TRANSITION PROBABILITY OF A CERTAIN STOCHASTIC PROCESS

Dedicated to Professor T. Otsuki on his 77th Birthday

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§1. Introduction

We are concerned with a stochastic process $(X_t, \mathcal{F}_t^X)$, which is defined by the equality

$$X_t(\omega) = \int_0^t (-1)^{N_s(\omega)} ds,$$

where $(N_t, \mathcal{F}_t^N)$ is the Poisson process. The Poisson process $(N_t, \mathcal{F}_t^N)$ with intensity $\lambda > 0$ is defined by following two conditions.

- (1) Random variables $N_{t_{i+1}} - N_{t_i}$, $i = 0, 1, 2, 3, \dots, n-1$, are independent for any $0 \leq t_0 < t_1 < t_2 < t_3 < \dots < t_{n-1} < t_n$ and $N_0 = 0$ a.s..
- (2) The equality

$$\mathcal{P}(\omega | N_t(\omega) - N_s(\omega) = m) = \frac{e^{-\lambda(t-s)}}{m!} (t-s)^m \lambda^m,$$

holds for any two real numbers t, s such that $t > s \geq 0$ and any nonnegative integer m .

Kac [8] researches the telegrapher's equation by using the Poisson process and the stochastic process $(X_t, \mathcal{F}_t^X)$ appears in works of many authors (Gaveau-Jacobson-Kac-Schulman [1], Ichinose [2], [3], Ichinose-Tamura [4], [5], [7]) related with path integral representations for two dimensional space-time Dirac equations for the past ten years.

In this paper, the author gives an explicit form of transition probability of X_t , that is

$$\mathcal{P}(\omega | X_t(\omega) \in [x, x + dx]) = p_t(x) dx,$$

and some fundamental properties of this measure. We shall show that the function $p_t(x)$ is given by the following equality;

(1-1)

$$\begin{aligned} e^{\lambda t} p_t(x) &= \delta(t - x) \\ &+ \left\{ \sum_{n=0}^{\infty} \left(\frac{\lambda}{2} \right)^{2n+1} \frac{1}{(n!)^2} (t - x)^n (t + x)^n \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \left(\frac{\lambda}{2} \right)^{2n+2} \frac{1}{n!(n+1)!} (t - x)^n (t + x)^{n+1} \right\} \theta(t^2 - x^2), \end{aligned}$$

where θ is the Heaviside function. Throughout the paper, we fix a probability space $(\mathcal{P}, \mathcal{F}, \Omega)$ and use a symbol ω for a sample point i.e. $\omega \in \Omega$.

§2. Transition Probability of X_t

We define integral-valued functions $\mathcal{L}_2(t, \omega)$ and $\mathcal{N}(t, \omega)$ by equalities;

$$\mathcal{L}_2(t, \omega) = \lim_{\varepsilon \rightarrow 0+} \frac{1}{2} |\dot{X}(t + \varepsilon, \omega) - \dot{X}(t - \varepsilon, \omega)| = \begin{cases} 0 \\ 1 \end{cases}$$

and

$$\mathcal{N}(t, \omega) = \#(s | \mathcal{L}_2(s, \omega) = 1, 0 \leq s \leq t),$$

where we set

$$\dot{X}(t, \omega) = \frac{dX(t, \omega)}{dt}.$$

We define sets $\Omega_n(t) \subseteq \Omega$ by $\Omega_n(t) = (\omega | \mathcal{N}(t, \omega) = n)$ and sets $\Omega_n(s, t) \subseteq \Omega$ by $\Omega_n(s, t) = (\omega | \mathcal{N}(t, \omega) - \mathcal{N}(s, \omega) = n)$ for any $0 \leq s \leq t$, and we get

LEMMA 2.1. *We have*

$$\mathcal{P}(\Omega_0(t)) = e^{-\lambda t}.$$

PROOF OF LEMMA 2.1. For any $t > s \geq 0$, we have

$$\mathcal{P}(\omega | N_t(\omega) - N_s(\omega) = m) = \frac{e^{-\lambda(t-s)}}{m!} (t-s)^m \lambda^m$$

and from this equality, we get

$$\mathcal{P}(\omega | N_t(\omega) - N_s(\omega) = \text{even}) = e^{-\lambda(t-s)} \cos(i\lambda(t-s)).$$

$\Omega_0(t)$ is redefined by

$$\Omega_0(t) = (\omega | N_{t_1}(\omega) - N_{t_2}(\omega) = \text{even}, \text{ for any } t \geq t_1 \geq t_2 \geq 0)$$

and we get

$$\begin{aligned} \Omega_0(0, t) &= \Omega_0\left(0, \frac{t}{2}\right) \cap \Omega_0\left(\frac{t}{2}, t\right) \\ &= \Omega_0\left(0, \frac{t}{4}\right) \cap \Omega_0\left(\frac{t}{4}, \frac{2t}{4}\right) \cap \Omega_0\left(\frac{2t}{4}, \frac{3t}{4}\right) \cap \Omega_0\left(\frac{3t}{4}, t\right) \\ &= \bigcap_{k=1}^{k=2^n} \Omega_0\left(\frac{k-1}{2^n}t, \frac{k}{2^n}t\right), \end{aligned}$$

where n is a positive integer. We have

$$\begin{aligned} \mathcal{P}(\omega | N_{\frac{k}{2^n}t}(\omega) - N_{\frac{k-1}{2^n}t}(\omega) = 0) &\leq \mathcal{P}\left(\Omega_0\left(\frac{k-1}{2^n}t, \frac{k}{2^n}t\right)\right) \\ &\leq \mathcal{P}(\omega | N_{\frac{k}{2^n}t}(\omega) - N_{\frac{k-1}{2^n}t}(\omega) = \text{even}), \end{aligned}$$

and these inequalities enable us to get

$$e^{-\frac{\lambda}{2^n}t} \leq \mathcal{P}\left(\Omega_0\left(\frac{k-1}{2^n}t, \frac{k}{2^n}t\right)\right) \leq e^{-\frac{\lambda}{2^n}t} \cos\left(i\frac{\lambda t}{2^n}\right).$$

Using these enaqualities, we get

$$\prod_{k=1}^{2^n} e^{-\frac{\lambda}{2^n}t} \leq \prod_{k=1}^{2^n} \mathcal{P}\left(\Omega_0\left(\frac{k-1}{2^n}t, \frac{k}{2^n}t\right)\right) \leq \prod_{k=1}^{2^n} e^{-\frac{\lambda}{2^n}t} \cos\left(i\frac{\lambda t}{2^n}\right).$$

Letting n to infinity in above inequalities, we get

$$\mathcal{P}(\Omega_0(0, t)) = e^{-\lambda t}.$$

This is what we want to prove. $\square$

We want to determine the explicit form of the transition probability density function $p_t(x)$, which is given by the following

THEOREM 2.1. Suppose that a stochastic process $(X_t, \mathcal{F}_t^X)$ is defined by

$$X_t(\omega) = \int_0^t (-1)^{N_s(\omega)} ds,$$

where $(N_t, \mathcal{F}_t^N)$ is the Poisson stochastic process with intensity λ . Then the density function $p_t(x)$ of X_t is given by (1-1).

PROOF OF THEOREM 2.1. We must prove

$$\begin{aligned} (\mathcal{A}_n) \quad & \mathcal{P}(\Omega_{2n+1}(t) \cap (\omega | X_t(\omega) \in [x, x + dx])) \\ &= e^{-\lambda t} \left\{ \left(\frac{\lambda}{2} \right)^{2n+1} \frac{1}{(n!)^2} (t-x)^n (t+x)^n \right\} \theta(t^2 - x^2) \end{aligned}$$

and

$$\begin{aligned} (\mathcal{B}_n) \quad & \mathcal{P}(\Omega_{2n+2}(t) \cap (\omega | X_t(\omega) \in [x, x + dx])) \\ &= e^{-\lambda t} \left\{ \left(\frac{\lambda}{2} \right)^{2n+2} \frac{1}{n!(n+1)!} (t-x)^n (t+x)^{n+1} \right\} \theta(t^2 - x^2). \end{aligned}$$

These equalities are established by induction.

Figure 1.

Figure 2.

Step 1. First of all, we must prove $(\mathcal{A}_0)$. By the definition of the Poisson process, we have

$$\mathcal{P}(\omega | N_{t_2}(\omega) - N_{t_1}(\omega) = 1) = e^{-\lambda(t_2 - t_1)} \lambda(t_2 - t_1),$$

for any $t_2 \geq t_1 \geq 0$. Using this equality, we get

$$\begin{aligned} & \mathcal{P}(\Omega_1(0, t) \cap (\omega | X_t(\omega) \in [x, x + \xi])) \\ &= \mathcal{P}\left(\Omega_0(0, s) \cap \Omega_1\left(s, s + \frac{\xi}{2}\right) \cap \Omega_0\left(s + \frac{\xi}{2}, t\right)\right) = e^{-\lambda t} \left(\frac{\lambda}{2}\right) \xi, \end{aligned}$$

for any ξ and x such that $0 < \xi \leq t - |x|$ and $0 \leq |x| \leq t$, where we set $s = \frac{t+x}{2}$. From these equalities, we get $(\mathcal{A}_0)$, that is

$$\mathcal{P}(\Omega_1(0, t) \cap (\omega | X_t(\omega) \in [x, x + dx])) = e^{-\lambda t} \left(\frac{\lambda}{2}\right) \theta(t^2 - x^2) dx.$$

Step 2. We want to derive $(\mathcal{B}_n)$ from $(\mathcal{A}_n)$. We define $\Omega_n^X(0, t, x, x + dx)$ by

$$(2-1) \quad \Omega_n^X(0, t, x, x + dx) = \Omega_n(0, t) \cap (\omega | X_t(\omega) \in [x, x + dx])$$

From figure 1, we have

$$\begin{aligned} & \mathcal{P}(\Omega_{2n+2}^X(0, t, x, x + dx)) \\ &= \int_s^t \mathcal{P}(\Omega_{2n+1}^X(0, t', x', x' + dx)) \mathcal{P}(\Omega_1(t', t' + dt')) \mathcal{P}(\Omega_0(t' + dt', t)) \\ &= \lambda \int_s^t e^{-\lambda(t-t')} \mathcal{P}(\Omega_{2n+1}^X(0, t', x', x' + dx)) dt' \\ &= \frac{1}{(n!)^2} \left(\frac{\lambda}{2}\right)^{2n+1} e^{-\lambda t} \lambda \left\{ \int_s^t (2t' - t + x)^n (t - x)^n dt' \right\} dx \\ &= \frac{1}{n!(n+1)!} \left(\frac{\lambda}{2}\right)^{2n+2} e^{-\lambda t} (t+x)^{n+1} (t-x)^n dx, \end{aligned}$$

where $t' - x' = t - x$, $2s = t - x$ and $0 \leq |x| \leq t$. From these equalities, we get

$$\begin{aligned} & \mathcal{P}(\Omega_{2n+2}^X(0, t, x, x + dx)) \\ &= \frac{1}{n!(n+1)!} \left(\frac{\lambda}{2}\right)^{2n+2} e^{-\lambda t} (t+x)^{n+1} (t-x)^n \theta(t^2 - x^2) dx. \end{aligned}$$

Step 3. Using figure 2, we can derive $(\mathcal{A}_{n+1})$ from $(\mathcal{B}_n)$ by the same argument as Step 2.

Summarizing Step 1, Step 2 and Step 3, we complete the proof of Theorem 2.1. $\square$

The following Theorem 2.2 gives the explicit forms of the expectation value $E(X_t)$ and the variance $V(X_t)$ of X_t .

THEOREM 2.2. *Let $(X_t, \mathcal{F}_t^X)$ be a stochastic process, which is defined by*

$$X_t(\omega) = \int_0^t (-1)^{N_s(\omega)} ds,$$

where $(N_t, \mathcal{F}_t^N)$ is the Poisson process with intensity λ . Then we have

$$E(X_t) = \frac{e^{-\lambda t}}{\lambda} \sinh(\lambda t)$$

and

$$V(X_t) = 4\lambda \int_0^t (E(X_s))^2 ds.$$

PROOF OF THEOREM 2.2. By easy calculations, we can get the following formulae of integral.

$$(2-2) \quad \int_{-t}^t x(t+x)^n(t-x)^n dx = 0,$$

$$(2-3) \quad \int_{-t}^t x(t+x)^{n+1}(t-x)^n dx = \frac{1}{n + \frac{3}{2}} \frac{(2n)!!}{(2n+1)!!} t^{2n+3},$$

$$(2-4) \quad \int_{-t}^t x^2(t+x)^n(t-x)^n dx = \frac{1}{n + \frac{3}{2}} \frac{(2n)!!}{(2n+1)!!} t^{2n+3},$$

$$(2-5) \quad \int_{-t}^t x^2(t+x)^{n+1}(t-x)^n dx = \frac{1}{n + \frac{3}{2}} \frac{(2n)!!}{(2n+1)!!} t^{2n+4},$$

where !! means the double factorial. Using (1-1), (2-2) and (2-3), we get

$$E(X_t) = \int_{-\infty}^{\infty} x p_t(x) dx = \frac{e^{-\lambda t}}{\lambda} \sum_{n=0}^{\infty} \frac{(\lambda t)^{2n+1}}{(2n+1)!} = \frac{e^{-\lambda t}}{\lambda} \sinh(\lambda t).$$

These complete the proof of the first part of Theorem 2.2. For the second part of Theorem 2.2, we remark that

$$(2-6) \quad V(X_t) = E(X_t^2) - (E(X_t))^2.$$

Using (2-4) and (2-5), we get

$$\begin{aligned} (2-7) \quad E(X_t^2) &= t^2 e^{-\lambda t} \left\{ 1 + \sum_{n=0}^{\infty} \frac{(\lambda t)^{2n+1}}{(2n+1)!} \frac{1}{2n+3} + \sum_{n=1}^{\infty} \frac{(\lambda t)^{2n}}{(2n+1)!} \right\} \\ &= t^2 e^{-\lambda t} \left\{ \sum_{n=0}^{\infty} \frac{(\lambda t)^{2n+1}}{(2n+1)!} \frac{1}{2n+3} + \sum_{n=0}^{\infty} \frac{(\lambda t)^{2n}}{(2n+1)!} \right\} \\ &= t^2 \frac{e^{-\lambda t}}{\lambda} \frac{d}{dt} \left\{ \frac{1}{\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right\} + t \frac{e^{-\lambda t}}{\lambda} \sum_{n=0}^{\infty} \frac{(\lambda t)^{2n+1}}{(2n+1)!} \\ &= t^2 \frac{e^{-\lambda t}}{\lambda} \frac{d}{dt} \left(\frac{\sinh(\lambda t)}{\lambda t} \right) + t \frac{e^{-\lambda t}}{\lambda} \sinh(\lambda t) = -\frac{1}{2\lambda^2} + \frac{e^{-2\lambda t}}{2\lambda^2} + \frac{t}{\lambda}. \end{aligned}$$

On the other hand, we have

$$(2-8) \quad (E(X_t))^2 = \frac{e^{-2\lambda t}}{\lambda^2} \sinh^2(\lambda t) = \frac{1}{4\lambda^2} - \frac{e^{-2\lambda t}}{2\lambda^2} + \frac{e^{-4\lambda t}}{4\lambda^2}.$$

Using (2-6), (2-7) and (2-8), we get

$$V(X_t) = E(X_t^2) - (E(X_t))^2 = -\frac{3}{4\lambda^2} + \frac{t}{\lambda} + \frac{e^{-2\lambda t}}{\lambda^2} - \frac{e^{-4\lambda t}}{4\lambda^2}.$$

To get the second part of Theorem 2.2, we differentiate the above equality with respect to t and we have

$$(2-9) \quad \frac{d}{dt} (V(X_t)) = \frac{1}{\lambda} - 2 \frac{e^{-2\lambda t}}{\lambda} + \frac{e^{-4\lambda t}}{\lambda} = 4\lambda \frac{e^{-2\lambda t}}{\lambda^2} \sinh^2(\lambda t) = 4\lambda (E(X_t))^2.$$

We remark that

$$(2-10) \quad V(X_0) = 0$$

and integrating the above equality in consideration of (2-10), we get

$$V(X_t) = 4\lambda \int_0^t (E(X_s))^2 ds.$$

This is what we wanted to prove. $\square$

Remark 2.1. We find the formulae of $E(X_t)$ and $E(X_t^2)$ at (34) and (39) in Kac [8].

§3. Kolmogorov Equation of X_t

We define a function $p(t; y, x)$ as the density function of $x + X_t(\omega)$, that means

$$(3-1) \quad \mathcal{P}(\omega | x + X_t(\omega) \in [y, y + dy]) = p(t; y, x) dy,$$

and we have

$$(3-2) \quad \begin{aligned} \mathcal{P}(\omega | x + X_t(\omega) \in [y, y + dy]) \\ = \mathcal{P}(\omega | X_t(\omega) \in [y - x, y - x + dy]) = p_t(y - x) dy. \end{aligned}$$

By (3-1) and (3-2), we get

$$(3-3) \quad p(t; y, x) = p_t(y - x).$$

We define another stochastic process $Y_t(\omega)$ and functions $q_t(x)$, $q(t; y, x)$ by

$$Y_t(\omega) = \int_0^t (-1)^{\mathcal{N}_s(\omega)+1} ds = -X_t(\omega),$$

$$\mathcal{P}(\omega | Y_t(\omega) \in [y, y + dy]) = q_t(y) dy,$$

and

$$(3-4) \quad \mathcal{P}(\omega | x + Y_t(\omega) \in [y, y + dy]) = q(t; y, x) dy$$

and get the following equalities.

$$(3-5) \quad \begin{aligned} \mathcal{P}(\omega | Y_t(\omega) \in [y, y + dy]) &= \mathcal{P}(\omega | -X_t(\omega) \in [y, y + dy]) \\ &= \mathcal{P}(\omega | X_t(\omega) \in [-y, -y + dy]) = p_t(-y) dy \end{aligned}$$

and

$$\begin{aligned}
 (3-6) \quad & \mathcal{P}(\omega | x + Y_t(\omega) \in [y, y + dy]) \\
 &= \mathcal{P}(\omega | X_t(\omega) \in [x - y, x - y + dy]) \\
 &= p_t(x - y) dy = p(t; x, y) dy.
 \end{aligned}$$

By (3-2), (3-4) and (3-6), we have

$$q_t(y) = p_t(-y), \quad q(t; y, x) = p(t; x, y).$$

We define other functions $p_t^+(y)$, $p_t^-(y)$, $q_t^+(y)$ and $q_t^-(y)$ by

$$(3-7) \quad \sum_{k=even} \mathcal{P}(\Omega_k^X(0, t, y, y + dy)) = p_t^+(y) dy,$$

$$(3-8) \quad \sum_{k=odd} \mathcal{P}(\Omega_k^X(0, t, y, y + dy)) = p_t^-(y) dy,$$

$$(3-9) \quad \sum_{k=even} \mathcal{P}(\Omega_k^Y(0, t, y, y + dy)) = p_t^+(y) dy$$

and

$$(3-10) \quad \sum_{k=odd} \mathcal{P}(\Omega_k^Y(0, t, y, y + dy)) = p_t^-(y) dy$$

and $p^+(t; y, x)$, $p^-(t; y, x)$, $q^+(t; y, x)$ and $q^-(t; y, x)$ by

$$(3-11) \quad p^+(t; y, x) = p_t^+(y - x), \quad p^-(t; y, x) = p_t^-(y - x)$$

and

$$(3-12) \quad q^+(t; y, x) = q_t^+(y - x), \quad q^-(t; y, x) = q_t^-(y - x).$$

We remark that the following equalities hold good.

$$\begin{aligned}
 (3-13) \quad & \sum_{k=even} \mathcal{P}(\Omega_k^X(0, t, y, y + dy)) + \sum_{k=odd} \mathcal{P}(\Omega_k^X(0, t, y, y + dy)) \\
 &= \mathcal{P}(\omega | X_t(\omega) \in [y, y + dy]) = p_t(y) dy
 \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=\text{even}} \mathcal{P}(\Omega_k^Y(0, t, y, y+dy)) + \sum_{k=\text{odd}} \mathcal{P}(\Omega_k^Y(0, t, y, y+dy)) \\ &= \mathcal{P}(\omega | Y_t(\omega) \in [y, y+dy]) = q_t(y) dy. \end{aligned}$$

By (3-7)–(3-13), we get

$$\begin{aligned} p_t(y) &= p_t^+(y) + p_t^-(y), \quad q_t(y) = q_t^+(y) + q_t^-(y), \\ p(t; y, x) &= p^+(t; y, x) + p^-(t; y, x), \\ q(t; y, x) &= q^+(t; y, x) + q^-(t; y, x), \end{aligned}$$

and

$$(3-14) \quad p^+(t; y, x) = q^+(t; x, y), \quad p^-(t; y, x) = q^-(t; x, y).$$

By $(\mathcal{A}_n), (\mathcal{B}_n), (2-1)$ and (3-7)–(3-13), explicit forms of $p^+(t; y, x)$ and $p^-(t; y, x)$ are given by the following equalities.

$$\begin{aligned} (3-15) \quad e^{\lambda t} p^+(t, y, x) &= \delta(t - (y - x)) \\ &+ \sum_{n=0}^{\infty} \left(\frac{\lambda}{2} \right)^{2n+2} \frac{1}{n!(n+1)!} (t - (y - x))^n (t + (y - x))^{n+1} \theta(t^2 - (y - x)^2) \end{aligned}$$

and

$$\begin{aligned} (3-16) \quad e^{\lambda t} p^-(t, y, x) &= \sum_{n=0}^{\infty} \left(\frac{\lambda}{2} \right)^{2n+1} \frac{1}{(n!)^2} (t - (y - x))^n (t + (y - x))^n \theta(t^2 - (y - x)^2). \end{aligned}$$

Now, we define a $M(2, \mathbf{R})$ valued function $\mathcal{Q}(t; y, x)$ by

$$\mathcal{Q}(t; y, x) = \begin{pmatrix} p^+(t; y, x) & p^-(t; x, y) \\ p^-(t; y, x) & p^+(t; x, y) \end{pmatrix}$$

and get the following

THEOREM 3.1. *Let $\mathcal{Q}(t; y, x)$ be a $M(2, \mathbf{R})$ valued function, which is defined by*

$$\mathcal{Q}(t; y, x) = \begin{pmatrix} p^+(t; y, x) & p^-(t; x, y) \\ p^-(t; y, x) & p^+(t; x, y) \end{pmatrix}.$$

Then the function $\mathcal{Q}(t; y, x)$ has the following property:

$$\mathcal{Q}(t + s; z, x) = \int_{-\infty}^{+\infty} \mathcal{Q}(t; z, y) \mathcal{Q}(s; y, x) dy.$$

PROOF OF THEOREM 3.1. A proof is established by straightforward calculation by using (3-15) and (3-16). But we will take another way to prove Theorem 3.1. Because of the proof of Theorem 2.1, we have the following equalities:

$$p^+(t + s; z, x) = \int_{-\infty}^{+\infty} \{p^+(t; z, y)p^+(s; y, x) + q^-(t; z, y)p^-(s; y, x)\} dy,$$

$$p^-(t + s; z, x) = \int_{-\infty}^{+\infty} \{p^-(t; z, y)p^+(s; y, x) + q^+(t; z, y)p^-(s; y, x)\} dy,$$

$$q^-(t + s; z, x) = \int_{-\infty}^{+\infty} \{p^+(t; z, y)q^-(s; y, x) + q^-(t; z, y)q^+(s; y, x)\} dy$$

and

$$q^+(t + s; z, x) = \int_{-\infty}^{+\infty} \{p^-(t; z, y)q^-(s; y, x) + q^+(t; z, y)q^+(s; y, x)\} dy.$$

We arrange these equalities in a matrix form as

(3-17)

$$\begin{pmatrix} p^+(t + s; z, x) & q^-(t + s; z, x) \\ p^-(t + s; z, x) & q^+(t + s; z, x) \end{pmatrix} = \int_{-\infty}^{+\infty} \begin{pmatrix} p^+(t; z, y) & q^-(t; z, y) \\ p^-(t; z, y) & q^+(t; z, y) \end{pmatrix} \cdot \begin{pmatrix} p^+(s; y, x) & q^-(s; y, x) \\ p^-(s; y, x) & q^+(s; y, x) \end{pmatrix} dy.$$

In the first part of this section, we have shown the following equalities:

$$(3-14) \quad p^+(t; x, y) = q^+(t; y, x), \quad p^-(t; x, y) = q^-(t; y, x).$$

Using (3-14) and (3-17), we get

$$\begin{pmatrix} p^+(t+s; z, x) & p^-(t+s; x, z) \\ p^-(t+s; z, x) & p^+(t+s; x, z) \end{pmatrix} \\
= \int_{-\infty}^{+\infty} \begin{pmatrix} p^+(t; z, y) & p^-(t; y, z) \\ p^-(t; z, y) & p^+(t; y, z) \end{pmatrix} \cdot \begin{pmatrix} p^+(s; y, x) & p^-(s; x, y) \\ p^-(s; y, x) & p^+(s; x, y) \end{pmatrix} dy.$$

This is what we wanted to prove. $\square$

THEOREM 3.2. *The function $\mathcal{Q}(t; y, x)$ has the following property:*

$$\frac{\partial \mathcal{Q}}{\partial t}(t; y, x) = (\mathcal{H} \mathcal{Q})(t; y, x),$$

where the operator $\mathcal{H}$ is defined by

$$(\mathcal{H} \mathcal{Q})(t; y, x) = \begin{pmatrix} -\left(\lambda + \frac{\partial}{\partial y}\right) & \lambda \\ \lambda & -\left(\lambda - \frac{\partial}{\partial y}\right) \end{pmatrix} \cdot \begin{pmatrix} q_{11}(t; y, x) & q_{12}(t; y, x) \\ q_{21}(t; y, x) & q_{22}(t; y, x) \end{pmatrix}.$$

PROOF OF THEOREM 3.2. Explicit forms of $p^+(t; y, x)$ and $p^-(t; y, x)$ are given by

$$\begin{aligned}
(3-18) \quad & e^{\lambda t} p^+(t, y, x) = \delta(t - (y - x)) \\
& + \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^{2n+2} \frac{1}{n!(n+1)!} (t - (y - x))^n (t + (y - x))^{n+1} \theta(t^2 - (y - x)^2)
\end{aligned}$$

and

$$\begin{aligned}
(3-19) \quad & e^{\lambda t} p^-(t, y, x) \\
& = \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^{2n+1} \frac{1}{(n!)^2} (t - (y - x))^n (t + (y - x))^n \theta(t^2 - (y - x)^2).
\end{aligned}$$

Differentiating the both sides of the equality (3-18) by a variable t , we have

$$\lambda e^{\lambda t} p^+(t, y, x) + e^{\lambda t} \frac{\partial p^+(t, y, x)}{\partial t}$$

$$\begin{aligned}
&= -\frac{\partial \delta(t-(y-x))}{\partial y} \\
&\quad - \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^{2n+2} \frac{1}{n!(n+1)!} \frac{\partial}{\partial y} \{(t-(y-x))^n (t+(y-x))^{n+1} \theta(t^2-(y-x)^2)\} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^{2n+1} \lambda \frac{1}{(n!)^2} (t-(y-x))^n (t+(y-x))^n \theta(t^2-(y-x)^2)
\end{aligned}$$

Dividing the above equality by $e^{\lambda t}$ and using (3-18) and (3-19), we get

$$(3-20) \quad \frac{\partial p^+(t, y, x)}{\partial t} = -\lambda p^+(t, y, x) - \frac{\partial p^+(t, y, x)}{\partial y} + \lambda p^-(t, y, x).$$

To complete the proof, we differentiate the both sides of (3-19) by a variable t and we have

$$\begin{aligned}
&\lambda e^{\lambda t} p^-(t, y, x) + e^{\lambda t} \frac{\partial p^-(t, y, x)}{\partial t} \\
&= \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^{2n+1} \frac{1}{(n!)^2} \frac{\partial}{\partial y} \{(t-(y-x))^n (t+(y-x))^n \theta(t^2-(y-x)^2)\} \\
&\quad + \lambda \delta(t-(y-x)) \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^{2n+2} \lambda \frac{1}{n!(n+1)!} (t-(y-x))^n (t+(y-x))^{n+1} \theta(t^2-(y-x)^2).
\end{aligned}$$

Dividing the above equality by $e^{\lambda t}$ and using (3-18) and (3-19), we get

$$(3-21) \quad \frac{\partial p^-(t, y, x)}{\partial t} = -\lambda p^-(t, y, x) + \frac{\partial p^-(t, y, x)}{\partial y} + \lambda p^+(t, y, x).$$

By (3-20) and (3-21), we complete the proof of Theorem 3.2. $\square$

Remark 3.1. By setting $v = a = \lambda$ on the equations (11) and (12) in Kac [8], we get the equation of Theorem 3.2.

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