THE CLASSIFICATION OF COMPLETE HYPERSURFACES WITH NON-ZERO CONSTANT MEAN CURVATURE OF SPACE FORM OF DIMENSION 4

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1. Introduction

An important problem in Differential Geometry is to investigate isometric immersions of the complete Riemannian manifolds with constant mean curvature into space forms. The problem involves analysis on manifolds equipped with the moving frames introduced by E. Cartan. The classical Bernstein theorem states that a complete minimal surface expressed as a graph of a plane in $\mathbf{R}^3$ is isometric to the Euclidean plane. As a sphere version of the Bernstein problem on the complete minimal surface in Euclidean plane, Myers [10] proved that if a closed Riemannian manifold M is minimally immersed into a closed hemi-sphere of the unit sphere $S^m(1)$ of constant curvature 1, then it is contained entirely in the great sphere. The result is also generalized to complete Riemannian manifolds by the author [2]. The idea of proof of Myers' Theorem is that the distance from every point on M to the boundary of the closed hemi-sphere is a superharmonic function on M and the maximum principle then implies that the distance function is identically zero on M . This type of idea was used by Simons [15] in the later. He gave an important contribution in the investigations of closed minimal submanifolds in the sphere of constant curvature. The linear elliptic second order partial differential equation satisfied by the second fundamental form of

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minimal submanifolds is derived by him. Making use of it the following statement was proved

Simons Theorem. *Let M be an n -dimensional closed minimal submanifolds in the unit sphere $S^{n+p}(1)$ and let S denote the squared norm of the second fundamental form of M . Then*

$$\int [(2 - 1/p)S - n] S dv \geq 0,$$

where dv is the volume elements of M .

It then follows that if $S \leq n/(2 - 1/p)$, we have either $S = 0$ or $S = n/(2 - 1/p)$. The complete determination of the latter case was obtained by Chern, do Carmo and Kobayashi [8] as follows:

Theorem CDK. *The Veronese surface in $S^4(1)$ and the Clifford torus in $S^{n+1}(1)$ are the only closed minimal submanifolds of dimension n in $S^{n+p}(1)$ satisfying $S = n/(2 - 1/p)$.*

Making use of Simons' type formulas, many authors studied the minimal submanifolds in the unit sphere and generalized the results due to Simons and Chern, do Carmo and Kobayashi to complete minimal submanifolds in the unit sphere (for examples: [3], [6], [7], [9] and [14]). On the other hand, Nomizu and Smyth [11] characterized the complete hypersurfaces with constant mean curvature and non-negative sectional curvature in space form $M^{n+1}(c)$ if the scalar curvature is constant. Yano and Ishihara [16] and Yau [18] generalized the results due to Nomizu and Smyth to submanifolds with higher codimension.

In this paper, we give the complete classification of 3-dimensional complete hypersurfaces with constant mean curvature H of the space form $M^4(c)(c \geq 0)$ if the scalar curvatures are not less than $-2.5542H^2$ constant. That is, we obtain the following statement:

Main Theorem. *Let M be a 3-dimensional complete hypersurface with constant mean curvature H of a space form $M^4(c)(c \geq 0)$. If the scalar curvature of M is not less than $-2.5542H^2$ constant, then M is one of the following manifolds:*

- (1) $S^3(c_1)$, $c_1 \geq c$;

- (2) $\mathbf{R}^k \times S^{3-k}(c_1)$, $k = 1, 2, 3$;
- (3) $S^1(c_1) \times S^2(c_2)$;
- (4) the tube of the Veronese surface $P^2\mathbf{R}$.

Remark 1. When $c = 1$, the part results have been obtained in [4] by the author.

Remark 2. Nomizu and Smyth [11] gave a classification under the assumptions which the sectional curvature is non-negative and the scalar curvature is constant. Hence our theorem can also be seen as a generalization of results of Nomizu and Smyth when $n = 3$.

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2. Basic concepts and local formulas

Throughout this paper, all manifolds are assumed to be smooth and connected without boundary. We discuss in smooth category. Let $M^{n+1}(c)$ be a simply connected Riemannian manifold of constant curvature c , which is called a *space form of constant curvature c* . Let M be an n -dimensional hypersurface in the $n + 1$ -dimensional space form $M^{n+1}(c)$ of constant curvature c . We choose a local field of orthonormal frames $e_1, \dots, e_{n+1}$ in $M^{n+1}(c)$ adapted to the Riemannian metric of $M^{n+1}(c)$ and the dual coframe $\omega_1, \dots, \omega_{n+1}$ in such a way that $e_1, \dots, e_n$ are tangent to M . Then connection forms $\{\omega_{AB}\}$ of $M^{n+1}(c)$ are characterized by the structure equations

$$(2.1) \quad \begin{cases} d\omega_A = -\sum \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} = -\sum \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \\ \Omega_{AB} = -\frac{1}{2} \sum R'_{ABCD} \omega_C \wedge \omega_D. \end{cases}$$

$$(2.2) \quad R'_{ABCD} = c(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD})$$

where Ω_{AB} (resp. R'_{ABCD}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of $M^{n+1}(c)$. Therefore the components of Ricci curvature tensor Ric' and the scalar curvature r' are given as

$$R'_{AB} = cn\delta_{AB}, \quad r' = n(n+1)c.$$

In the sequel, we use the following convention on the range of indices unless otherwise stated:

$$A, B, C, \dots = 1, 2, \dots, n+1; i, j, k, \dots = 1, 2, \dots, n.$$

We agree the repeated indices under a summation sign without indication are summed over the respective range. The canonical forms $\{\omega_A\}$ and the connection forms $\{\omega_{AB}\}$ restricted to M are also denoted by the same symbols. We then have

$$(2.3) \quad \omega_{n+1} = 0.$$

we see that $e_1, \dots, e_n$ is a local field of orthonormal frames on M and $\omega_1, \dots, \omega_n$ is a local field of its dual coframes on M . It follows from (2.1), (2.3) and Cartan's Lemma that

$$(2.4) \quad \omega_{n+1i} = \sum h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

The second fundamental form α and the mean curvature H of M are defined by

$$\alpha = \sum h_{ij}\omega_i\omega_j, \quad H = \frac{1}{n}\sum h_{ii} = h/n.$$

Let $S = \sum h_{ij}^2$ denote the squared norm of the second fundamental form of M . We call that M is by definition a *minimal hypersurface* if the mean curvature of M is identically zero. M is said to be *totally umbilic* if the h_{ij} can be expressed as $h_{ij} = H\delta_{ij}$. The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations of M .

$$(2.5) \quad \begin{cases} d\omega_i = -\sum \omega_{ij} \wedge \omega_j, & \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \\ \Omega_{ij} = -\frac{1}{2}\sum R_{ijkl}\omega_k \wedge \omega_l, \end{cases}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of M . Therefore, the Gauss equation is given by, from (2.1) and (2.5),

$$(2.6) \quad R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + (h_{il}h_{jk} - h_{ik}h_{jl}).$$

The covariant derivative $\nabla\alpha$ of the second fundamental form α of M with

components h_{ijk} is given by

$$(2.6') \quad \sum h_{ijk} \omega_k = dh_{ij} + \sum h_{jk} \omega_{ik} + \sum h_{ik} \omega_{jk}.$$

Then the exterior derivative of (2.4) together with the structure equation (2.5) gives

$$\sum h_{ijk} \omega_j \wedge \omega_k = 0.$$

Thus we can obtain the Codazzi equation

$$(2.7) \quad h_{ijk} = h_{ikj}.$$

Similarly, we have the covariant derivative $\nabla^2 \alpha$ of $\nabla \alpha$ with components h_{ijkl} as follows

$$(2.7') \quad \sum h_{ijkl} \omega_l = dh_{ijk} + \sum h_{ijk} \omega_{il} + \sum h_{ilk} \omega_{jl} + \sum h_{ijl} \omega_{kl}.$$

Then taking the exterior derivative of (2.6') we obtain the Ricci formulas:

$$(2.8) \quad h_{ijkl} - h_{ijlk} = - \sum h_{im} R_{mjkl} - \sum h_{jm} R_{mikl}.$$

Similarly we also have

$$(2.9) \quad h_{ijklm} - h_{ijkml} = - \sum h_{ijk} R_{tilm} - \sum h_{itk} R_{tjlm} - \sum h_{ijt} R_{tklm},$$

where h_{ijklm} denote the components of the covariant differential $\nabla^3 \alpha$ of $\nabla^2 \alpha$. The components of the Ricci curvature Ric and the scalar curvature r are given by

$$(2.10) \quad R_{ij} = (n-1)c\delta_{ij} + nHh_{ij} - \sum h_{ik}h_{kj},$$

$$(2.11) \quad r = n(n-1)c + n^2H^2 - \sum h_{ij}^2.$$

The Laplacian Δh_{ij} of the components h_{ij} of the second fundamental form α is given by

$$\Delta h_{ij} = \sum h_{ijkk}.$$

From (2.8), we obtain

$$\Delta h_{ij} = \sum h_{kkij} - \sum h_{km} R_{mijk} - \sum h_{im} R_{mkjk}.$$

Now we compute some local formulae under the assumption that the mean curvature of M is constant. For arbitrary fixed point p in M we choose a local frame field $e_1, \dots, e_n$ such that

$$h_{ij} = \lambda_i \delta_{ij}.$$

Let $\mu_i = \lambda_i - H$, $S = \sum \lambda_i^2$, $f_k = \sum_i \lambda_i^k$, $B_k = \sum_i \mu_i^k$. Then

$$\begin{aligned} \sum \mu_i &= 0, \quad B = B_2 = S - h^2/n, \\ \frac{1}{2} \Delta S &= \sum h_{ijk}^2 + \sum h_{ij} \Delta h_{ij} \\ &= \sum h_{ijk}^2 + (nc - S)S - ch^2 + hf_3 \quad (\text{by Ricci formulae}), \\ \frac{1}{2} \Delta \sum h_{ijk}^2 &= \sum h_{ijkl}^2 + \sum h_{ijk} \Delta h_{ijk}. \end{aligned}$$

By a direct computation we have

$$\begin{aligned} \Delta h_{ijk} &= - \sum_{t,l} h_{tjl} R_{tikl} - \sum_{t,l} h_{itl} R_{tjkl} \\ &\quad - \sum_{t,l} h_{ijt} R_{tlkl} - \sum_{m,l} h_{mjl} R_{mikl} - \sum_{m,l} h_{iml} R_{mjkl} \\ &\quad - \sum_{m,l} h_{mlk} R_{mijl} - \sum_{m,l} h_{imk} R_{mljk} - \sum_{m,l} h_{mj} \Delta_l R_{mikl} \\ &\quad - \sum_{m,l} h_{im} \Delta_l R_{mjkl} - \sum_{m,l} h_{ml} \Delta_k R_{mijl} - \sum_{m,l} h_{im} \Delta_k R_{mljl}, \end{aligned}$$

where $\Delta_l R_{ijkm}$ denote the components of the covariant derivative of the Riemannian curvature tensor of M . Hence we obtain

$$\begin{aligned} \frac{1}{2} \Delta \sum h_{ijk}^2 &= \sum h_{ijkl}^2 + \{(2n+3)c - S\} \sum h_{ijk}^2 \\ &\quad - 3 \{ \sum \lambda_i^2 h_{ijk}^2 - 2 \sum \lambda_i \lambda_j h_{ijk}^2 \} \\ &\quad + 3h \sum \lambda_i h_{ijk}^2 - \frac{3}{2} |\text{grad } S|^2, \\ \Delta f_3 &= 3 \sum \lambda_i^2 h_{ijmm} + 6 \sum \lambda_i h_{ijk}^2 \\ &= 3(nc - S)f_3 + 3hf_4 - 3chS + 6 \sum \lambda_i h_{ijk}^2, \\ \Delta f_4 &= 4 \sum \lambda_i^3 h_{ijmm} + 8 \sum \lambda_i^2 h_{ijk}^2 + 4 \sum \lambda_i \lambda_j h_{ijk}^2 \end{aligned}$$

$$\begin{aligned}
&= 4(nc - S)f_4 + 4hf_5 - 4chf_3 + 8\sum \lambda_i^2 h_{ijk}^2 \\
&\quad + 4\sum \lambda_i \lambda_j h_{ijk}^2.
\end{aligned}$$

Thus we have proved the following statements:

Proposition 2.1. *Let M be an n -dimensional hypersurface with constant mean curvature of a space form $M^{n+1}(c)$. Then we have*

$$(2.12) \quad \frac{1}{2}\Delta S = \sum h_{ijk}^2 + (nc - S)S - ch^2 + hf_3,$$

$$\begin{aligned}
(2.13) \quad \frac{1}{2}\Delta \sum h_{ijk}^2 &= \sum h_{ijkl}^2 + \{(2n + 3)c - S\} \sum h_{ijk}^2 \\
&\quad - 3\{\sum \lambda_i^2 h_{ijk}^2 - 2\sum \lambda_i \lambda_j h_{ijk}^2\} \\
&\quad + 3h \sum \lambda_i h_{ijk}^2 - \frac{3}{2}|\text{grad } S|^2,
\end{aligned}$$

$$(2.14) \quad \Delta f_3 = 3(nc - S)f_3 + 3hf_4 - 3chS + 6\sum \lambda_i h_{ijk}^2,$$

$$\begin{aligned}
(2.15) \quad \Delta f_4 &= 4(nc - S)f_4 + 4hf_5 - 4chf_3 \\
&\quad + 8\sum \lambda_i^2 h_{ijk}^2 + 4\sum \lambda_i \lambda_j h_{ijk}^2.
\end{aligned}$$

Next we only consider the case that $n = 3$ and $S = \text{constant}$. Thus

$$(2.16) \quad B_3 = f_3 - hS + \frac{2}{9}h^3,$$

$$(2.17) \quad f_4 = -h^2S + \frac{1}{6}h^4 + \frac{4}{3}hf_3 + \frac{1}{2}S^2,$$

$$(2.18) \quad f_5 = \frac{5}{6}(S + h^2)f_3 + \frac{1}{6}h^5 - \frac{5}{6}h^3S.$$

Proposition 2.2. *Let M be a 3-dimensional hypersurface in a space form $M^4(c)$ with constant mean curvature. If S is constant, then we have*

$$\begin{aligned}
(2.19) \quad \frac{1}{3}\Delta \sum h_{ijk}^2 &= -2h\sum \mu_i h_{ijk}^2 \\
&\quad - (S - 3c - 2h^2/3)\sum h_{ijk}^2 + (S - h^2/3)(S - 3c - 3h^2/4)
\end{aligned}$$

$$\begin{aligned}
& + \sqrt{h^4 + 8ch^2/4}(S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2/4}), \\
(2.20) \quad & \sum h_{ijkl}^2 = -\frac{5}{3} \sum (\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 \\
& - 8h \sum \mu_i h_{ijk}^2 + \frac{3}{2} (S - 3c - 2h^2/3) \sum h_{ijk}^2 \\
& + \frac{3}{2} (S - h^2/3) (S - 3c - 3h^2/4 + \sqrt{h^4 + 8ch^2/4}) \\
& \times (S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2/4}).
\end{aligned}$$

Proof. From (2.12) and (2.17) we have

$$\Delta f_4 = \frac{4}{3} h \Delta f_3 = -\frac{4}{3} \Delta \sum h_{ijk}^2.$$

(2.15) and (2.21) imply

$$\begin{aligned}
(2.22) \quad & 2 \sum \lambda_i^2 h_{ijk}^2 + \sum \lambda_i \lambda_j h_{ijk}^2 \\
& = -\frac{1}{3} \Delta \sum h_{ijk}^2 + (S - 3c) f_4 - h f_5 + c h f_3.
\end{aligned}$$

According to $\mu_i = \lambda_i - H$ and the symmetry of h_{ijk} , we have

$$\begin{aligned}
(2.23) \quad & \sum \lambda_i^2 h_{ijk}^2 + 2 \sum \lambda_i \lambda_j h_{ijk}^2 \\
& = \frac{1}{3} h^2 \sum h_{ijk}^2 + 2h \sum \mu_i h_{ijk}^2 \\
& + \frac{1}{3} \sum (\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2.
\end{aligned}$$

It follows from (2.12), (2.14), (2.17) and (2.21) that

$$\begin{aligned}
& \frac{1}{3} \Delta \sum h_{ijk}^2 = h(S - 3c) f_3 - 2h \sum \mu_i h_{ijk}^2 \\
& - \frac{2}{3} h^2 \sum h_{ijk}^2 - h^2 f_4 + ch^2 S \\
& = -2h \sum \mu_i h_{ijk}^2 - (S - 3c + 2h^2/3) \sum h_{ijk}^2
\end{aligned}$$

$$\begin{aligned}
& + (S - 3c) \{ (S - 3c)S + ch^2 \} \\
& - h^2 (-h^2S + h^4/6 + 4hf_3/3 + S^2/2) + ch^2S \\
& = -2h \sum \mu_i h_{ijk}^2 - (S - 3c - 2h^2/3) \sum h_{ijk}^2 \\
& + (S - h^2/3)(S - 3c - 3h^2/4 + \sqrt{h^4 + 8ch^2}/4) \\
& \times (S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2}/4),
\end{aligned}$$

Thus we prove the equality (2.19). From (2.13) and (2.19) we have

$$\begin{aligned}
\sum h_{ijkl}^2 &= \frac{1}{2} \Delta \sum h_{ijk}^2 + (S - 9c) \sum h_{ijk}^2 \\
&- h^2 \sum h_{ijk}^2 - 3h \sum \mu_i h_{ijk}^2 - 5 \{ \sum \lambda_i^2 h_{ijk}^2 \\
&+ 2 \sum \lambda_i \lambda_j h_{ijk}^2 \} + 4 \{ 2 \sum \lambda_i^2 h_{ijk}^2 + \sum \lambda_i \lambda_j h_{ijk}^2 \} \\
&= \frac{1}{2} \Delta \sum h_{ijk}^2 + (S - 9c - h^2) \sum h_{ijk}^2 \\
&- 3h \sum \mu_i h_{ijk}^2 - 5 \left\{ \frac{h^2}{3} \sum h_{ijk}^2 + 2h \sum \mu_i h_{ijk}^2 \right. \\
&+ \left. \frac{1}{3} \sum (\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 \right\} + 4 \left\{ -\frac{1}{3} \Delta \sum h_{ijk}^2 \right. \\
&+ \left. (S - 3c)f_4 - hf_5 + chf_3 \right\} \text{ (by (2.22) and (2.23))} \\
&= -\frac{5}{3} \sum (\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 - 13h \sum \mu_i h_{ijk}^2 \\
&+ (S - 9c - 8h^2/3) \sum h_{ijk}^2 - \frac{5}{6} \Delta \sum h_{ijk}^2 \\
&+ 4(S - 3c)(-h^2S + h^4/6 + 4hf_3/3 + S^2/2) \\
&- 4h \{ 5(S + h^2)f_3/6 + h^5/6 - 5h^3S/6 \} + 4chf_3 \\
&= -\frac{5}{3} \sum (\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 - 8h \sum \mu_i h_{ijk}^2 \\
&- \frac{5}{2} (S - h^2/3)(S - 3c - 3h^2/4 + \sqrt{h^4 + 8ch^2}/4)
\end{aligned}$$

$$\begin{aligned}
& \times (S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2}/4) \\
& + (7S/2 - 33c/2 - 13h^2/3) \sum h_{ijk}^2 \\
& + 4(S - 3c)(-h^2S + h^4/6 + S^2/2) - 4h(h^5/6 - 5h^3S/6) \\
& + \left\{ \frac{16}{3}h(S - 3c) + 4ch - \frac{10}{3}h(S + h^2) \right\} f_3 \\
& = -\frac{5}{3} \sum (\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 \\
& - 8h \sum \mu_i h_{ijk}^2 + \frac{3}{2}(S - 3c - 2h^2/3) \sum h_{ijk}^2 \\
& + \frac{3}{2}(S - h^2/3)(S - 3c - 3h^2/4 + \sqrt{h^4 + 8ch^2}/4) \\
& \times (S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2}/4).
\end{aligned}$$

Hence we prove that Proposition 2.2 is true.

The following Generalized Maximum Principle due to Omori [13] and Yau [19], Cartan's equality on the principal curvatures of the isoparametric hypersurfaces and an inequality will be often used in this paper.

Generalized Maximum Principle (cf. Omori [13] and Yau [19]). *Let M be an n -dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2 -function bounded from above on M , then there exists a sequence $\{p_m\}$ of points in M such that*

$$\lim_{m \rightarrow \infty} F(p_m) = \sup F, \quad \lim_{m \rightarrow \infty} |\text{grad} F(p_m)| = 0, \quad \lim_{m \rightarrow \infty} \sup \Delta F(p_m) \leq 0.$$

Lemma 2.1. (cf. Cartan [1]). *Let M be an isoparametric hypersurface in a space form $M^{n+1}(c)$. Let $\lambda_1, \dots, \lambda_p$ be all distinct constant principal curvatures with multiplicities $m_1, \dots, m_p$, respectively. Then we have, for any i ,*

$$\sum_{j \neq i} m_j \frac{c + \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0.$$

Thus we see that $p \leq 2$ if $c \leq 0$.

Sublemma (cf. Okumura [12]). *Let $a_1, \dots, a_n$ be real numbers satisfying*

$\sum a_i = 0$ and $\sum a_i^2 = k^2$ for $k > 0$. Then

$$|\sum a_i^3| \leq \frac{n-2}{\{n(n-1)\}^{1/2}} k^3.$$

The equality holds if and only if $n-1$ of them are equal with each other.

3. Proof of Main Theorem

This section is devoted to the proof of Main Theorem. In order to prove the Main Theorem, we prove the following Theorem 3.1 and Theorem 3.2 at first.

Theorem 3.1. *A complete hypersurface M of $M^4(c)$ ($c \geq 0$) with nonzero constant mean curvature H and constant scalar curvature is isoparametric if*

$$3c + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8ch^2} \leq S \leq 1.2838h^2 + 6c,$$

where S denote the squared norm of the second fundamental form and $h = 3H$.

Proof. Since H is nonzero constant, we may suppose $H > 0$. From (2.11) we know that the scalar curvature is constant if and only if S is constant. According to (2.10) we get that the Ricci curvature is bounded from below since S is constant. If B_3 is constant, then (2.16) means that f_3 is constant. Because H and S are constant, we obtain that the principal curvatures of M are constant. Hence M is isoparametric. Next we shall assert B_3 and f_3 must be constant. Otherwise we will obtain a contradiction. In order to prove the assertion, we consider $\inf B_3 \cdot \sup B_3$ in the following three cases:

- (1) $\inf B_3 \cdot \sup B_3 = 0$;
- (2) $\inf B_3 \cdot \sup B_3 > 0$;
- (3) $\inf B_3 \cdot \sup B_3 < 0$.

In the first case $\inf B_3 \cdot \sup B_3 = 0$. If $\inf B_3 = \sup B_3 = 0$, then $B_3 = 0$ and (2.16) means that f_3 is constant. Hence the assertion is true. Next without loss of generality, we may suppose $\sup B_3 = 0$. In fact if $\inf B_3 = 0$, we can give the same proof by using $\inf B_3 = 0$. Applying the Generalized Maximum Principle due to Omori and Yau in section 2 to B_3 , there exists a sequence $\{p_m\}$ of points in M with the following properties:

$$(3.1) \quad \lim_{m \rightarrow \infty} B_3(p_m) = \sup B_3 = 0, \quad \lim_{m \rightarrow \infty} |\text{grad } B_3(p_m)| = 0,$$

$$(3.2) \quad \lim_{m \rightarrow \infty} \sup \Delta B_3(p_m) \leq 0.$$

By (2.16) we have

$$(3.3) \quad \lim_{m \rightarrow \infty} f_3(p_m) = \sup f_3 = hS - \frac{2}{9}h^3, \quad \lim_{m \rightarrow \infty} |\text{grad } f_3(p_m)| = 0,$$

$$(3.4) \quad \lim_{m \rightarrow \infty} \sup \Delta f_3(p_m) \leq 0.$$

Since λ_i , h_{ijk} and h_{ijkl} are all bounded from $S = \text{constant}$, (2.12) and Proposition 2.2, we may consider, by taking a subsequence of $\{p_m\}$ if necessary, that

$$(3.5) \quad \begin{cases} \lim_{m \rightarrow \infty} \lambda_i(p_m) = \bar{\lambda}_i, \\ \lim_{m \rightarrow \infty} h_{ijk}(p_m) = \bar{h}_{ijk}, \\ \lim_{m \rightarrow \infty} h_{ijkl}(p_m) = \bar{h}_{ijkl}. \end{cases}$$

Thus we get

$$(3.6) \quad \lim_{m \rightarrow \infty} \mu_i(p_m) = \bar{\lambda}_i - H =: \bar{\mu}_i,$$

$$(3.7) \quad \sum \bar{\mu}_i = 0, \quad \sum \bar{\mu}_i^2 = S - h^2/3, \quad \sum \bar{\mu}_i^3 = 0.$$

Setting $B = \sum \bar{\mu}_i^2$, we have

$$(3.8) \quad \bar{\mu}_1 = -\sqrt{\frac{B}{2}}, \quad \bar{\mu}_2 = 0, \quad \bar{\mu}_3 = \sqrt{\frac{B}{2}}.$$

Here we assume that $\bar{\mu}_1 \leq \bar{\mu}_2 \leq \bar{\mu}_3$. From $\sum \mu_i = 0$ and $\sum \mu_i^2 = S - h^2/3$ being constant we get

$$(3.10) \quad \sum \bar{h}_{iik} = 0 \quad \text{for every } k,$$

$$(3.11) \quad \sum \bar{\mu}_i \bar{h}_{iik} = 0 \quad \text{for every } k.$$

According to (3.1), we have

$$(3.12) \quad \sum \bar{\mu}_i^2 \bar{h}_{iik} = 0 \quad \text{for every } k,$$

Since $\bar{\mu}_1$, $\bar{\mu}_2$ and $\bar{\mu}_3$ are distinct, we get from (3.10), (3.11) and (3.12)

$$(3.13) \quad \bar{h}_{iik} = 0 \quad \text{for any } i \text{ and } k$$

(3.7) and (3.13) yield

$$(3.14) \quad \sum \bar{\mu}_i \bar{h}_{ijk}^2 = 0, \quad \sum (\bar{\mu}_i + \bar{\mu}_j + \bar{\mu}_k)^2 \bar{h}_{ijk}^2 = 0.$$

According to (2.16) and (2.21) we have

$$-h\Delta B_3 = \Delta \sum h_{ijk}^2.$$

Thus Proposition 2.2, (3.2), (3.14) and the above equality imply

$$\begin{aligned} 0 \leq & -(S - 3c - 2h^2/3) \sum \bar{h}_{ijk}^2 \\ & + (S - h^2/3)(S - 3c - 3h^2/4 + \sqrt{h^4 + 8ch^2}/4) \\ & \times (S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2}/4). \end{aligned}$$

On the other hand since $\lim_{m \rightarrow \infty} f_3(p_m) = \sup f_3 = hS - \frac{2}{9}h^3$, we get

$$\sum \bar{h}_{ijk}^2 = \left(S - \frac{1}{3}h^2\right) \left(S - 3c - \frac{2}{3}h^2\right).$$

Hence we conclude

$$\begin{aligned} 0 \leq & -\left(S - \frac{1}{3}h^2\right) (S - 3c - 2h^2/3)^2 \\ & + (S - h^2/3)(S - 3c - 3h^2/4 + \sqrt{h^4 + 8ch^2}/4) \\ & \times (S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2}/4) \\ = & (S - h^2/3) \left[(S - 3c - 3h^2/4)^2 - \frac{1}{16}(h^4 + 8ch^2) \right] \\ & - \left(S - \frac{1}{3}h^2\right) (S - 3c - 2h^2/3)^2 \\ < & -(S - h^2/3) \frac{1}{16}(h^4 + 8ch^2) < 0. \end{aligned}$$

This is a contradiction. This completes the proof in the first case. In this second case $\inf B_3 \cdot \sup B_3 > 0$. Hence $\inf B_3$ and $\sup B_3$ have the same sign. Without loss of generality, we may assume that $\sup B_3 < 0$. Indeed, if $\sup B_3 > 0$, then $\inf B_3 > 0$. Making use of $\inf B_3 > 0$ we can give the same

proof. From the Sublemma we see that

$$(3.15) \quad 0 > \sup B_3 > -\frac{1}{\sqrt{6}} \left(S - \frac{1}{3} h^2 \right)^{3/2}.$$

Applying the Generalized Maximum Principle due to Omori and Yau to B_3 , there exists a sequence $\{p_m\}$ of points in M with the following properties:

$$\lim_{m \rightarrow \infty} B_3(p_m) = \sup B_3, \quad \lim_{m \rightarrow \infty} |\text{grad } B_3(p_m)| = 0,$$

$$\lim_{m \rightarrow \infty} \sup \Delta B_3(p_m) \leq 0.$$

By the same discussion as developed in the first case of proof of Theorem 3.1, we see that (3.4), (3.13) and (3.14) are valid from (3.15) and Sublemma. From (2.12), (2.14) and (2.16) we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \sup \Delta f_3(p_m) &\geq 6 \sum \bar{\mu}_i \bar{h}_{ijk}^2 \\ &\quad - 3 \left(S - 3c - \frac{2}{3} h^2 \right) \left(hS - \frac{2}{9} h^3 \right) \\ &\quad + h \left(\frac{7}{2} S^2 - 9cS + 2c^2 h^2 - 3h^2 S + \frac{1}{2} h^4 \right). \end{aligned}$$

This means from (3.4) and (3.14) that

$$0 \geq h \left(S - \frac{1}{3} h^2 \right)^2 > 0.$$

This is impossible. That is, the second case does not occur. In the final case $\inf B_3 \cdot \sup B_3 < 0$. Then there exists a point p in M such that $B_3(p) = 0$. From (2.16) we have

$$(3.16) \quad f_3(p) = \left(hS - \frac{2}{9} h^3 \right).$$

At point p we have

$$\sum \mu_i = 0, \quad \sum \mu_i^2 = S - \frac{1}{3} h^2, \quad \sum \mu_i^3 = 0.$$

Hence we obtain

$$(3.17) \quad \begin{aligned} \mu_1 &= -\sqrt{\frac{B}{2}}, \mu_2 = 0, \mu_3 = \sqrt{\frac{B}{2}}. \\ \lambda_1 &= H - \sqrt{\frac{B}{2}}, \lambda_2 = H, \lambda_3 = H + \sqrt{\frac{B}{2}}. \end{aligned}$$

It follows from $\sum \lambda_i = 3H$ and $\sum \lambda_i^2 = S$ that

$$\begin{aligned} \sum h_{iik} &= 0 \quad \text{for every } k, \\ \sum \lambda_i h_{iik} &= 0 \quad \text{for every } k, \end{aligned}$$

Thus we get

$$(3.18) \quad h_{11k} = h_{33k}, \quad h_{22k} = -2h_{11k} \quad \text{for any } k.$$

On the other hand, because $B_3(p) = 0$ we have

$$(3.19) \quad \sum h_{ijk}^2 = \left(S - \frac{1}{3}h^2\right) \left(S - 3c - \frac{2}{3}h^2\right).$$

Then (3.18) and (3.19) imply that

$$\begin{aligned} \sum h_{ijk}^2 &= 6h_{123}^2 + 16h_{111}^2 + \frac{5}{2}h_{222}^2 + 16h_{333}^2 \\ &= \left(S - \frac{h^2}{3}\right) \left(S - 3c - \frac{2}{3}h^2\right). \end{aligned}$$

Therefore we have

$$\begin{aligned} (3.19') \quad \sum h_{ij2}^2 &= 2h_{123}^2 + 8h_{111}^2 + \frac{3}{2}h_{222}^2 + 8h_{333}^2 \\ &\geq \frac{1}{3} \sum h_{ijk}^2 + \frac{8}{3}h_{111}^2 = \frac{1}{3} \left(S - \frac{h^2}{3}\right) \left(S - 3c - \frac{2}{3}h^2\right) + \frac{8}{3}h_{111}^2. \end{aligned}$$

Letting $t_{ij} = h_{ijij} - h_{jiji}$. From Ricci formula we have

$$t_{ij} = h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)(c + \lambda_i \lambda_j).$$

$$(3.19'') \quad \sum t_{ij}^2 = \sum (h_{ijij} - h_{jiji})^2$$

$$\begin{aligned}
&= \sum (\lambda_i - \lambda_j)^2 (c + \lambda_i \lambda_j)^2 \\
&= 2 \left(S - \frac{1}{3} h^2 \right) \left(\frac{1}{2} S^2 - \frac{1}{2} h^2 S \right. \\
&\quad \left. + \frac{4}{27} h^4 - 2cS + \frac{4}{3} ch^2 + 3c^2 \right),
\end{aligned}$$

$$\begin{aligned}
(3.19''') \quad & \frac{1}{4} \sum_{i \neq j} (h_{ijij} + h_{jiij})^2 \\
& \geq \frac{1}{2} \{ (h_{1212} + h_{2121})^2 + (h_{2323} + h_{3232})^2 \} \\
& \geq \left[h_{1212} - h_{3232} - t_{23} - \frac{1}{2} (t_{12} - t_{23}) \right]^2.
\end{aligned}$$

By differentiating $S = \sum h_{ij}^2$, we obtain

$$\sum h_{ijk}^2 + \sum \lambda_i h_{iikk} = 0 \quad \text{for every } k.$$

Accordingly,

$$\sum h_{ij2}^2 = -\sqrt{\frac{B}{2}} (h_{3322} - h_{1212}),$$

$$t_{12} - t_{23} = \frac{1}{3} hB.$$

$$t_{23} = -\left(c \sqrt{\frac{B}{2}} + \frac{1}{9} h^2 \sqrt{\frac{B}{2}} + \frac{1}{6} hB \right).$$

Thus we get

$$\begin{aligned}
(3.19''') \quad & \frac{1}{4} \sum_{i \neq j} (h_{ijij} + h_{jiij})^2 \\
& \geq \left[h_{1212} - h_{3232} - t_{23} - \frac{1}{2} (t_{12} - t_{23}) \right]^2 \\
& \geq 2B \left(\frac{1}{3} S - \frac{1}{2} c - \frac{1}{6} h^2 \right)^2 + \frac{32}{9} \left(B - \frac{3}{2} c - \frac{h^2}{6} \right) h_{111}^2.
\end{aligned}$$

From (3.19'') and (3.19'''), we have

(3.20)

$$\begin{aligned}
& \sum h_{ijkl}^2 \\
& \geq \frac{3}{4} \sum_{i \neq j} (h_{ijij} + h_{jiij})^2 + \frac{3}{4} \sum (h_{ijij} - h_{jiij})^2 \\
& = \frac{3}{4} \sum_{i \neq j} (h_{ijij} + h_{jiij})^2 + \frac{3}{4} \sum t_{ij}^2 \\
& \geq \frac{3}{2} \left(S - \frac{1}{3} h^2 \right) \left(\frac{1}{2} S^2 - \frac{1}{2} h^2 S + \frac{4}{27} h^4 \right. \\
& \quad \left. - 2cS + \frac{4}{3} ch^2 + 3c^2 \right) + 6B \left(\frac{1}{3} S - \frac{1}{2} c - \frac{1}{6} h^2 \right)^2 + \frac{32}{3} \left(B - \frac{3}{2} c - \frac{h^2}{6} \right) h_{111}^2.
\end{aligned}$$

(3.19), (3.20) and Proposition 2.2 yield

$$\begin{aligned}
(3.20') \quad 0 & \leq -\frac{5}{3} \sum (\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 - 8h \sum \mu_i h_{ijk}^2 - \frac{32}{3} \left(B - \frac{3}{2} c - \frac{h^2}{6} \right) h_{111}^2 \\
& \quad + \left[\frac{19}{12} S^2 - \left(\frac{17}{6} h^2 + 13c \right) S + \frac{37}{36} h^4 + 9ch^2 + 21c^2 \right] B.
\end{aligned}$$

From (3.17) and (3.18), we have

$$\begin{aligned}
& -\frac{5}{3} \sum (\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 \\
& = -\frac{5}{3} 3 \sum_{i \neq j} (2\mu_i + \mu_j)^2 h_{iij}^2 - \frac{5}{3} 9 \sum \mu_i^2 h_{iii}^2 \\
& \leq -20Bh_{111}^2, \\
& -8h \sum \mu_i h_{ijk}^2 = -8h \sum \frac{1}{3} (\mu_i + \mu_j + \mu_k) h_{ijk}^2 \\
& = -8h \sum_{i \neq j} (2\mu_i + \mu_j) h_{iij}^2 - 8h \sum \mu_i h_{iii}^2 \\
& \leq 32h \sqrt{\frac{B}{2}} h_{111}^2.
\end{aligned}$$

Hence

$$\begin{aligned}
& -\frac{5}{3}\sum(\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 - 8h\sum\mu_i h_{ijk}^2 - \frac{32}{3}\left(8 - \frac{3}{2}c - \frac{h^2}{6}\right)h_{111}^2 \\
& \leq 0
\end{aligned}$$

Thus we have

$$\frac{19}{12}S^2 - \left(\frac{17}{6}h^2 + 13c\right)S + \frac{37}{36}h^4 + 9ch^2 + 21c^2 \geq 0.$$

Define

$$F(S) := \frac{19}{12}S^2 - \left(\frac{17}{6}h^2 + 13c\right)S + \frac{37}{36}h^4 + 9ch^2 + 21c^2.$$

As a function of S , $F(S)$ reaches its only minimum at $S = \frac{6}{19}(\frac{17}{6}h^2 + 13c)$. Without loss of generality we may assume

$$3c + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8ch^2} \leq \frac{6}{19}\left(\frac{17}{6}h^2 + 13c\right) \leq 1.2838h^2 + 6c.$$

Because $F(S)$ has only one critical point $S = \frac{6}{19}(\frac{17}{6}h^2 + 13c)$ at which F get its minimum, then F get its maximum at boundary points. On the other hand, since $F(3c + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8ch^2}) < 0$ and $F(1.2838h^2 + 6c) < 0$, we obtain $F < 0$ if

$$3c + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8ch^2} \leq S \leq 1.2838h^2 + 6c.$$

This is a contradiction. Hence f_3 is constant. Thus we complete the proof of Theorem 3.1.

Theorem 3.2. *A complete hypersurface M of $M^4(c)$ ($c \geq 0$) with nonzero constant mean curvature H and constant scalar curvature is isoparametric if*

$$S < 3c + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8ch^2},$$

where S denote the squared norm of the second fundamental form and $h = 3H$.

Proof. Since H is nonzero constant, we may suppose $H > 0$. From (2.11) we know that the scalar curvature is constant if and only if S is constant. If

$$S < 3c + \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8ch^2},$$

then the theorem due to Cheng and Nakagawa [5] implies that M is totally umbilic. Hence we only consider the case that

$$(3.20) \quad \begin{aligned} 3c + \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8ch^2} \\ \leq S < 3c + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8ch^2}. \end{aligned}$$

If f_3 is constant, then M is of constant principal curvatures. Hence M is isoparametric. Next we will prove that f_3 must be constant. If f_3 is not constant, we will conclude a contradiction. In fact, since the Ricci curvature is bounded from below from $S = \text{constant}$, we can apply the Generalized Maximum Principle to $-\sum h_{ijk}^2$. Then there exists a sequence $\{p_m\}$ of points in M with the following properties:

$$(3.21) \quad \begin{cases} \lim_{m \rightarrow \infty} \sum h_{ijk}^2(p_m) = \inf \sum h_{ijk}^2, \\ \lim_{m \rightarrow \infty} |\text{grad} \sum h_{ijk}^2(p_m)| = 0, \\ \lim_{m \rightarrow \infty} \inf \Delta \sum h_{ijk}^2(p_m) \geq 0. \end{cases}$$

From (2.12) and (2.16) we have

$$\begin{aligned} \lim_{m \rightarrow \infty} B_3(p_m) &= \sup B_3, \quad \lim_{m \rightarrow \infty} |\text{grad} B_3(p_m)| = 0, \\ \lim_{m \rightarrow \infty} \sup \Delta B_3(p_m) &\leq 0. \end{aligned}$$

and

$$\begin{aligned} \lim_{m \rightarrow \infty} f_3(p_m) &= \sup f_3, \quad \lim_{m \rightarrow \infty} |\text{grad} f_3(p_m)| = 0, \\ \lim_{m \rightarrow \infty} \sup \Delta f_3(p_m) &\leq 0. \end{aligned}$$

Suppose that $\lim_{m \rightarrow \infty} B_3(p_m) = -\frac{1}{\sqrt{6}}(S - \frac{1}{3}h^2)^{3/2}$. Then Sublemma implies that $B_3 \equiv -\frac{1}{\sqrt{6}}(S - \frac{1}{3}h^2)^{3/2}$. Hence f_3 is constant. This is a contradiction. This argument shows that $\lim_{m \rightarrow \infty} B_3(p_m) > -\frac{1}{\sqrt{6}}(S - \frac{1}{3}h^2)^{3/2}$. Then we shall assert that $\lim_{m \rightarrow \infty} B_3(p_m) = \frac{1}{\sqrt{6}}(S - \frac{1}{3}h^2)^{3/2}$. To show this assertion we suppose that

$\lim_{m \rightarrow \infty} B_3(p_m) < \frac{1}{\sqrt{6}}(S - \frac{1}{3}h^2)^{3/2}$. Thus since λ_i , h_{ijk} and h_{ijkl} are all bounded from $S = \text{constant}$, (2.12) and Proposition 2.2, we may consider, by taking a subsequence of $\{p_m\}$ if necessary, that

$$\begin{cases} \lim_{m \rightarrow \infty} \lambda_i(p_m) = \bar{\lambda}_i, \\ \lim_{m \rightarrow \infty} h_{ijk}(p_m) = \bar{h}_{ijk}, \\ \lim_{m \rightarrow \infty} h_{ijkl}(p_m) = \bar{h}_{ijkl}. \end{cases}$$

Hence $\bar{\lambda}_i$ are all distinct from Sublemma. Making use of the same proof technique developed in the first case in Theorem 3.1, we can get

$$\sum \bar{\mu}_i \bar{h}_{ijk}^2 = 0. \quad \sum (\bar{\mu}_i + \bar{\mu}_j + \bar{\mu}_k)^2 \bar{h}_{ijk}^2 = 0.$$

Thus (3.21) and Proposition 2.2 imply

$$\begin{aligned} 0 \leq & -(S - 3c - 2h^2/3) \sum \bar{h}_{ijk}^2 + (S - h^2/3)(S - 3c - 3h^2/4 \\ & + \sqrt{h^4 + 8ch^2}/4)(S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2}/4). \end{aligned}$$

Hence

$$\begin{aligned} (3.22) \quad \sum \bar{h}_{ijkl}^2 &= \frac{3}{2}(S - 3c - 2h^2/3) \sum \bar{h}_{ijk}^2 \\ &+ \frac{3}{2}(S - h^2/3)(S - 3c - 3h^2/4 + \sqrt{h^4 + 8ch^2}/4) \\ &\times (S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2}/4) \\ &\leq 3(S - h^2/3)(S - 3c - 3h^2/4 + \sqrt{h^4 + 8ch^2}/4) \\ &\times (S - 3c - 3h^2/4 - \sqrt{h^4 + 8ch^2}/4) \leq 0, \end{aligned}$$

from the assumption of (3.20). On the other hand,

$$\begin{aligned} \sum h_{ijkl}^2 &\geq \frac{3}{4} \sum_{i \neq j} (h_{ijij} + h_{jiji})^2 + \frac{3}{4} \sum (h_{ijij} - h_{jiji})^2 \\ &\geq \frac{3}{4} \sum (h_{ijij} - h_{jiji})^2. \end{aligned}$$

Hence we have

$$\begin{aligned}\sum \bar{h}_{ijkl}^2 &\geq \frac{3}{4} \sum_{i \neq j} (\bar{h}_{ijij} - \bar{h}_{jiij})^2 \\ &= \frac{3}{4} \sum (\lambda_i - \lambda_j)^2 (c + \lambda_i \lambda_j)^2 > 0\end{aligned}$$

since $\bar{\lambda}_i$ are all distinct. This is in contradiction with (3.22). Hence we have

$$(3.23) \quad \lim_{m \rightarrow \infty} B_3(p_m) = \frac{1}{\sqrt{6}} \left(S - \frac{1}{3} h^2 \right)^{3/2}.$$

Thus (2.12), (2.16) and (3.23) yield

$$\begin{aligned}\sum \bar{h}_{ijk}^2 &= S(S - 3c) + h^2 c - h \lim_{m \rightarrow \infty} f_3(p_m) \\ &= \left(S - \frac{1}{3} h^2 \right) \left\{ \sqrt{S - \frac{h^2}{3}} - \frac{h}{2\sqrt{6}} + \sqrt{\frac{3}{8} h^2 + 3c} \right\} \\ &\quad \times \left\{ \sqrt{S - \frac{h^2}{3}} - \frac{h}{2\sqrt{6}} - \sqrt{\frac{3}{8} h^2 + 3c} \right\}.\end{aligned}$$

From the assumption of Theorem 3.2, we have

$$\sqrt{S - \frac{h^2}{3}} - \frac{h}{2\sqrt{6}} - \sqrt{\frac{3}{8} h^2 + 3c} < 0.$$

Therefore we obtain

$$\sqrt{S - \frac{h^2}{3}} - \frac{h}{2\sqrt{6}} + \sqrt{\frac{3}{8} h^2 + 3c} \leq 0$$

Thus we have

$$S \leq 3c + 3h^2/4 - \sqrt{h^4 + 8ch^2}/4.$$

Hence

$$S = 3c + 3h^2/4 - \sqrt{h^4 + 8ch^2}/4$$

and

$$\lim_{m \rightarrow \infty} B_3(p_m) = \frac{1}{\sqrt{6}} \left(S - \frac{1}{3} h^2 \right)^{3/2}.$$

If

$$B_3 = \frac{1}{\sqrt{6}} \left(S - \frac{1}{3} h^2 \right)^{3/2},$$

then f_3 is constant. Hence this is a contradiction. If $\inf B_3 < 0$, then there exists a point p such that $B_3(p) = 0$. Hence we have

$$\sum h_{ijk}^2(p) = \left(S - \frac{1}{3} h^2 \right) \left(S - 3c - \frac{2}{3} h^2 \right).$$

Thus we have from

$$S = 3c + \frac{3}{4} h^2 - \frac{1}{4} \sqrt{h^4 + 8ch^2}$$

$$\sum h_{ijk}^2(p) = \left(S - \frac{1}{3} h^2 \right) \left(\frac{1}{12} h^2 - \frac{1}{4} \sqrt{h^4 + 8ch^2} \right) < 0.$$

This is impossible. If $\inf B_3 = 0$, applying the Generalized Maximum Principle due to Omori and Yau to $-B_3$, there exists a sequence $\{p_m\}$ of points in M with the following properties:

$$\lim_{m \rightarrow \infty} B(p_m) = \inf B_3 = 0, \quad \lim_{m \rightarrow \infty} |\text{grad } B_3(p_m)| = 0,$$

$$\lim_{m \rightarrow \infty} \inf \Delta B_3(p_m) \geq 0.$$

Making use of the same proof as that in the first case of Theorem 3.1, we can obtain

$$\begin{aligned} \sum \bar{h}_{ijk}^2(p) &= \left(S - \frac{1}{3} h^2 \right) \left(S - 3c - \frac{2}{3} h^2 \right) \\ &= \left(S - \frac{1}{3} h^2 \right) \left(\frac{1}{12} h^2 - \frac{1}{4} \sqrt{h^4 + 8ch^2} \right) < 0 \end{aligned}$$

This is also impossible. If $\inf B_3 > 0$, applying the Generalized Maximum Principle due to Omori and Yau to $-B_3$, there exists a sequence $\{p_m\}$ of points in M with the following properties:

$$\lim_{m \rightarrow \infty} B_3(p_m) = \inf B_3, \quad \lim_{m \rightarrow \infty} |\text{grad } B_3(p_m)| = 0,$$

$$\lim_{m \rightarrow \infty} \inf \Delta B_3(p_m) \geq 0.$$

Making use of (2.12) and (2.16), we can obtain that

$$(3.24) \quad \begin{cases} \lim_{m \rightarrow \infty} \sum h_{ijk}^2(p_m) = \sup \sum h_{ijk}^2, \\ \lim_{m \rightarrow \infty} |\text{grad} \sum h_{ijk}^2(p_m)| = 0, \\ \lim_{m \rightarrow \infty} \sup \Delta \sum h_{ijk}^2(p_m) \leq 0 \end{cases}$$

and (3.5) hold. Making use of the same proof as in the first case of Theorem 3.1, we can obtain that (3.14) is valid since $\bar{\mu}_i$ are all distinct from $\inf B_3 > 0$ and Sublemma. Proposition 2.2, (3.14) and (3.24) imply

$$S - 3c - \frac{2}{3}h^2 \geq 0.$$

Hence

$$\frac{1}{12}h^2 - \frac{1}{4}\sqrt{h^4 + 8ch^2} \geq 0.$$

This is a contradiction. Hence B_3 must be constant. Thus we complete the proof of Theorem 3.2.

Proof of Main Theorem. Since the scalar curvature is not less than $-2.5542H^2$ if and only if $S \leq 1.2838h^2 + 6c$ from (2.11). If $H = 0$, we know that M is minimal. According to the results due to Chang [20], Cheng [3], we know that Theorem is true. If $H \neq 0$, Theorem 3.1 and Theorem 3.2 yield that M is isoparametric. From the classifications on the Isoparametric hypersurfaces with at most three distinct principal curvatures due to Cartan, we know that Main Theorem is true.

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