \)-th coordinate \( z_i \) of a point \( z = (z_1, z_2, z_3, \ldots, z_m) \) from the others, adopt notations \( z'_i = z_i, z''_i = (z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m) \) and \( D'_i = D_i, D''_i = D_1 \times D_2 \times \cdots \times D_{i-1} \times D_{i+1} \times \cdots \times D_m \) in (1.27)–(1.29) and identify the point \( z \) of \( D \) with \((z'_i, z''_i)\) of \( D'_i \times D''_i \).

Let \( X_i \) be a simply connected subdomain of \( D_i \) which is not necessarily simply connected. Let \( z^0_i \) be any point of \( X_i \) and \( f(z''_i, r) = (f_1, f_2, \ldots, f_m) \) be any element of \( H(D''_i \times M) \). We consider the solution of the initial value problem

\begin{equation}
\begin{aligned}
&T_i u = 0 \\
&u(z^0_i, z''_i, r) = f(z''_i, r)
\end{aligned}
\end{equation}

Replacing \( 1 \) by \( i \) in the proof of Proposition 1, we see that the problem (2.1) has a unique single-valued holomorphic solution in \( X_i \times D''_i \times M \), which is the solution \( u = (u_1, u_2, \ldots, u_m) \) of the integral equation
\[ u_j(z^r_i, z^\alpha_i, r) = f_j(z^\alpha_i, r) \]
\[ - \int_{z^r_i}^{z^\alpha_i} \left( \sum_{k=1}^{m} a^j_{jk}(s_i, z^\alpha_i, r) u_k(s_i, z^\alpha_i, r) \right) ds_i \quad (1 \leq j \leq m) \quad (2.2) \]

and is obtained by the method of successive approximation

\[ u_j(1, z^r_i, z^\alpha_i, r) = f_j(z^\alpha_i, r) \quad (2.3) \]

\[ u_j(p, z^r_i, z^\alpha_i, r) = f_j(z^\alpha_i, r) \]
\[ - \int_{z^r_i}^{z^\alpha_i} \left( \sum_{k=1}^{m} a^j_{jk}(s_i, z^\alpha_i, r) u_k(p-1, s_i, z^\alpha_i, r) \right) ds_i \quad (1 \leq j \leq m, p = 2, 3, 4, \ldots) \quad (2.4) \]

Since our \( u(z, r) \) given by (2.2) or equivalently by (2.3)-(2.4) is linear with respect to the initial value \( f_j(z^\alpha_i, r) \in H(D^\alpha_i \times M) \), we give recurrence formula of the matrix-valued functions

\[ M_j^i(p, z^r_i, z^\alpha_i, r) = (M_j^i(p, z^r_i, z^\alpha_i, r)), \]
\[ u_j(p, z^r_i, z^\alpha_i, r) = \sum_{k=1}^{m} M_j^i(p, z^r_i, z^\alpha_i, r) f_k(z^\alpha_i, r) \quad (2.5) \]

inductively as follows:

By (2.5), we put

\[ M_j^i(1, z^r_i, z^\alpha_i, r) = 1 \quad (2.6) \]

Substituting (2.5) for \( p - 1 \) into (2.4), we have

\[ u_j(p, z^r_i, z^\alpha_i, r) = f_j(z^\alpha_i, r) \]
\[ - \int_{z^r_i}^{z^\alpha_i} \left( \sum_{k=1}^{m} a^j_{jk}(s_i, z^\alpha_i, r) \sum_{k=1}^{m} M_k^i(p-1, s_i, z^\alpha_i, r) f_k(z^\alpha_i, r) \right) ds_i \quad (2.7) \]

Hence we have

\[ M_j^i(p, z^r_i, z^\alpha_i, r) = \delta_{jk} \]
\[ - \int_{z^r_i}^{z^\alpha_i} \left( \sum_{k=1}^{m} a^j_{jk}(s_i, z^\alpha_i, r) M_k^i(p-1, s_i, z^\alpha_i, r) \right) ds_i \quad (2.8) \]

where
\[ \delta_{jk} = 0 \text{ if } j \neq k, \quad \delta_{jj} = 1 \]  \hspace{1cm} (2.9)

We denote the matrix given as the limit of (2.8) by the left hand side of

\[ M^i(z_i; z_i^0; z_i^{''}, r; X_i) = \exp \left( - \int_{z_i^0}^{z_i} a^i(s_i, z_i^{''}, r) d s_i; X_i \right) \]  \hspace{1cm} (2.10),

\[ u(z_i^1, z_i^{''}, r) = M^i(z_i; z_i^0; z_i^{''}, r; X_i) f(z_i^{''}, r) \]  \hspace{1cm} (2.11).

In case that \( m = 1 \), it is represented by the right hand side of (2.10), which we use when we want to recognize the dependence on the coefficient \( a^i(z_i^1, z_i^{''}, r) \) even in case \( m > 1 \). As we know from above, for any \((z_i^{''}, r)\) fixed in \( D_i^{''} \times M \), each element of the \( m \times m \) matrix \( M^i(z_i; z_i^0; z_i^{''}, r; X_i) \) of (2.10) is a single-valued holomorphic function in \( X_i \) which is omitted in (2.10) or (2.11) when there occurs no confusion. Often, we omit the parameter \((z_i^{''}, r)\) belonging to \( D_i^{''} \times M \) in the matrices \( M \) and \( L \) below too.

Then for any point \( z_i \) fixed in \( X_i \), \( M^i(z_i; z_i^0; z_i^{''}, r; X_i) \) is a linear mapping of the linear space \( H(D_i^{''} \times M) \) regarded as the linear space of initial values into the same \( H(D_i^{''} \times M) \) regarded as the linear space of values \( u(z_i^1, z_i^{''}, r) \) at \( z_i \) of solutions to the homogeneous equation \( T_i u = 0 \). By the uniqueness of the initial value problems, the mapping (2.10) is bijective and the matrix (2.10) is regular.

Let \( z_i^1 \) be any point of the simply connected subdomain \( X_i \). For \( u \) defined by (2.1), \( u(z_i^1, z_i^{''}, r) \) is its value at the point \( z_i^1 \). So, by definition (2.11) the right hand side of

\[ u(z, r) = M^i(z_i; z_i^1; X_i) u(z_i^1, z_i^{''}, r) \]  \hspace{1cm} (2.12)

is the value at a point \( z_i \) of the solution to the initial value problem

\[ \begin{align*}
T_i u &= 0 \\
u &= u(z_i^1, z_i^{''}, r) \quad \text{for} \quad z_i = z_i^1
\end{align*} \]  \hspace{1cm} (2.13).

By the uniqueness of solutions to initial value problems, it coincides with the original \( u \) by the meaning of (2.10) and (2.11), for the pair of initial point \( z_i = z_i^1 \) and initial value \( u(z_i^1, z_i^{''}, r) \) instead of \( z_i^0 \) and \( f(z_i^{''}, r) \). Substituting (2.12) into (2.11), we have the matrix relation

\[ M^i(z_i; z_i^0; X_i) = M^i(z_i; z_i^1; X_i) M^i(z_i^1, z_i^0; X_i) \]  \hspace{1cm} (2.14).

So we can regard \( M^i \) as non commutative exponentials.
Now, we consider the case that the $i$-th domain $D_i$ is not simply connected. We take two simply connected domains $D_i \text{d}$ and $D_i \text{g}$ which cover $D_i$ and denote their intersection by $E_i$. The character $\text{d}$ stands for right, i.e., droit in French. Because English $r$ is already taken for a point $r$ of the Stein manifold $M$, we use French $\text{d}$. Thus, the characters $\text{g}, \text{b}, \text{h}$ in the style of Plain Text stand for left = gauch, bottom = bas, top = haut. We take a starting point $z_i^b$ in the simply connected domain $D_i \text{d}$. Let

$$K_i = \{z_i = k_i(t); \ 0 \leq t \leq 1\}$$

be a closed Jordan curve in $D_i$ with counter-clock wise orientation and with $z_i^b$ as the starting and end point. We assume that the part

$$K_i \text{d} = \{z_i = k_i(t); \ 0 \leq t \leq 1/2\}$$

is contained in $D_i \text{d}$, but the whole $K_i$ is not contained in $D_i \text{d}$, the part

$$K_i \text{g} = \{z_i = k_i(t); \ 1/2 \leq t \leq 1\}$$

is contained in $D_i \text{g}$, the relatively compact connected component of $D_i - K_i$ is not contained in $D_i$ and $K_i$ is one element of a homology base of $D_i$.

Let $f(z_i^r, r)$ be any element of $H(D_i^r \times M)$. For any $(z_i^r, r) \in D_i^r \times M$, we consider an initial value problem

$$T_i u = 0$$

$$u(z_i, z_i^r, r) = f(z_i^r, r) \quad \text{for} \quad z_i = z_i^b$$

(2.18)

in $D_i \text{d}$ and continue it analytically along the closed curve $K_i$. After analytic prolongation of the solution $u = (u_1, u_2, \ldots, u_m)$ to the initial problem (2.18) along the first half $K_i \text{d}$ of the closed curve $K_i$, we arrive at a relay point

$$z_i^b = k_i(1/2) \in E_i$$

(2.19).

By the definition (2.11), its value at $z_i^b$ is represented by

$$u(z_i^b, z_i^r, r) = M'(z_i^b; z_i^r; D_i \text{d}) f(z_i^r, r)$$

(2.20).

We consider another initial value problem

$$T_i u = 0$$

$$u(z_i^b, z_i^r, r) = M'(z_i^b; z_i^r; D_i \text{d}) f(z_i^r, r)$$

(2.21)

in the simply connected domain $D_i \text{g}$. The continuation of (2.21) along $K_i \text{g}$
is regarded as a succession of the continuation of (2.18) along \( K_i \). By the uniqueness of initial value problems and (2.11), the value of \( u \) given by (2.21) is the right hand-side of

\[
\text{Monod}(K_i; \ z_i^b; z_i^r, r) f(z_i^r, r) = M^i(z_i^b; z_i^b; D_t, g) M^i(z_i^b; z_i^b; D_t, d) f(z_i^r, r)
\]

and is denoted by the left hand-side of (2.22). Then the matrix

\[
\text{Monod}(K_i; z_i^b; z_i^r, r) = M^i(z_i^b; z_i^b; D_t, g) M^i(z_i^b; z_i^b; D_t, d)
\]

denote the result of analytic prolongation round the closed curve \( K_i \) in \( D_t \). Since the two matrices in the right hand-side of (2.23) are regular as we show it above, the matrix \( \text{Monod}(K_i; z_i^b; z_i^r, r) \) is also regular for any \((z_i^r, r)\) of \( D_t \times M \).

We investigate \( \text{Monod}(K_i; z_i^b; z_i^r, r) \) directly. We return to the initial value problem (2.21). We reconsider its value at the initial point \( z_i^b \) and the terminal point \( z_i^b \). The solution is obtained as the limit of the method of successive approximation as in (2.4), \( z_i^b \) being replaced by \( z_i^b \). In the \( p \)-th step, the result after the continuation round the closed curve is represented by the contour integral round \( K_i \)

\[
u_j(p, z_i^b, z_i^r, r) = f_j(z_i^r, r) - \int_{K_i} \left( \sum_{k=1}^m a_{jk}(s_i, z_i^r, r) u_k(p - 1, s_i, z_i^r, r) \right) ds_i
\]

Its limit is the linear transformation, \( \text{Monod}(K_i; z_i^b; z_i^r, r) f(z_i^r, r) \), of the linear space \( H(D_t \times M) \) into itself. We denote it by the right hand-side of

\[
\text{Monod}(K_i; z_i^b; z_i^r, r) f(z_i^r, r) = \exp \left( - \int_{K_i} a(s_i', z_i^r, r) ds_i \right) f(z_i^r, r)
\]

as result of the linear transformation \( \text{Monod}(K_i; z_i^b; z_i^r, r) \). In case that \( m = 1 \), it is represented by the right hand-side of (2.25), which we use when we want to remark its dependence on the coefficients. This \( \text{Monod}(K_i; z_i^b; z_i^r, r) \) is regarded as a \( m \times m \) matrix whose elements belong to \( H(D_t \times M) \) and invertible. We put

\[
L^i(K_i; z_i^b; z_i^r, r) = \text{Monod}(K_i; z_i^b; z_i^r, r) - 1
\]
where \(1\) denotes the \(m \times m\) unit matrix. Then the \(m \times m\) matrix (2.26) explains the fluctuation of values of the initial value problem (2.18) after the analytic prolongation along the closed Jordan curve \(K_i\).

3. Connectivity of \(D_i\) under the assumption \(H^1(D \times M, \text{Ker} T_i) = 0\)

**Proposition 6.** Under the notation of the preceding paragraph, if there holds

\[
H(D \times M) = T_i(iH(D \times M))
\]

that is,

\[
H^1(D \times M, \text{Ker} T_i) = 0
\]

then either \(D_i\) is a simply connected domain or \(D_i\) is a doubly connected domain with regular \(L^1(K_i; z_i^b; z_i^\prime, r)\) for a homology base \(K_i\).

**Proof.** Assume that \(D_i\) were neither simply nor doubly connected. Then there exist two closed Jordan curves \(Kb\) and \(Kh\):

\[
Kq = \{z_i = kq(t); 0 \leq t \leq 1\} \quad (q = b, h)
\]

and simply connected subdomains \(D_i d\) and \(D_i g\) of \(D_i\) satisfying the following conditions:

(a) \(\{D_i d, D_i g\}\) is an open covering of \(D_i\).

(b) The orientations of \(Kb\) and \(Kh\) with increasing \(t\) are counter-clock wise. \(Kb\) and \(Kh\) belong to a homology base of \(D_i\). \(kh(1/2) = kb(1/2)\). \(Khg = \{kh(t); 0 \leq t \leq 1/2\}\) and \(Kbd = \{kb(t); 0 \leq t \leq 1/2\}\) are contained, respectively, in \(D_i g\) and \(D_i d\). \(Khd = \{kh(t); 1/2 \leq t \leq 1\}\) and \(Kbg = \{kb(t); 1/2 \leq t \leq 1\}\) are contained, respectively, in \(D_i d\) and \(D_i g\).

(c) The connected components of the intersection \(E_i\) of \(D_i d\) and \(D_i g\) consist of simply connected components of an open subset \(E_i 0\), simply connected subdomains \(E_i h\) and \(E_i b\). The open set \(E_i 0\) contains \(Kh(0) = kh(1)\), the simply connected domain \(E_i h\) contains

\[
z_i^b = kh(1/2) = kb(1/2)
\]

and the simply connected domain \(E_i b\) contains

\[
z_i^b = kb(0) = kb(1).
\]
We put

\[ Dd = D'_i d \times D''_i, \]
\[ Dg = D'_i g \times D''_i \]  \hspace{1cm} (3.6)

and

\[ E = E_i \times D''_i, \]
\[ Eh = E_i h \times D''_i, \]
\[ Eb = E_i b \times D''_i, \]
\[ E0 = E_i 0 \times D''_i \]  \hspace{1cm} (3.7)

Then \( U = \{Dd \times M, Dg \times M\} \) is an open covering of \( D \times M \). By (3.2), we have

\[ H^1(U, \mathrm{Ker} T_i) = 0 \]  \hspace{1cm} (3.8)

For any element \( h \) of \( H^0(Eb \times M, \mathrm{Ker} T_i) \), we attach the element \( D_i \) of \( H^0(Eb \times M, \mathrm{Ker} T_i) \) to \( Eb \times M \), the element 0 of \( H^0(Eh \times M, \mathrm{Ker} T_i) \) to \( Eh \times M \), the element 0 of \( H^0(E0 \times M, \mathrm{Ker} T_i) \) to \( E0 \times M \) and regard this as a cocycle belonging to \( Z^1(U, \mathrm{Ker} T_i) \). By (3.8), this is a coboundary, i.e., there exist \( hd \) of \( H^0(Ed \times M, \mathrm{Ker} T_i) \) and \( hg \) of \( H^0(Eg \times M, \mathrm{Ker} T_i) \) such that

\[ hg - hd = h \]  \hspace{1cm} (3.9)

on \( Eb \times M \) and

\[ hg = hd \]  \hspace{1cm} (3.10)

on \( Eh \times M \) and \( E0 \times M \). For any fixed point \((z''_i, r)\) of \( D''_i \times M \), starting from the point \( z''_i \) in \( E_i b \), \( hd \) can be analytically continuous along the closed curve \( Kh \) as we have discussed in Paragraph 2. Moreover, (3.10) claims that \( hg \) is a successor at a relay point \( z''_i \) to the result of this analytic continuation and (3.9) claims that the difference coincides with the just given \( h \).

Now, we take any element \( g(z''_i, r) \) and \( f(z''_i, r) \) of \( H(D''_i \times M) \) adopt the solution in \( Dd \times M \) of the initial value problem (2.18) as \( h \) and adopt the solution in \( Dd \times M \) of the problem of the same differential equation \( T_i u = 0 \) with the initial data \( f(z''_i, r) \) as \( hd \). Then, substituting \( z_i = z''_i \) in (3.9) and taking into account of the definition (2.26), we have
\[ L^i(K_i b; z_i^0; z_i^r, r) f(z_i^r, r) = g(z_i^r, r) \]  
(3.11).

Since \( g(z_i^r, r) \) is arbitrary, the linear transformation \( L^i(K_i b; z_i^0; z_i^r, r) \) is surjective. Since we are now discussing the finite dimensional matrix \( L^i(K_i b; z_i^0; z_i^r, r) \), it is invertible, i.e., regular matrix.

Thus, if we are given the set \( H(D_i^r \times M) \) of all initial datum \( g(z_i^r, r) \)'s at the starting point \( z_i^0 \), then the set of all \( f(z_i^r, r) \)'s determined by (3.11) coincides with the full set \( H(D_i^r \times M) \). So we may assume that the initial data \( f(z_i^r, r) \) at the starting point \( z_i^0 \) is arbitrary. Then the continuation of \( hd \) along \( Kbd \) supplies us the set of \( M^i(z_i^0; z_i^0; D_i d) f(z_i^r, r) \). Since \( M^i(z_i^0; z_i^0; D_i d) \) is surjective, this set coincides with the full set \( H(D_i^r \times M) \) too. So we may assume any element \( hd \) is given at the point \( z_i^0 \).

Now, we see the other closed curve \( K_h \). Since \( h g = h d \) holds on \( E h \times M \) and \( E 0 \times M \), \( h g \) is a direct continuation of \( h d \) to \( D g \times M \). This means that any solution \( u \) of the homogeneous equation \( T_i u = 0 \) is single valued along the closed curve \( K_h \).

Because the two closed curves \( K_b \) and \( K_h \) in \( D_i \) are logically equal, this is a contradiction.

Thus we have proved that \( D_i \) is either simply connected or doubly connected with \( H^0(D \times M, \text{Ker} T_i) = 0 \). q.e.d.

### 4. Inhomogeneous solutions

Let \( g(z, r) \) be any element of \( H(D \times M) \). We discuss the zero initial value problem of the inhomogeneous equation

\[
\begin{cases}
T_i u = g \\
u(x_i^0, z_i^r, r) = 0
\end{cases}
\]  
(4.1)

and the method of successive approximation

\[
u_j(p, z_i^r, z_i^r, r) = \int_{z_i^0}^{z_i^r} (g_j(s_i, z_i^r, r) \\
- \sum_{k=1}^{\infty} a_{jk}(s_i, z_i^r, r) u_k(p - 1, s_i, z_i^r)) d s_i \quad (1 \leq j \leq m).
\]  
(4.2)

We give recurrence formula of the matrix-valued kernel functions \( N^i(p, z_i^r, z_i^r, r) = (N_{jk}^i(p, z_i^r, z_i^r, r)) \),
Global existence of holomorphic solutions

\[ u_j(p, z_i^1, z_i^r, r) = \int_{z_i^0}^{z_i} \left( \sum_{k=1}^{m} N_{jk}^i(p - 1, s_i, z_i^r, r) g_k(s_i, z_i^r, r) \right) ds_i \]

(4.3)

inductively as follows:

By the zero initial data, we put

\[ N_{jk}^i(1, s_i; z_i^1, z_i^r, r) = 0 \]  

(4.4).

Substituting (4.3) for \( p - 1 \) into (4.2), we have

\[ u_j(p, z_i^1, z_i^r, r) = \int_{z_i^0}^{z_i} (g_j(s_i, z_i^r, r) - \sum_{k=1}^{m} a_{jk}^i(s_i, z_i^r, r)) \]

\[ \times \int_{z_i^0}^{z_i} \left( \sum_{k=1}^{m} N_{nk}^i(p - 1, t_i, z_i^r, r) g_k(t_i, z_i^r, r) dt_i \right) ds_i \]

(4.5)

Hence we have the recurrence formula:

\[ N_{jk}^i(p, t_i, z_i^r, r) = \delta_{jk} \]

\[ - \int_{z_i^0}^{z_i} \left( \sum_{n=1}^{m} a_{jn}^i(s_i, z_i^r, r) N_{nk}^i(p - 1, s_i, z_i^r, r) \right) ds_i \]

(4.6).

We denote the matrix given as the limit of (4.6) by the first term of the right hand-side of the following (4.7) and represent the solution \( u \) to (4.1) with this kernel

\[ u(z_i^1, z_i^r, r) = \int_{z_i^0}^{z_i} (N^i(s_i, z_i^1, r; X_i)g(s_i, z_i^r, r)) ds_i \]

(4.7).

Thus we have the following proposition by the unicity of solutions of initial value problems:

**Proposition 7.** Let \( X_i \) be a simply connected subdomain of \( D_i \), \( f \) be an element of \( H(D_i^+ \times M) \) and \( g \) be an element of \( H(D_i \times M) \). Then the function

\[ u(z_i^1, z_i^r, r) = M^i(z_i^1, z_i^r; z_i^1, r; X_i)f(z_i^r, r) \]

\[ + \int_{z_i^0}^{z_i} (N^i(s_i, z_i^r, r; X_i)g(s_i, z_i^r, r)) ds_i \]

(4.8)

is the single-valued solution in \( X_i \times D_i^+ \times M \) to the initial value problem.
\[ T_i u = g \]
\[ u(z_i^1, z_i^0, r) = f(z_i^0, r) \quad \text{for } z_i = z_i^0 \]

(4.9)

5. Necessary and sufficient condition for \( H^1(D \times M, \text{Ker } T_i) = 0 \)

At first we consider the case that the target \( i \)-th domain \( D_i \) is doubly connected and let \( z_i^b \) be a point in \( D_i \) and

\[ K = \{ z_i = k(t); 0 \leq t \leq 1 \} \]

(5.1)

be a closed Jordan curve in \( D_i \) with the starting and end point \( z_i^b \) which is a homology base of \( D_i \). As in the proof of Proposition 6, we associate to the above closed curve \( K \) in \( D_i \) two subdomains \( D_i d \) and \( D_i g \) of the domain \( D_i \) satisfying the following conditions:

(a) \( \{ D_i d, D_i g \} \) is an open covering of \( D_i \).

(b) The orientation of \( K \) with increasing \( t \) is counterclockwise. \( K \) is a homology base of \( D_i \).

(c) The connected components of the intersection of \( D_i d \) and \( D_i g \) consist of two simply connected subdomains \( E_i b \) and \( E_i h \). \( E_i b \) and \( E_i h \) contain, respectively,

\[ k(0) = z_i^b \]

(5.2)

and

\[ z_i^b = k(1/2) \]

(5.3).

We put

\[ D d = D_i d \times D_i^0, \]

\[ D g = D_i g \times D_i^0 \]

(5.4)

and

\[ E = E_i \times D_i^0, \]

\[ E b = E_i b \times D_i^0, \]

\[ E h = E_i h \times D_i^0 \]

(5.5)

Then \( U = \{ D d \times M, D g \times M \} \) is an open covering of \( D \times M \).

Since \( D \times M \) is the union of \( D d \times M \) and \( D g \times M \) and since \( E \times M \) is
the intersection of $Dd \times M$ and $Dg \times M$, we can adopt the exact Mayer-Vietoris cohomology sequence

\[ H^0(D \times M, \text{Ker } T_i) \longrightarrow H^0(Dd \times M, \text{Ker } T_i) \oplus H^0(Dg \times M, \text{Ker } T_i) \longrightarrow \]
\[ H^0(E \times M, \text{Ker } T_i) \longrightarrow H^1(D \times M, \text{Ker } T_i) \longrightarrow \]
\[ H^1(Dd \times M, \text{Ker } T_i) \oplus H^1(Dg \times M, \text{Ker } T_i) \longrightarrow \] (5.6)

where the homomorphism

\[ \Xi: H^0(Dd \times M, \text{Ker } T_i) \oplus H^0(Dg \times M, \text{Ker } T_i) \longrightarrow H^0(E \times M, \text{Ker } T_i) \] (5.7)

is the canonical subtraction

\[ \Xi[h_d, h_g] = h_d - h_g, \]
\[ [h_d, h_g] \in H^0(Dd \times M, \text{Ker } T_i) \oplus H^0(Dg \times M, \text{Ker } T_i) \] (5.8).

**Proposition 8.** Assume that the $i$-th domain $D_i$ is doubly connected. Then the following conditions are equivalent:

(a) \[ H^0(D \times M, (O_{D \times M})^\times) = T_i(H^0(D \times M, (O_{D \times M})^\times)) \] (5.9).

(b) \[ H^1(D \times M, \text{Ker } T_i) = 0 \] (5.10).

(c) The homomorphism $\Xi$ is surjective

(d) The matrix $L'(K_i; z^r_i; z^r_i, r)$ defined by (2.26) is invertible at each point $(z, r)$ of $D \times M$

(e) \[ H^0(D \times M, \text{Ker } T_i) = 0 \] (5.12).

**Proof.** By Proposition 2, (a) and (b) are equivalent. Since $D_i d$ and $D_i g$ are simply connected,

\[ H^1(Dd \times M, \text{Ker } T_i) \oplus H^1(Dg \times M, \text{Ker } T_i) = 0 \] (5.14)

by Proposition 1, and Proposition 2. By the exactness of (5.6), we have

\[ H^1(D \times M, \text{Ker } T_i) = H^0(E \times M, \text{Ker } T_i)/\text{Im } \Xi \] (5.15).

Hence (b) and (c) are equivalent.

We have already shown in the proof of Proposition 6 that (c) implies (d).

We show that (d) implies (c). Assume that $L'(K_i; z^r_i; z^r_i, r)$ is a regular matrix at any point $(z^r_i, r)$ of $D^r_i \times M$. We take any element $u = [ub, uh] \in H^0(E \times M, \text{Ker } T_i)$ and seek $[ud, ug] \in H^0(Dd \times M, \text{Ker } T_i) \oplus H^0(Dg \times M, \text{Ker } T_i)$ with $\Xi[ud, ug] = u$ on $E \times M$ what is equivalent to the following
simultaneous equations

\[ ug - ud = ub \]  \hspace{1cm} (5.16)

in \( Eb \times M \) and

\[ ug - ud = uh \]  \hspace{1cm} (5.17)

in \( Eh \times M \) because \( E \) is divided into \( Eb \) and \( Eh \). We put

\[ g(z_i^{\prime\prime}, r) = (M^i(z_i^h; z_i^b; D_i d))^{-1} uh(z_i^h, z_i^{\prime\prime}, r) \]  \hspace{1cm} (5.18)

and

\[ f(z_i^{\prime\prime}, r) = (L^i(K_i; z_i^h; z_i^{\prime\prime}, r))^{-1}(ub(z_i^h; z_i^{\prime\prime}, r) - g(z_i^{\prime\prime}, r)) \]  \hspace{1cm} (5.19)

in \( D_i^{\prime\prime} \times M \). We put

\[ ud(z_i, z_i^{\prime\prime}, r) = M^i(z_i^h; z_i^b; D_i d)(f - g)(z_i^{\prime\prime}, r) \]  \hspace{1cm} (5.20)

in \( Dd \times M \) and

\[ ug(z_i, z_i^{\prime\prime}, r) = M^i(z_i^h; z_i^b; D_i g) \text{ Monod}(K_i; z_i^h; z_i^{\prime\prime}, r) f(z_i^{\prime\prime}, r) \]  \hspace{1cm} (5.21)

in \( Dg \times M \). We have

\[ ug - ud = M^i(z_i^h; z_i^b; D_i g)(L^i(K_i; z_i^h; z_i^{\prime\prime}, r) f(z_i^{\prime\prime}, r) + g) \]  \hspace{1cm} (5.22)

in \( Eb \times M \). By (5.21) and the relations (2.23) and (2.14) we have

\[ ug(z_i, z_i^{\prime\prime}, r) \]

\[ = M^i(z_i^h; z_i^b; D_i g) M^i(z_i^h; z_i^b; D_i d)f(z_i^{\prime\prime}, r) \]

\[ = M^i(z_i^h; z_i^b; D_i d) M^i(z_i^h; z_i^b; D_i d)f(z_i^{\prime\prime}, r) \]

\[ = M^i(z_i^h; z_i^b; D_i d) M^i(z_i^h; z_i^b; D_i d)f(z_i^{\prime\prime}, r) \]  \hspace{1cm} (5.23)

in \( Eh \times M \). By (2.14), we have

\[ ud(z_i, z_i^{\prime\prime}, r) = M^i(z_i^h; z_i^b; D_i d) M^i(z_i^h; z_i^b; D_i d)(f - g)(z_i^{\prime\prime}, r) \]  \hspace{1cm} (5.24)

in \( Eh \times M \). Hence we have

\[ ug - ud = M^i(z_i^h; z_i^b; D_i d) M^i(z_i^h; z_i^b; D_i d) g(z_i^{\prime\prime}, r) \]  \hspace{1cm} (5.25)

in \( Eh \times M \). By (5.25), one of the surjectivity condition (5.17) is equivalent to the equality
\[ M^i(z_i^1; z_i^b_1; D_i^d) M^i(z_i^b_1; z_i^b_1; D_i^d) g(z_i^b_1, r) = u h(z_i, z_i^b_1, r) \]  
(5.26)

in \( Eh \times M \). By the uniqueness of the initial value problem, it suffices to get (5.26) at \( z_i = z_i^b_1 \). By (5.22), another surjectivity condition (5.16) is equivalent to the equality

\[ M^i(z_i^1; z_i^b_1; D_i^g)(L^i(K_i; z_i^b_1; z_i^b_1, r)f(z_i^b_1, r) + g) = u b(z_i, z_i^b_1, r) \]  
(5.27)
in \( Eb \times M \). This is the reason why we put (5.19). Since we have

\[ M^i(z_i^1; z_i^b_1; D_i^d) = M^i(z_i^1; z_i^b_1; D_i^g) \]  
(5.28)
in the simply connected subdomain \( E_i h \) contained in both \( D_i^d \) and \( D_i^g \), we have

\[ u g - u d = M^i(z_i^1; z_i^b_1; D_i^d) g(z_i^b_1, r) \]  
(5.29)
in \( Eh \times M \). (5.26) and (5.29) imply (5.17).

Equivalence of (d) and (e). Let \( f(z_i^b_1, r) \) be any initial value at \( z_i = z_i^b_1 \).

The necessary and sufficient condition that the solution is single-valued along \( K_i \) is that

\[ L^i(K_i; z_i^b_1; z_i^b_1, r)f(z_i^b_1, r) = 0 \]  
(5.30)

Hence, the condition (e) is equivalent to the injectivity of \( L^i(K_i; z_i^b_1; z_i^b_1, r) \).

Since the dimension \( m \) is finite, this is equivalent to (d).

**Theorem 1.** The necessary and sufficient condition that \( H(D \times M) = T_i(H(D \times M)) \) is that either \( D_i \) is simply connected or \( D_i \) is doubly connected and \( H^0(D \times M, \text{Ker } T_i) = 0 \).

**Proof.** By Propositions 1, 6 and 8 lead Theorem 1.

In this occasion, we give an integral representation of the above unique solution in the following proposition:

**Proposition 9.** Assume that the \( i \)-th domain \( D_i \) is double connected and there holds the condition (5.12). Let \( g(z_i, z_i^b_1, r) \) be any element of \( H(D \times M) \). Then the function \( T_i^{-1}g \) defined by

\[ T_i^{-1}g(z_i, z_i^b_1, r) \]
\[ = \int_{z_i^b_1}^{z_i} (N^i(s_i, z_i^b_1, r; D_i^d) g(s_i, z_i^b_1, r)) \, ds_i \]
is the unique global solution to $T_i u = g$ on $D \times M$.

Proof. Let $u$ be an element of $H(D \times M)$ with $T_i u = g$. At first we adopt an inhomogeneous solution

$$v(z_i, z_i^\ast, r) = \int_{z_i^b}^{z_i^a} (N^i(s_i, z_i^\ast, r; D_i d)g(s_i, z_i^\ast, r)) ds_i$$

(5.32)

with zero initial data (4.7). Then $h = u - v$ is a solution to the homogeneous problem

$$T_i h = 0$$

$$h(z_i, z_i^\ast, r) = h(z_i^b, z_i^\ast, r) \text{ for } z_i = z_i^b$$

(5.33).

Starting from the initial point $z_i^b$, we continue $u$ round the closed curve $K_i$ and return to $z_i^b$ which is the terminal point. In this moment, we denote the result at $z_i^b$ of the analytic prolongation of (5.32) round $K_i$ by

$$\int_{K_i} (N^i(s_i, z_i^\ast, r; D_i d)g(s_i, z_i^\ast, r)) ds_i$$

(5.34).

Then the result at $z_i^b$ of analytic prolongation of $u = v + h$ round $K_i$ is denoted by

$$\text{Monod } (K_i; z_i^b, z_i^\ast, r) h(z_i^b, z_i^\ast, r) + (5.34)$$

(5.35).

Hence, the necessary and sufficient condition that $u$ is single value round $K_i$ is

$$L^i(K_i; z_i^b, z_i^\ast, r) h(z_i^b, z_i^\ast, r) = -(5.34)$$

(5.36)

which has a unique solution by (5.12) and which gives the second term of the right hand-side of (5.31).

6. General solution $u$ of $S_2 u = 0$

In this paragraph, we separate two variables $z_1$ and $z_2$ from the other $m - 2$ variables and put
\[ z_3^{(3)} = (z_3, z_4, \ldots, z_m) \] (6.1)

We also assume that the first domain \( D_1 \) is simply connected but the second domain \( D_2 \) is not necessarily simply connected. So, we consider any simply connected subdomain \( X_2 \) of \( D_2 \). We put
\[ D_3^{(3)} = D_3 \times D_4 \times \cdots \times D_m \] (6.2).

Let \( f^{(1)} = f^{(1)}(z_2, z_3^{(3)}, r) = (f_1^{(1)}(z_2, z_3^{(3)}, r), f_2^{(1)}(z_2, z_3^{(3)}, r), \ldots, f_m^{(1)}(z_2, z_3^{(3)}, r)) \) be any element of \( H(D_2 \times D_3^{(3)} \times M) \) and \( f^{(0)} = f^{(0)}(z_1, z_3^{(3)}, r) = (f_1^{(0)}(z_1, z_3^{(3)}, r), f_2^{(0)}(z_1, z_3^{(3)}, r), \ldots, f_m^{(0)}(z_1, z_3^{(3)}, r)) \) be any element of \( H(D_1 \times D_3^{(3)} \times M) \). Let \( z_1^0 \) and \( z_2^0 \) be, respectively, points of \( D_1 \) and \( X_2 \), and \( u \) be an element of \( H(D_1 \times X_2 \times D_3^{(3)} \times M) \) a solution to the problem
\[
S_2 u = T_1(T_2 u) = 0 \\
u(z_1, z_2^0, z_3^{(3)}, r) = f^{(0)}(z_1, z_3^{(3)}, r) \\
T_2 u(z_1^0, z_2, z_3^{(3)}, r) = f^{(1)}(z_2, z_3^{(3)}, r) 
\] (6.3).

Since \( v = T_2 u \) satisfies the initial value problem
\[
\begin{align*}
T_1 v &= 0 \\
v(z_1^0, z_2, z_3^{(3)}, r) &= f^{(1)}(z_2, z_3^{(3)}, r)
\end{align*}
\] (6.4),

by (2.11) \( u \) is a solution to the initial value problem
\[
\begin{align*}
T_2 u &= M^1(z_1; z_2^0; D_1) f^{(1)}(z_2, z_3^{(3)}, r) \\
u(z_1, z_2^0, z_3^{(3)}, r) &= f^{(0)}(z_1, z_3^{(3)}, r)
\end{align*}
\] (6.5)

and, from (4.8), is represented by
\[
u(z_1, z_2, z_3^{(3)}, r) = M^2(z, r; X_2) f^{(0)}(z_1, z_3^{(3)}, r) \\
+ \int_{z_2^0}^{z_2} (N^2(z_1, s_2, z_3^{(3)}, r; X_2) M^1(z_1; z_2^0; D_1) f^{(1)}(z_2, z_3^{(3)}, r)) ds_2
\] (6.6).

7. Connectivity of \( D_2 \) under the assumption \( H^1(D \times M, \text{Ker} S) = 0 \)

**Proposition 10.** If \( D_1 \) is doubly connected and if we have
\[
H(D \times M) = S(H(D \times M))
\] (7.1),

then we have \( H(D \times M) = T_2(H(D \times M)) \).
Proof. By Proposition 4, we have $H^1(D \times M, \text{Ker } T_1) = 0$. By Proposition 2, we have $H(D \times M) = T_1(H(D \times M))$. By Proposition 8 we have $H^0(D \times M, \text{Ker } T_1) = 0$. Let $g$ be any element of $H(D \times M)$. By (7.1), there exists an element $u$ of $H(D \times M)$ such that

$$Su = T_1 T_2 \cdots T_n u = T_1 g$$  \hspace{1cm} (7.2).

We put

$$v = T_2 T_3 \cdots T_n u$$  \hspace{1cm} (7.3).

Since $v - g$ belongs to $H^0(D \times M, \text{Ker } T_1) = 0$ and since $D_1$ is doubly connected, we have $v = g$. Since $g$ is arbitrary, we have

$$H(D \times M) = T_2 \cdots T_n (H(D \times M))$$  \hspace{1cm} (7.4).

By the same argument as in the proof of Proposition 4, we have

$$H(D \times M) = T_2 (H(D \times M))$$  \hspace{1cm} (7.5).

Proposition 11. If $D_1$ is simply connected and if we have (7.1), then $D_2$ is either a simply connected domain or a doubly connected domain with invertible $L^2(K_2; z_1, z_1^{(N)}, r)$ for its homology base $K_2$.

Proof. Let $g$ be any element of $H(D \times M)$. By (7.1) there exists an element $u$ of $H(D \times M)$ satisfying (7.2). Then $v = T_2 T_3 \cdots T_n u$ satisfies $S_2 v = T_1 T_2 v = g$. Hence we have

$$H(D \times M) = S_2 (H(D \times M))$$  \hspace{1cm} (7.6).

By Proposition 2, we have

$$H^1(D \times M, \text{Ker } S_2) = 0$$  \hspace{1cm} (7.7).

We use the notation

$$D_2^w = D_1 \times D_3^{(3)}$$  \hspace{1cm} (7.8)

and regard $D_2 \times D_2^w$ as $D$.

Now assume that $D_2$ were neither simply connected nor doubly connected. Then there exist two closed Jordan curves $Kb$ and $Kh$;

$$Ki = \{z_1 = ki(t); 0 \leq t \leq 1\} (i = b, h)$$  \hspace{1cm} (7.9)

in $D_2$ and simply connected subdomains $D_2d$ and $D_2g$ of $D_2$ satisfying the
following conditions:

(a) \( \{D_2 \text{d}, D_2 \text{d}\} \) is an open covering of \( D_2 \).

(b) The orientations of \( K \text{b} \) and \( K \text{h} \) with increasing \( t \) are counter-clockwise. \( K \text{b} \) and \( K \text{h} \) belong to the homology base of \( D_2 \). \( k \text{b}(1/2) = k \text{h}(1/2) \).

(c) The connected components of the intersection \( E_2 \) of \( D_2 \text{d} \) and \( D_2 \text{g} \) consist of simply connected components of an open subset \( E_2 \text{b} \), simply connected subdomains \( E_2 \text{h} \) and \( E_2 \text{b} \). \( E_2 \text{b} \), \( E_2 \text{h} \) and \( E_2 \text{b} \) contain, respectively, \( K \text{h}(0) = K \text{h}(1) \), \( z_2^\text{h} = k \text{h}(1/2) = k \text{b}(1/2) \) and \( z_2^\text{b} = k \text{b}(0) \).

Now, \( U = \{D \text{d} \times M, D \text{g} \times M\} \) is an open covering of \( D \times M \). By (7.7), we have

\[
H^1(U, \text{Ker} S_2) = 0
\]  

For any element \( h \) of \( H^0(Eb \times M, \text{Ker} S_2) \), we attach the element 0 of \( H^0(E0, \text{Ker} T_1) \) to \( E0 \times M \), the element 0 of \( H^0(Eh \times M, \text{Ker} S_2) \) to \( Eh \times M \) and the element \( h \) of \( H^0(Eb \times M, \text{Ker} S_2) \) to \( Eb \times M \) and regard this attachment as a cocycle belonging to \( Z^1(U, \text{Ker} S_2) \). By (7.10), this is a coboundary, i.e., there exist \( hd \) of \( H^0(Dd \times M, \text{Ker} S_2) \) and \( hg \) of \( H^0(Dg \times M, \text{Ker} S_2) \) such that

\[
hd - hg = h
\]  

on \( Eb \) and

\[
hg = hd
\]  

on \( E0 \times M \) and \( Eh \times M \). Operating the operator \( T_2 \) on both of (7.11) and (7.12), we have

\[
T_2 hg - T_2 hd = T_2 h
\]  

on \( Eb \) and

\[
T_2 hg = T_2 hd
\]  

on \( E0 \times M \) and \( Eh \times M \) too.

Let \( g^{(1)} = g^{(1)}(z_2, z_3^{(1)}, r) = (g_1^{(1)}(z_2, z_3^{(3)}, r), g_2^{(1)}(z_2, z_3^{(3)}, r), \ldots, g_m^{(1)}(z_2, z_3^{(3)}, r)) \) be any element of \( H(E_2 \times D_3^{(3)} \times M) \) and \( g^{(0)} = g^{(0)}(z_1, z_3^{(3)}, r) = (g_1^{(0)}(z_1, z_3^{(3)}, r), g_2^{(0)}(z_1, z_3^{(3)}, r), \ldots, g_m^{(0)}(z_1, z_3^{(3)}, r)) \) be any element of \( H(E_2 \times D_3^{(3)} \times M) \). We take any point \( z_1^0 \) in the simply connected domain \( D_1 \). For the solution

\[
T_2 h(z_1, z_2, z_3^{(3)}, r) = M_1(z_1^0, z_1, z_2, z_3^{(3)}, r; D_1)g^{(1)}(z_2, z_3^{(3)}, r)
\]  

(7.15).
\[ h(z_1, z_2, z_3^{(3)}, r) = M^2(z_2; z_2^{(3)}, r; E_2 b)g^{(0)}(z_1, z_3^{(3)}, r) \]
\[ + \int_{z_2^{(3)}}^{z_2} (N^2(z_1, s_2, z_3^{(3)}, r; E_2 b)M^1(z_1; z_1^{(3)}, D_1)g^{(1)}(z_2, z_3^{(3)}, r))ds_2 \quad (7.16) \]
of a problem
\[ \begin{aligned}
S_2 u &= T_1(T_2 u) = 0 \\
u(z_1, z_2^{(3)}, r) &= g^{(0)}(z_1, z_3^{(3)}, r) \\
T_2 u(z_1, z_2, z_3^{(3)}, r) &= g^{(1)}(z_2, z_3^{(3)}, r)
\end{aligned} \quad (7.17) \]
corresponds to an element \( f^{(1)} = f^{(1)}(z_2, z_3^{(3)}, r) = (f_1^{(1)}(z_2, z_3^{(3)}, r), f_2^{(1)}(z_2, z_3^{(3)}, r), \ldots, f_m^{(1)}(z_2, z_3^{(3)}, r)) \) of \( H(D_2 \times D_3^{(3)} \times M) \) and an element \( f^{(0)}(z_1, z_3^{(3)}, r) = (f_1^{(0)}(z_1, z_3^{(3)}, r), f_2^{(0)}(z_1, z_3^{(3)}, r), \ldots, f_m^{(0)}(z_1, z_3^{(3)}, r)) \) of \( H(D_1 \times D_3^{(3)} \times M) \) which lead the solution
\[ T_2 h d(z_1, z_2, z_3^{(3)}, r) = M^1(z_1; z_1^{(3)}, r; z_2, z_3^{(3)}, r; D_1)f^{(1)}(z_2, z_3^{(3)}, r) \quad (7.18) \]
\[ h d(z_1, z_2, z_3^{(3)}, r) = M^2(z_2; z_2^{(3)}, r; z_1, z_3^{(3)}, r; D_2 d)f^{(0)}(z_1, z_3^{(3)}, r) \]
\[ + \int_{z_2^{(3)}}^{z_2} (N^2(z_1, s_2, z_3^{(3)}, r; D_2 d)M^1(z_1; z_1^{(3)}, r; z_2, z_3^{(3)}, r; D_1)f^{(1)}(z_2, z_3^{(3)}, r))ds_2 \quad (7.19) \]

We put
\[ K^2(z_1, z_2, z_3^{(3)}, r; D_2 d) = N^2(z_1, s_2, z_3^{(3)}, r; D_2 d)M^1(z_1; z_1^{(3)}, r; z_2, z_3^{(3)}, r; D_1) \quad (7.20) \]
in \( D \times D_3^{(3)} \times M \).

Now, for any fixed \((z_1, z_3^{(3)}, r)\) in \( D_1 \times D_3^{(3)} \times M\), we continue the solution \( h d(z_1, z_2, z_3^{(3)}, r) \) given in \( D_2 d \) along the closed curve \( K b \) from the starting point \( z_2^{(3)} \) to the terminal point \( z_2^b \). Then the relation (7.12) asserts that the function \( h g(z_1, z_2, z_3^{(3)}, r) \), given in \( D_2 g \), succeeds to the solution \( h d \) at the relay point \( z_2^b \). The relation (7.11) at \( z_2^b \) gives
\[ L^2(Kb; z_1, z_3^{(3)}, r)f^{(0)}(z_1, z_3^{(3)}, r) \]
\[ + \int_{Kb} (f^{(1)}(s_2, z_3^{(3)}, r)K^2(z_1, s_2, z_3^{(3)}, r; D_2 d)ds_2 = g^{(0)}(z_1, z_3^{(3)}, r) \quad (7.21) \]
in $D_1 \times D_3^{(3)} \times M$ where
\begin{equation}
L^2(Kb; z_1, z_3^{(3)}, r) = M^2(z_2^b, z_2^b; z_1, z_3^{(3)}, r; D_2 d) - 1 \tag{7.22}
\end{equation}
in $D_1 \times D_3^{(3)} \times M$. Similarly, for the continuation of $T_2 h d$, from (7.13), (7.14) and (7.15), and by the single-valuedness and inversibility of the matrix $M^2(z_2^b; z_2^b; z_1, z_3^{(3)}, r; D_2 d)$, we have
\begin{equation}
\left( \int_{Kb} + \int_{Z_2^b} \right) \left( \frac{\partial}{\partial \theta} f^{(1)}(s_2, z_3^{(3)}, r) \right) ds_2 = g^{(1)}(z_2, z_3^{(3)}, r) \tag{7.23}
\end{equation}
in $E_3 b \times D_3 \times \ldots \times D_m$.

Now assume that $L^2(Kb; z_1, z_3^{(3)}, r)$ were not invertible at a point in $D_1 \times D_3^{(3)} \times M$. Then its image $H$ is a proper subspace of $C^m$ for any point belonging to the analytic set
\begin{equation}
A = \{ (z_1, z_3^{(3)}, r) \in D_1 \times D_3^{(3)} \times M; \det L^2(Kb; z_1, z_3^{(3)}, r) = 0 \} \tag{7.24}
\end{equation}
We choose 0 as $g^{(1)}(z_2, z_3^{(3)}, r)$. We take an element $g^{(0)}(z_1, z_3^{(3)}, r)$ of $H(D_1 \times D_3^{(3)} \times M)$ whose value in the analytic set $A$ does not belong to $H$ and whose domain of holomorphy is just the domain $D_1 \times D_3^{(3)} \times M$. In the analytic set $A$, the first term of (7.21) takes only a value in $H$. Since the growth near $\partial D_1$ of the second term of the left hand of (7.21), regarded as a compact operator with respect to the variable $z_1$, is constant times of the kernel $K^2$, the left hand of (7.21) can not supply the growth near $\partial D_1$ along the analytic set $A$ of the right hand of (7.21). So the right hand of (7.21) can not be such an image of the left hand of (7.21). This is a contradiction. Hence $L^2(Kb; z_1, z_3^{(3)}, r)$ is invertible at each point of $D_1 \times D_3^{(3)} \times M$. Since $L^2(Kb; z_1, z_3^{(3)}, r)$ is invertible at each point of $D_1 \times D_3^{(3)} \times M$, any non trivial solution $u$ of the homogeneous equation $S_2 u = 0$ is not single-valued along the closed curve $Kb$ in $D_2$. Since the other closed curve $Kh$ is logically equivalent to $Kb$, we interchange $Kh$ and $Kb$ and we see that any non trivial solution $u$ of the equation $S_2 u = 0$ is not single-valued along the closed curve $Kb$ too. But this contradicts to (7.12).

q.e.d.

**Proposition 12.** If $H^1(D \times M, \text{Ker} T) = 0$, then $D_i$ is either a simply connected domain or a doubly connected domain with invertible $L_i(K_i; z_i^n, r)$ for its homology base $K_i$ for $i = 1, 2$.

**Proof.** By Propositions 4 and 6, $D_1$ is either a simply connected domain
or a doubly connected domain with invertible $L^1(K_1; z_1^r, r)$ for its homology base $K_1$. Then, in the former case by Propositions 10 and 6, and in the latter case by Proposition 11, the assertion of the proposition holds for $i = 2$.

q.e.d.

8. Necessary and sufficient condition for $H^1(D \times M, \text{Ker } S_2) = 0$

Theorem 2. The necessary and sufficient condition for $H(D \times M) = S_2(H(D \times M))$ is that $D_1$ is either a simply connected domain or a doubly connected domain in $D_i$ with invertible $L^1(K_i; z_i^r, r)$ for its homology base for $i = 1, 2$.

Proof. By Proposition 12, it suffices to prove the sufficiency and four cases may occur concerning connectivity of $D_1$ and $D_2$.

Let $w(z_1, z_2, \ldots, z_m, r)$ be any element of $H(D \times M)$. By Theorem 1, there exists a solution $v(z_1, z_2, \ldots, z_m, r)$ in $D \times M$ of the equation $T_1 v = w$. When $D_2$ is simply connected, by Theorem 1, there exists a solution $u(z_1, z_2, \ldots, z_m, r)$ in $D \times M$ of the equation $T_2 u = v$.

We consider the case that $D_2$ is doubly connected. Let

$$K_2 = \{z_2 = k(t); 0 \leq t \leq 1\} \quad (8.1)$$

be a homology base of $D_2$ which has counterclockwise orientation. Then there exist two simply connected subdomains $D_2 d$ and $D_2 g$ of $D_2$ such that $\{D_2 d, D_2 g\}$ is an open covering of $D_2$ and that $D_2 d$ and $D_2 g$ contains the initial and terminal point $z_2^b = k(0) = k(1)$.

We put

$$D d = D_1 \times D_2 d \times D_3^{(3)}$$
$$D g = D_1 \times D_2 g \times D_3^{(3)} \quad (8.2)$$

By Theorem 1, there exists a solution $u d(z_1, z_2, \ldots, z_m, r)$ in $D d \times M$ to the equation $T_2 u d = v$. We continue it round the closed curve $K_2$ and return to the point $z_2^b$. We denote the successor of $u d$ in $D g \times M$ by $u g$. By the theorem of identity, we have $T_2 u g = v$ in $D g \times M$. Then the difference $h = u g - u d$ is a solution to the homogeneous equation in the intersection $E \times M$ of $D d \times M$ and $D g \times M$. We define an element of $H(D_1 \times D_2^{(3)} \times M)$ by putting
Global existence of holomorphic solutions

\[ f(z_1, z_3, \ldots, z_m, r) = L^2(K_2; z_2^b; z_1, z_3, \ldots, z_m, r)^{-1}(-h(z_1, z_3, \ldots, z_m, r)) \]  \hspace{1cm} (8.3)

and consider the initial value problem

\[
\begin{align*}
T_2 y &= 0 \\
y(z_1, z_2^b, z_3, \ldots, z_m, r) &= f(z_1, z_3, \ldots, z_m, r)
\end{align*}
\]  \hspace{1cm} (8.4)

and the function \( ud + y \) in \( Dd \times M \) which satisfies \( T_2(ud + y) = v \). For any point \((z_1, z_3, \ldots, z_m, r)\) in \( D_1 \times D_3 \times \cdots \times D_m \times M \), we continue \( ud + y \) round the closed curve \( K_2 \) starting from the point \( z_2^b \) in \( Dd \times M \). When we return to the point \( z_2^b \), its value is \( \text{Monod}(K_2; z_2^b; z_1, z_3, \ldots, z_m, r)f(z_1, z_3, \ldots, z_m, r) + u \text{g} \). The necessary and sufficient condition that \( ud + y \) is single-valued round the closed curve \( K_2 \) is that

\[ L^2(K_2; z_2^b; z_1, z_3, \ldots, z_m, r)f(z_1, z_3, \ldots, z_m, r) = -h(z_1, z_3, \ldots, z_m, r) \]  \hspace{1cm} (8.5)

which is the reason why we have put (8.3). Thus we have obtained a single-valued \( ud + y \) in \( D \times M \) such that \( T_1(T_2(ud + y)) = T_1(v) = w \). q.e.d.

9. Elements of \( H^0(D \times M, \text{Ker} \, Sp) \).

Let \( p \) be a positive integer with \( 1 \leq p \leq m \) and assume that \( D_i \) is either a simply connected domain or a doubly connected domain in \( D_i \) with invertible \( L^i(K_i; z_i^0, r) \) for its homology base for \( i = 1, 2, \ldots, p \). Let \( I_1 \) or \( I_2 \) be, respectively, the set of elements \( i \) of \( l = \{1, 2, \ldots, p\} \) such that \( D_i \) is simply or doubly connected. For any element \( i \) of \( I_1 \) or \( I_2 \), let \( z_i^0, z_i^b \) be, respectively, a point of \( D_i \).

**Proposition 13.** For any element \( q \) of \( I_1 \), let \( g^{(p-q+1)}(z_1, \ldots, z_{q-1}, z_{q+1}, \ldots, z_m, r) \) be an element of \( H(D_1 \times D_2 \times \cdots \times D_{q-1} \times D_{q+1} \times \cdots \times D_m \times M) \), then there exists uniquely an element \( u \) of \( H(D \times M) \) satisfying

\[
\begin{align*}
S_p u &= 0 \\
T_{q+1}(T_{q+2} \cdots (T_p u((z_1, \ldots, z_{q-1}, O, z_{q+1}, \ldots, z_m, r)))) &= (q = 1, 2, \ldots, p)
\end{align*}
\]  \hspace{1cm} (9.1)

where the left side of the second equation of (9.1) means \( u \) when \( q = p \).

**Proof.** We have
in $D \times M$ where
\[
T_2(T_3(\cdots(T_p u))) = u^{(1)}(z_1, z_2, \ldots, z_m, r)
\]
(9.2)
in $D \times M$ if $D_1$ is simply connected and
\[
u^{(1)}(z_1; z_2, \ldots, z_m, r) = \text{the unique solution (5.31)}
\]
(9.4)
if $D_i$ is doubly connected. As the assumption of the induction, we assume that, for $q$ with $1 \leq q \leq p$, we have
\[
T_q(T_{q+1}(\cdots(T_p u))) = u^{(q-1)}(z_1, z_2, \ldots, z_m)
\]
(9.5)
in $D \times M$ where
\[
u^{(q-1)}(z_1, z_2, \ldots, z_m) = M^{q-1}(z_{q-1}; z_{q-1}; z_1, z_2, \ldots, z_{q-2}, z_q, \ldots, z_m, r; D_{q-1})
\]
\[
g^{(p-q+2)}(z_1, z_2, \ldots, z_{q-2}, z_{q}, \ldots, z_m, r)
\]
\[
+ \int_{z_{q-1}}^{z_1} (N^{q-1}(z_1, z_2, \ldots, z_{q-2}, s_q, \ldots, z_m; D_{q-1})
\]
\[
- u^{(q-2)}(z_1, z_2, \ldots, z_{q-2}, s_q, \ldots, z_m) \, ds_q
\]
(8.6)
in $D \times M$ if $D_{q-1}$ is simply connected and $u^{(q-1)}(z_1, z_2, \ldots, z_m)$ is determined uniquely by the method in (8.3)–(8.5) if $D_{q-1}$ is doubly connected.

Then, if $D_q$ is simply connected, we have
\[
T_{q+1}(\cdots(T_p u)) = u^q(z_1, z_2, z_3, \ldots, z_m, r)
\]
(9.7)
in $D \times M$ where
\[
u^q(z_1, z_2, z_3, \ldots, z_m, r)
\]
\[
= M^q(z_q; z_q^0; z_1, \ldots, z_{q-1}, z_{q+1}, \ldots, z_m, r) g^{(p-q+1)}(z_1, \ldots, z_{q-1}, z_{q+1}, \ldots, z_m, r)
\]
\[
+ \int_{z_q}^{z_{q-1}} (N^q(z_1, z_2, \ldots, z_{q-1}, s_q, z_{q+1}, \ldots, z_m; D_q)
\]
\[
\times u^{(q-1)}(z_1, z_2, \ldots, z_{q-1}, s_q, z_{q+1}, \ldots, z_m) \, ds_q
\]
(9.8)
in $D \times M$.

When $D_q$ is doubly connected, we use notations in the preceding
paragraph. Since $D_q$ is simply connected, by exchanging the variable $z_1$ by the variable $z_q$ in Proposition 11, we have a solution $ud(z_1, z_2, \ldots, z_m, r)$ in $Dd \times M$ to the equation

$$T_q ud = u^{(q-1)}(z_1, \ldots, z_2, \ldots, z_m)$$

(9.9)

with the initial value 0. Since the initial value is holomorphic in $D_1 \times D_2 \times \cdots \times D_{q-1} \times D_{q+1} \times \cdots \times D_m \times M$, and the right hand-side of (9.9) and the coefficients of $T_q$ in the left hand-side of (9.9) are holomorphic in $D \times M$, the solution $ud$ is holomorphically and unlimitedly continued in $D \times M$. The continued solution is not necessarily single-valued there. We continue it round the closed curve $K_q$ and return to the point $z^b_q$. We denote by $ug$ the successor of $ud$ in $Dg \times M$. By the argument in the preceding paragraph, for

$$f(z_1, z_2, z_{q-1}, z_{q+1}, \ldots, z_m, r)$$

$$= (L^q(K_q; z^b_q; z_1, z_2, z_{q-1}, z_{q+1}, \ldots, z_m, r))^{-1}$$

$$(ud - ug)(z_1, z_2, z_{q-1}, z_{q+1}, \ldots, z_m, r),$$

(9.10)

$$u^q(z_1, z_2, z_3, \ldots, z_m, r) = ud$$

$$+ M^q(z_q; z^b_q, z_1, \ldots, z_{q-1}, z_{q+1}, \ldots, z_m, r) f(z_1, z_2, z_{q-1}, z_{q+1}, \ldots, z_m, r)$$

(9.11)

is the unique global solution to (9.9) in $D \times M$. There holds (9.7) in this case too.

10. Necessary and sufficient condition for $H^1(D \times M, \text{Ker} S_p) = 0$.

Theorem 3p. The necessary and sufficient condition for $H(D \times M) = S_p(H(D \times M))$ is that $D_i$ is either a simply connected domain or a doubly connected domain in $D_i$ with invertible $L^i(K_i; z^\prime; r)$ for its homology base $K_i$ for $i = 1, 2, \ldots, p$.

Proof. We have proved the sufficiency of the theorem in Proposition 13. So we prove the necessity of the theorem 3p by induction with respect to $p$. We prove Theorem 3p under the assumption of validity of Theorem 3p-1. We divide the variable $z = (z_1, z_2, \ldots, z_m)$ into three parts:

$$z_p' = (z_1, z_2, \ldots, z_{p-1})$$

(10.1)
\[ z_p \quad \text{and} \quad \quad z'_p = (z_{p+1}, z_{p+2}, \ldots, z_m) \quad (10.2). \]

We use similar notations in the proof of Proposition 11, substituting \( p \) for \( 2 \). By Proposition 2, we have
\[ H^1(D \times M, \text{Ker} S_p) = 0 \quad (10.3). \]

Now assume that \( D_p \) were neither simply connected nor doubly connected. Then there exist two closed curves \( K_b \) and \( K_h \) in \( D_p \) satisfying that conditions, \( p \) being substituted for \( 2 \) in the proof of Proposition 11 and simply connected subdomains \( D_{p,d} \) and \( D_{p,g} \) of \( D_p \). We consider the open subset \( E_p \), simply connected component \( E_p,h \) and \( E_p,b \) of the intersection \( E_p \) of \( D_{p,d} \) and \( D_{p,g} \). \( E_p,0, E_p,h \) and \( E_p,b \) contain, respectively, \( z^0_p = k(0), z^h_p = kh(1/2) = kb(1/2) \) and \( z^b_p = kb(0) \). For the open covering \( U = \{Dd \times M, Dg \times M\} \) of \( D \times M \), we also consider the Mayer-Vietoris exact cohomology sequence
\[
\begin{align*}
H^0(D \times M, \text{Ker} S_p) & \longrightarrow H^0(Dd \times M, \text{Ker} S_p) \oplus H^0(Dg \times M, \text{Ker} S_p) \\
H^0(Dg \times M, \text{Ker} S_p) & \longrightarrow H^0(E \times M, \text{Ker} S_p) \\
H^1(D \times M, \text{Ker} S_p) & \longrightarrow H^1(Dd \times M, \text{Ker} S_p) \oplus H^1(Dg \times M, \text{Ker} S_p) \longrightarrow \cdots 
\end{align*}
\] (10.4)

where the homomorphism
\[ \Xi : H^0(Dd \times M, \text{Ker} S_p) \oplus H^0(Dg \times M, \text{Ker} S_p) \longrightarrow H^0(E \times M, \text{Ker} S_p) \quad (10.5) \]

is the canonical substraction
\[ \Xi [hd, hg] = hd - hg, \]
for
\[ [hd, hg] \in H^0(Dd \times M, \text{Ker} S_p) \oplus H^0(Dg \times M, \text{Ker} S_p) \quad (10.6). \]

Since \( D_{p,d} \) and \( D_{p,g} \) are simply connected subdomains of \( D_p \), we have
\[ H^1(Dd \times M, \text{Ker} S_p) \oplus H^1(Dg \times M, \text{Ker} S_p) = 0 \quad (10.7) \]

by Theorem 3_{p-1}. By (10.7) and (10.3) we have
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\[ H^0(\mathbb{E} \times M, \text{Ker} S_p) = \text{Im} \mathcal{E} \]  
\[(10.8).\]

Now, let \( h \) be any element of \( H^0(\mathbb{E} \times M, \text{Ker} S_p) \). By (10.8) there exist \( h_d \) of \( H^0(D_d \times M, \text{Ker} S_p) \) and \( h_g \) of \( H^0(D_g \times M, \text{Ker} S_p) \) such that

\[ h_g - h_d = h \]  
\[(10.9).\]
on \( \mathbb{E} \times M \) and \( \mathbb{E} \times M \). Then \( h_d \) is an element of \( H^0(D_d \times M, \text{Ker} S_p) \) and the element \( h_g \) of \( H^0(D_g \times M, \text{Ker} S_p) \) is its successor in its continuation round the closed curve \( \mathcal{K} \mathbb{E} \mathcal{B} \) in \( D_p \). By the theorem of identity, \( h_g \) belongs to \( H^0(D_g \times M, \text{Ker} S_{p-1}) \).

Here, we turn to the notations, \( I_s, I_d \) and so on, in Proposition 13. For any element \( q \) of \( I_s \), let \( g^{(p-q+1)}(z_1, \ldots, z_{q-1}, z_{q+1}, \ldots, z_m, r) \) be an element of \( H(D_1 \times D_2 \times \cdots \times D_{q-1} \times D_{q+1} \times \cdots \times D_m \times M) \). Then there exists a unique element \( h \) of \( H^0(D \times M, (\mathcal{O}_D \times M)^n) \) satisfying

\[
\begin{aligned}
S_p h &= 0 \\
T_{q+1}(T_{q+2} \cdots (T_p u((z_1, \ldots, z_{q-1}, 0, z_{q+1}, \ldots, z_m, r)))) &= g^{(p-q+1)}(z_1, \ldots, z_{q-1}, z_{q+1}, \ldots, z_m, r) \quad (q = 1, 2, \ldots, p)
\end{aligned}
\]

\[(10.11).\]

where the left side of the second equation of (10.11) means \( h \) when \( q = p \). This solution \( h \) has the inductive representation similar to (9.2)–(9.8) in the proof of Proposition 13. Especially

\[
h(z_1, z_2, z_3, \ldots, z_m, r)
\]
\[
= M^p(z_p; z_0^0, z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r)g^{(1)}(z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r) \\
+ \int_{z_p^0}^{z_p} (N^p(z_1, z_2, \ldots, z_{p-1}, s_p, z_{p+1}, \ldots, z_m; E_p h) \\
\quad h b^{(p-1)}(z_1, \ldots, z_{p-1}, s_p, z_{p+1}, \ldots, z_m, r)) ds_p
\]

in \( \mathbb{E} \times M \) where the \( p - 1 \) th \( h b^{(p-1)}(z_1, \ldots, z_{p-1}, z_p, z_{p+1}, \ldots, z_m, r) \) is holomorphic in \( \mathbb{E} \times M \). For this \( h \), the \( h d \) has a similar representation

\[
h d(z_1, z_2, z_3, \ldots, z_m, r)
\]
\[
= M^p(z_p; z_0^0, z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r)f^{(1)}(z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r)
\]
+ \int_{\mathbb{C}^p} (N^p(z_1, \ldots, z_{p-1}, s_p, z_{p+1}, \ldots, z_m; D_p d) \\
\times h^d(p-1)(z_1, \ldots, z_{p-1}, s_p, z_{p+1}, \ldots, z_m, r)) d\sigma_p.

By (10.9), we have

\[ L^p(K b; z^0_p; z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r) f^{(1)}(z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r) \]

\[ + \int_{\mathbb{C}^p} (N^p(z_1, \ldots, z_{p-1}, s_p, z_{p+1}, \ldots, z_m; D_p d) \\
\times h^d(p-1)(z_1, \ldots, z_{p-1}, s_p, z_{p+1}, \ldots, z_m, r)) d\sigma_p \]

\[ + \int_{\mathbb{C}^p} (N^p(z_1, \ldots, z_{p-1}, s_p, z_{p+1}, \ldots, z_m; D_p g) \\
\times h^g(p-1)(z_1, \ldots, z_{p-1}, s_p, z_{p+1}, \ldots, z_m, r)) d\sigma_p = g^{(1)}(z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r). \]

Since the second and third terms of the left hand-side are holomorphic in the whole \( D_1 \times D_2 \times \cdots \times D_{q-1} \times D_{q+1} \times \cdots \times D_m \times M \) and since the left hand-side supplies an element \( f^{(1)}(z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r) \) of \( H(D_1 \times D_2 \times \cdots \times D_{q-1} \times D_{q+1} \times \cdots \times D_m \times M) \) for any element \( g^{(1)}(z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r) \) whose domain of holomorphy is just \( D_1 \times D_2 \times \cdots \times D_{q-1} \times D_{q+1} \times \cdots \times D_m \times M \), as in the proof of Proposition 13, \( L^p(K b; z^0_p; z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_m, r) \) is regular in \( D_1 \times D_2 \times \cdots \times D_{q-1} \times D_{q+1} \times \cdots \times D_m \times M \). By the same arguments in the proof of Proposition 13, \( D_p \) is either a simply connected domain or a doubly connected domain in \( C \) with invertible \( L^p(K_p; z^p, r) \) for its homology base \( K_p \).

q.e.d.

11. Main Theorem.

As Theorem 3_m, we have the following main theorem.

**Main Theorem.** Let \( m \) be an integer and, for each \( j \) with \( 1 \leq j \leq m \), \( D_j \) be a domain in the complex plane \( C \). We put

\[ D = D_1 \times D_2 \times \cdots \times D_m. \]

Let \( M \) be a Stein manifold. We consider the product manifold \( D \times M \). For each \( i \) with \( 1 \leq i \leq m \), let \( a^i_j = (a^i_{jk}(z, r)) \) be a square matrix of degree \( m \) whose element \( a^i_{jk}(z, r) \) are holomorphic functions in \( D \times M \). For each \( i \) with \( 1 \leq i \leq m \),
let $T_i$ be a differential operator defined by

$$T_i = \frac{\partial}{\partial z_i} + a^i.$$

Let $O_{D \times M}$ be the sheaf of germs of all holomorphic function on $D$. Then each differential operator $T_i$ defines a sheaf homomorphism

$$T_i: (O_{D \times M})^m \longrightarrow (O_{D \times M})^m.$$  

We define

$$S = T_1 T_2 \cdots T_m$$  

by putting

$$Su = T_1(T_2(\cdots T_m(u)))$$

for any $u$ of $(O_{D \times M})^m$.

Then, the necessary and sufficient condition for $H(D \times M) = S(H(D \times M))$ is that $D_i$ is either a simply connected domain or a doubly connected domain in $C$ with invertible $L(K_i; z_i^r, r)$ where $K_i$ is a homology base of $D_i$ and $L(K_i; z_i^r, r)$ is a matrix defined in (2.26) for $i = 1, 2, \ldots, m$.

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