THE RIEMANN PROBLEM FOR APPELL'S F_4

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Introduction

Let z be an unknown and $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, $p = z_x$, $q = z_y$. The following system of differential equations is called Appell's system of type F_4 :

$$(F_4) \quad \begin{cases} x(1-x)r - y^2t - 2xys + \{\gamma - (\alpha + \beta + 1)x\}p - (\alpha + \beta + 1) yq - \alpha\beta z = 0 \\ y(1-y)t - x^2r - 2xys + \{\gamma' - (\alpha + \beta + 1)y\}q - (\alpha + \beta + 1) xp - \alpha\beta z = 0 \end{cases}$$

This system has singularities along

$S := \{x=0\} \cup \{y=0\} \cup \{(x-y)^2 - 2(x+y) + 1 = 0\}$. Outside the singular locus S , (F_4) has four linearly independent holomorphic solutions.

Assume that $\gamma, \gamma' \neq 0, -1, -2, \dots$. Then the Appell's hypergeometric series

$$F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \sum \frac{(\alpha, m+n)(\beta, m+n)}{(\gamma, m)(\gamma', n)(1, m)(1, n)} x^m y^n$$

is a solution of (F_4) , where $(a, k) = a(a+1)\cdots(a+k-1)$.

Let $\mathcal{F}_4$ be the solution space of (F_4) , that is, the set of holomorphic functions on the universal covering space of $\mathbf{C}^2 - S$ satisfying (F_4) .

In the natural way, the system of the differential equations (F_4) can be extended to $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$. The extended system has singularities also along the line at infinity L_∞ , or $\{x = \infty\} \cup \{y = \infty\}$.

In this paper we prove that $\mathcal{F}_4$ is characterized by the "exponents" at each irreducible component of the singular locus in the following way.

For generic $\alpha, \beta, \gamma, \gamma'$,

$$\mathcal{F}_4 = P \left\{ \begin{array}{cccc} x=0 & y=0 & (x-y)^2 - 2(x+y) + 1 = 0 & L_\infty \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha \\ 1-\gamma & 1-\gamma' & 0 & \beta \\ 1-\gamma & 1-\gamma' & \gamma + \gamma' - \alpha - \beta - \frac{1}{2} & \beta \end{array} \right\}$$

on $\mathbf{P}^2$ and

$$\mathcal{F}_4 = P \left\{ \begin{array}{ccccc} x=0 & y=0 & (x-y)^2 - 2(x+y) + 1 = 0 & x=\infty & y=\infty \\ 0 & 0 & 0 & \alpha & \alpha \\ 0 & 0 & 0 & \beta & \beta \\ 1-\gamma & 1-\gamma' & 0 & 1+\alpha-\gamma' & 1+\alpha-\gamma \\ 1-\gamma & 1-\gamma' & \gamma+\gamma'-\alpha-\beta-\frac{1}{2} & 1+\beta-\gamma' & 1+\beta-\gamma \end{array} \right\}$$

on $\mathbf{P}^1 \times \mathbf{P}^1$.

For the meaning of “exponents”, see the section 2 and for the meaning of the “ P -function”, see the section 3.

The proof is divided into two steps. First (in the section 2) we determine the “exponents” of $\mathcal{F}_4$ at each irreducible component of the singular locus of (F_4) . Next (in the section 3) we show that any 4 dimensional vector space $\mathcal{F}$ consisting of holomorphic functions on the universal covering space of $\mathbf{C}^2 - S$ must be $\mathcal{F}_4$ if $\mathcal{F}$ has the same “exponents” as $\mathcal{F}_4$.

For Appell’s system $(F_1), (F_2), (F_3)$, the corresponding characterizations are known in Appell et Kampé de Fériet [1] and in Terada [3]. But (F_4) has rather complicated singularities and has no Pfaffian system only with logarithmic singularities equivalent to itself. So it has been difficult to determine the “exponents” at the singularities.

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1. Existence of the solutions of a Pfaffian system

Let $\Omega(x, y) = \Omega_1 dy + \Omega_2 dx$ be a holomorphic (n, n) matrix with $d\Omega = \Omega \wedge \Omega$. Then it is well known that $du = \Omega u$ has n -dimensional solution space. For later use we prove this existence theorem for $\Omega(x, y)$ with a logarithmic pole on $\{x = 0\}$.

1.1 Theorem Let $A(y), B(x, y), \Omega_2(x, y)$ be (n, n) matrices holomorphic on the closure of $D = \{(x, y) | |x| < r, |y - y_0| < s\}$. Put $\Omega_1 = \frac{A}{x} + B$, $\Omega = \Omega_1 dx + \Omega_2 dy$. Assume that $d\Omega = \Omega \wedge \Omega$ then $A(y)$ is conjugate to a constant matrix.

If moreover none of the eigen values of $A(y)$ is a positive integer then $du = \Omega u$ has m linearly independent holomorphic solutions on D , where m is the corank of $A(y)$.

Proof.

Put $Q(y) = \Omega_2(0, y)$. Then $d\Omega = \Omega \wedge \Omega$ restricted to $\{x = 0\}$ implies $A'(y) = [Q, A]$. Let $P(y)$ be a fundamental solution of $P' = QP$. Then $P^{-1}AP$ is a constant matrix. If we replace u by Pv then we obtain $dv = \tilde{\Omega}v$ where $\tilde{\Omega} = \left(\frac{P^{-1}AP}{x} + \tilde{B} \right) dx + \tilde{\Omega}_2 dy$. Hence A is conjugate to a constant matrix.

And we may assume that A is a constant matrix with $[A, Q] = 0$.

Let u be holomorphic in D . Then $du = \Omega u$ is equivalent to

$$(1.1.A) \quad u_x = \Omega_1 u, \quad u_y = \Omega_2 u$$

which is again equivalent to

$$(1.1.B1) \quad u_x = \Omega_1 u,$$

$$(1.1.B2) \quad u_y - \Omega_2 u|_{x=0} = 0.$$

We will prove this equivalence.

Let $u = \sum u_k(y)x^k$, $B = \sum B_k(y)x^k$. Then (1.1.B1), (1.1.B2) are equivalent to the following

$$(1.1.C1) \quad Au_0 = 0, \quad (k - A)u_k = \sum_{i=0}^{k-1} B_{k-1-i}u_i$$

$$(1.1.C2) \quad u'_0 = Q(y)u_0 \quad (Q(y) = \Omega_2(0, y)).$$

First note that the uniqueness property of (1.1.B1) holds. This means that if the "initial value" $u_0(y) = u(0, y)$ of a solution $u(x, y)$ of (1.1.B1) is zero then from (1.1.C1) all $u_k(y) = 0$ hence $u = 0$.

Assume (1.1.B1) and (1.1.B2). Put $v = u_y - \Omega_2 u$. Then $v|_{x=0} = 0$ and

$$\begin{aligned} v_x &= u_{xy} - \Omega_2 u_x - \Omega_{2x} u = (\Omega_1 u)_y - \Omega_2 \Omega_1 u - \Omega_{2x} u \\ &= \Omega_1(u_y - \Omega_2 u) + ([\Omega_1, \Omega_2] + \Omega_{1y} - \Omega_{2x})u \\ &= \Omega_1 v \quad (\text{by } d\Omega = \Omega \wedge \Omega). \end{aligned}$$

This proves that $v = 0$ that is (1.1.A).

Now we solve (1.1.C1), (1.1.C2) formally.

Since A is commutable with Q , $Au_0 = 0$ and $u'_0 = Qu_0$ has m linearly independent solutions. (1.1.C1) determines u_k for each u_0 .

We next show the convergence of the formal solutions. For any u_0 , let u_k be the solution of (1.1.C1) above. By the Cauchy's inequality we have

$\|B_k\| < L_1/r^k$ ($k = 0, 1, 2, \dots$) for some positive number L_1 . For some positive number L_2 , $\|(k-A)^{-1}\| < L_2/k$ holds for all k ($k = 1, 2, 3, \dots$). Put $M = \|u_0\|$ and $L = L_1 L_2$. Then using the second equality of (1.1.C1), we obtain the inequality

$$\|u_k\| < \frac{M}{r^k} (1 + rL) \left(1 + \frac{rL}{2}\right) \cdots \left(1 + \frac{rL}{k-1}\right) \frac{rL}{k}$$

by the induction on k . Hence we have $\|u_k\| = o(r_1^{-k})$ for any $r_1 < r$. This proves the convergence of u .

This completes the proof of the theorem 1.1.

2. The exponents of $\mathcal{F}_4$ along the singular sets

2.1 Exponent of $\mathcal{F}_4$ along $\{x = 0\}$.

Assume that γ is not an integer.

Let $\phi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a 2-sheeted covering defined by $\phi(X, Y) = (XY, (1-X)(1-Y))$. ϕ is locally biholomorphic at $X = 0, Y = Y_0 \neq 0$.

As is stated in Kato [2], the pull back $\phi^*(F_4)$ of (F_4) is equivalent to a Pfaffian system $dv = Pv$, where $v = (z, Xz_X, Yz_Y, XY(z_{XY} + \varepsilon(z_X - z_Y)/(X - Y)))$,

$\varepsilon = \gamma + \gamma' - \alpha - \beta - 1$ and $P = P_1 \frac{dX}{X} + P_2 \frac{dY}{Y} + P_3 \frac{dX}{X-1} + P_4 \frac{dY}{Y-1} + P_5 \frac{d(X-Y)}{X-Y}$ with constant $(4, 4)$ matrices

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 \\ 0 & \varepsilon & 0 & 1 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \varepsilon & 1 \\ 0 & 0 & 1-\gamma & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha\beta & -\gamma' & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(\alpha + \varepsilon)(\beta + \varepsilon) & -\gamma' \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta & \varepsilon & -\gamma' & 0 \\ 0 & -(\alpha + \varepsilon)(\beta + \varepsilon) & 0 & -\gamma' \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon & -\varepsilon & 0 \\ 0 & -\varepsilon & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Put $P = \left(\frac{A}{X} + B\right)dX + CdY$ and $Q = C(0, Y)$ then $A = P_1$ and $Q = (P_2 + P_5)$

$$\frac{1}{Y} + P_4 \frac{1}{Y-1}.$$

The solution v_0 of $Av_0 = 0$ and $v'_0 = Qv_0$ consists of $(z(0, 1-Y), 0, -Yz_Y(0, 1-Y), 0)$ with $z(0, y)$ being a solution of $(F(\alpha, \beta, \gamma'; y))$.

This means that for any solution $z_0(y)$ of $(F(\alpha, \beta, \gamma'; y))$, there exists a unique holomorphic solution $z(x, y)$ of (F_4) in a neighborhood of $(0, y_0)$ with $z(0, y) = z_0(y)$.

Next we know that $z = x^{1-\gamma}f(x, y)$ is a solution of $(F_4(\alpha, \beta, \gamma, \gamma'))$ if and only if $f(x, y)$ is a solution of $(F_4(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \gamma'))$ (Appell et Kampé de Fériet [1]).

Thus we have proved that if γ is not an integer then $\mathcal{F}_4$ has the exponent $(0, 0, 1 - \gamma, 1 - \gamma)$ along $\{x = 0\}$. This means, by definition, that at any point $(0, y_0)$ ($y_0 \neq 0, 1, \infty$), there exist holomorphic functions $g_1(x, y), g_2(x, y), g_3(x, y), g_4(x, y)$ so that $g_1, g_2, x^{1-\gamma}g_3, x^{1-\gamma}g_4$ form a basis of $\mathcal{F}_4$ and that both pairs $g_1(0, y), g_2(0, y)$ and $g_3(0, y), g_4(0, y)$ are linearly independent respectively.

In fact $\mathcal{F}_4$ has the following four special elements provided that neither of γ, γ' is an integer

(Appell et Kampé de Fériet [1]):

$$F_4(\alpha, \beta, \gamma, \gamma'; x, y),$$

$$x^{1-\gamma}F_4(1 + \alpha - \gamma, 1 + \beta - \gamma, 2 - \gamma, \gamma'; x, y),$$

$$y^{1-\gamma'}F_4(1 + \alpha - \gamma', 1 + \beta - \gamma', \gamma, 2 - \gamma'; x, y),$$

$$x^{1-\gamma}y^{1-\gamma'}F_4(2 + \alpha - \gamma - \gamma', 2 + \beta - \gamma - \gamma', 2 - \gamma, 2 - \gamma'; x, y).$$

2.2 Exponent along $\{y = 0\}$.

Assume that γ' is not an integer, then $\mathcal{F}_4$ has the exponent $(0, 0, 1 - \gamma', 1 - \gamma')$ along $\{y = 0\}$.

2.3 Exponent along L_∞ .

Assume that $\alpha - \beta$ is not an integer.

The direct calculations show that $z = y^{-\alpha}w(x/y, 1/y)$ is a solution of (F_4) if and only if $w(\xi, \eta)$ is a solution of $(F_4(\alpha, \alpha - \gamma' + 1, \gamma, \alpha - \beta + 1; \xi, \eta))$. So from 2.2, $\mathcal{F}_4$ has the exponent $(\alpha, \alpha, \beta, \beta)$ along L_∞ .

2.4 Exponent along $\{(x - y)^2 - 2(x + y) + 1 = 0\}$.

Assume that $\gamma + \gamma' - \alpha - \beta - \frac{1}{2}$ is not an integer. Put $D = (x - y)^2 - 2(x + y) + 1$.

As in the section 2.1, we put $\varepsilon = \gamma + \gamma' - \alpha - \beta - 1$ and consider the 2:1 mapping $\phi(X, Y) = (XY, (1 - X)(1 - Y))$. Then as is stated in 2.1, the pull back $\phi^*(F_4)$ of (F_4) is equivalent to

$$(2.4.A) \quad dv = Pv.$$

The pull back of $\{D = 0\}$ is $\{X - Y = 0\}$.

$$\text{Put } \xi = \frac{X - Y}{2}, \eta = \frac{X + Y}{2} \text{ and } u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} v$$

Then (2.4.A) takes the form

$$(2.4.B) \quad \begin{aligned} du &= \Omega u \\ &= \left\{ \left(\frac{1}{\xi} A + B \right) d\xi + \Omega_2 d\eta \right\} u. \end{aligned}$$

Where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = q(\xi, \eta) \left(Q_1 - \frac{\xi}{\eta} Q_2 \right) + q(\xi, \eta - 1) \left(Q_3 - \frac{\xi}{\eta - 1} Q_4 \right),$$

$$\Omega_2 = q(\xi, \eta) \left(Q_2 - \frac{\xi}{\eta} Q_1 \right) + q(\xi, \eta - 1) \left(Q_4 - \frac{\xi}{\eta - 1} Q_3 \right),$$

$$q(\xi, \eta) = \frac{\eta}{\eta^2 - \xi^2},$$

$$Q_1, Q_3 \text{ are constant matrices of the form } \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & * & * \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix},$$

$$Q_2, Q_4 \text{ are constant matrices of the form } \begin{pmatrix} * & 0 & * & * \\ 0 & * & 0 & 0 \\ * & 0 & * & * \\ * & 0 & * & * \end{pmatrix},$$

$$\text{More precisely } Q_2 \text{ is of the form } \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 2 \\ 0 & 0 & 0 & * \end{pmatrix},$$

$$Q_4 \text{ is of the form } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix}.$$

$$\text{Put } Q(\eta) = \Omega_2(0, \eta) = \frac{1}{\eta} Q_2 + \frac{1}{\eta - 1} Q_4.$$

For $u = \xi^\lambda w$, $w = w_0(\eta) + \xi w_1(\eta) + \xi^2 w_2(\eta) + \dots$, (2.4.B) is equivalent to the following (2.4.C1) and (2.4.C2).

$$(2.4.C1) \quad w_\xi = \left(\frac{1}{\xi} (A - \lambda E) + B \right) w$$

$$(2.4.C2) \quad w_\eta = Qw$$

which implies the following (2.4.EQ_k) ($k = 0, 1, 2, \dots$) and (2.4.EQ _{η}).

$$(2.4.EQ_k) \quad (\lambda + k - A)w_k = \\ \frac{1}{\eta} Q_1 \left(w_{k-1} + \frac{w_{k-3}}{\eta^2} + \frac{w_{k-5}}{\eta^4} + \dots \right) \\ + \frac{1}{\eta - 1} Q_3 \left(w_{k-1} + \frac{w_{k-3}}{(\eta - 1)^2} + \frac{w_{k-5}}{(\eta - 1)^4} + \dots \right) \\ - \frac{1}{\eta^2} Q_2 \left(w_{k-2} + \frac{w_{k-4}}{\eta^2} + \dots \right) - \frac{1}{(\eta - 1)^2} Q_4 \left(w_{k-2} + \frac{w_{k-4}}{(\eta - 1)^2} + \dots \right)$$

$$(2.4.EQ_\eta) \quad w_0(\eta)' = Qw_0.$$

Since $A - \lambda$ may have a positive integer as its eigen value, (2.4.EQ_k) and (2.4.EQ _{η}) does not imply (2.4.B) (but see the Lemma 2.4.2 below).

Corresponding to the eigen values 0 and 2ε of A , (2.4.EQ _{η}) has two invariant vector spaces V_0 and V_1 : $V_0 = \{(f_0(\eta), 0, g_0(\eta), h_0(\eta))\}$, $V_1 = \{(0, f_1(\eta), 0, 0)\}$, where f_0, g_0, h_0, f_1 are functions of η .

For $k = 0$ and 1 we have

$$(2.4.QV) \quad Q_i V_k \subset V_k \text{ for } i = 2, 4 \text{ and } Q_i V_k \subset V_{1-k} \text{ for } i = 1, 3.$$

2.4.1

We assume $\lambda = 0$ or 2ε . It is clear that

if $\varepsilon \neq 0$ and $\lambda = 0$ then (2.4.EQ₀) implies $w_0 \in V_0$,

if $\varepsilon \neq 0$ and $\lambda = 2\varepsilon$ then (2.4.EQ₀) implies $w_0 \in V_1$,

if $\varepsilon = 0$ and $\lambda = 0$ then (2.4.EQ₀) holds for any w_0 .

Lemma Assume $\varepsilon + \frac{1}{2}$ is not an integer.

Let $\lambda = 0$, then for any $w_0 \in V_0$ there exists a unique solution w_k of (2.4.EQ_k) $k = 1, 2, 3, \dots$ satisfying the following property:

$$(2.4.1.V0) \quad w_{2k} \in V_0 \quad \text{and} \quad w_{2k+1} \in V_1 \quad k \geq 0.$$

Let $\lambda = 2\varepsilon$, then for any $w_0 \in V_1$ there exists a unique solution w_k of (2.4.EQ_k) satisfying the following property:

$$(2.4.1.V1) \quad w_{2k} \in V_1 \quad \text{and} \quad w_{2k+1} \in V_0 \quad k \geq 0.$$

Proof This follows from 2.4.1.1 ~ 2.4.1.3 below.

2.4.1.1 Except for the case $\lambda = 0$ and 2ε is a positive integer or the case $\lambda = 2\varepsilon$ and 2ε is a negative integer, for any w_0 , (2.4.EQ $_k$) has the unique solution w_k . If $w_0 \in V_0$ then w_k satisfies (2.4.1.V0). If $w_0 \in V_1$ then w_k satisfies (2.4.1.V1).

Proof $(\lambda + n)E - A$ is singular only in the exceptional case stated above. From (2.4.QV), (2.4.1.V0) or (2.4.1.V1) holds inductively.

2.4.1.2 Assume $\lambda = 0$ and ε is a positive integer. Then for any $w_0 \in V_0$, (2.4.EQ $_k$) has the unique solution w_k if we require the property (2.4.1.V0).

Proof Put $\varepsilon = m$. Then (2.4.EQ $_k$) has the unique solution w_k up to $k = 2m - 1$ with the property (2.4.1.V0). Hence the right hand side of (2.4.EQ $_{2m}$) belongs to V_0 by (2.4.QV), and the left hand side is $(2mE - A)w_{2m}$. Thus (2.4.EQ $_{2m}$) has the unique solution w_{2m} if we restrict w_{2m} to V_0 ($w_{2m} + V_1$ is the general solution). For $k > 2m$ (2.4.EQ $_k$) has the unique solution w_k with the property (2.4.1.V0). This completes the proof.

2.4.1.3 Assume $\lambda = 2\varepsilon$ and ε is a negative integer. Then for any $w_0 \in V_1$, (2.4.EQ $_k$) has the unique solution w_k if we require the property (2.4.1.V1).

Proof Let $\lambda = 2\varepsilon = -2m$ for some positive integer m . Then (2.4.EQ $_k$) has the unique solution w_k up to $k = 2m - 1$ with the property (2.4.1.V1). Hence the right hand side of (2.4.EQ $_{2m}$) belongs to V_1 by (2.4.QV), and the left hand side is $-Aw_{2m}$. Thus (2.4.EQ $_{2m}$) has the unique solution w_{2m} if we restrict w_{2m} to V_1 ($w_{2m} + V_0$ is the general solution). For $k > 2m$ (2.4.EQ $_k$) has the unique solution w_k with the property (2.4.1.V1). This completes the proof.

2.4.2 Lemma

For $u = \xi^\lambda w$, $w = w_0(\eta) + \xi w_1(\eta) + \xi^2 w_2(\eta) + \dots$, (2.4.EQ $_k$) and (2.4.EQ $_\eta$) are sufficient conditions for (2.4.B) in the following sense.

In the case when $\lambda = 0$. If $w_0(\eta) \in V_0$ satisfies (2.4.EQ $_\eta$) and w_k satisfies (2.4.EQ $_k$) together with the property (2.4.1.V0), then $u = w$ satisfies (2.4.B).

In the case when $\lambda = 2\varepsilon$. If $w_0(\eta) \in V_1$ satisfies (2.4.EQ $_\eta$) and w_k satisfies (2.4.EQ $_k$) together with the property (2.4.1.V1), then $u = \xi^\lambda w$ satisfies (2.4.B).

Proof Recall that $Q = \left(\frac{1}{\xi} A + B \right) d\xi + \Omega_2 d\eta$ and that (2.4.B) is equivalent

to (2.4.C1) and (2.4.C2). Put $\psi = w_\eta - \Omega_2 w$. Then we have to show that $\psi|_{\xi=0} = 0$ implies $\psi = 0$.

By the integrability condition $d\Omega = \Omega \wedge \Omega$, ψ also satisfies (2.4.C1).

If $\lambda = 0$ then w_k satisfies (2.4.1.V0) so ψ also satisfies (2.4.1.V0).

If $\lambda = 2\varepsilon$ then w_k satisfies (2.4.1.V1) so ψ also satisfies (2.4.1.V1).

Hence we can apply the uniqueness property in the lemma 2.4.1 to ψ . Since $\psi|_{\xi=0} = 0$, we have $\psi = 0$. This completes the proof.

2.4.3 The formal existence of the solution of (2.4.B)

We first construct the formal solutions of (2.4.B) of the form $u = \xi^\lambda w$, $w = w_0(\eta) + \xi w_1(\eta) + \xi^2 w_2(\eta) + \dots$, $w_0 \neq 0$, $\lambda = 0$ or 2ε .

Let $\lambda = 0$. Since V_0 is an invariant subspace of $(2.4.EQ_\eta)$, $(2.4.EQ_\eta)$ has a 3 dimensional solution space in V_0 . Let $w_0 = (f, 0, g, h) \in V_0$ be any nonzero solution of $(2.4.EQ_\eta)$. If $f = 0$ then $(2.4.EQ_\eta)$ implies that $g = h = 0$. That is, $w_0 \neq 0$ means $f \neq 0$. By the lemma 2.4.1, there exists $w = w_0 + \xi w_1 + \dots$ satisfying $(2.4.EQ_k)$ and $(2.4.1.V0)$. Again by 2.4.2 $u = w$ is a solution of (2.4.B).

Let $\lambda = 2\varepsilon$. V_1 is also an invariant subspace of $(2.4.EQ_\eta)$. The solution $w_0 \in V_1$ of $(2.4.EQ_\eta)$ is of the form $(0, c\eta^\mu(\eta - 1)^\nu, 0, 0)$. By 2.4.1, there exists $w = w_0 + \xi w_1 + \dots$ satisfying $(2.4.EQ_k)$ and $(2.4.1.V1)$. Then by 2.4.2, $u = \xi^\lambda w$ is a solution of (2.4.B). We note $w_1 = \left(\frac{c}{2\varepsilon + 1} \eta^{\mu-1}(\eta - 1)^\nu, 0, *, * \right)$ for later use.

2.4.4 The Convergence of the power series w

Assume that $w_k(\eta)$ are holomorphic functions satisfying $(2.4.EQ_k)$. Then $w = w_0 + \xi w_1 + \xi^2 w_2 + \dots$ converges if $|\xi| < |\eta|$, $|1 - \eta|$.

Proof The proof is the same as that of Theorem 1.1. Note that $\xi = \pm \eta$, $\xi = \pm(1 - \eta)$ means that $XY = 0$, $(1 - X)(1 - Y) = 0$ respectively.

2.4.5 The exponent of $\mathcal{F}_4$ along $\{D=0\}$

Recall that $D = (x - y)^2 - 2(x + y) + 1$.

We will construct solutions of (F_4) from solutions of (2.4.B).

Let $\lambda = 0$. Then (2.4.B) has three independent solutions u_1, u_2, u_3 with the property (2.4.1.V0). Put $u_k = (g_k, *, *, *)$. Then the property (2.4.1.V0) guarantees the fact that g_k are even functions of ξ . The linealy independence of u_k $k = 1, 2, 3$ implies that of $u_k(0, \eta)$ $k = 1, 2, 3$ and that of $g_k(0, \eta)$ $k = 1, 2, 3$ (cf 2.4.1, 2.4.3). Since $\xi^2 = D/4$ and $\eta = (1 + x - y)/2$, g_k are holomorphic

functions of (x, y) which are also denoted by $g_k(x, y)$. Thus obtained $g_k(x, y)$ are solutions of (F_4) and are linearly independent when they are restricted to $\{D = 0\}$.

Let $\lambda = 2\varepsilon$. Then (2.4.B) has a solution $u = \xi^\lambda w$ with the property (2.4.1.V1). Put $u = {}^t(g, *, *, *)$. Then g is of the form $\xi^{\lambda+1} g_4$ where g_4 is an even function of ξ and $g_4(0, \eta) = \frac{c}{2\varepsilon + 1} \eta^{\mu-1} (\eta - 1)^v \neq 0$ as stated in

2.4.3. Again g_4 may be considered a holomorphic function of (x, y) . $g = D^{\varepsilon+1/2} g_4(x, y)$ is a solution of (F_4) and the restriction of g_4 to $\{D = 0\}$ is not zero.

Thus we have proved the following result.

If $\gamma + \gamma' - \alpha - \beta - \frac{1}{2} \left(\varepsilon + \frac{1}{2} \right)$ is not an integer, then at each point (a, b) with $D(a, b) = 0$, $ab \neq 0$, (F_4) has solutions $g_1(x, y)$, $g_2(x, y)$, $g_3(x, y)$, $D^{\gamma+\gamma'-\alpha-\beta-\frac{1}{2}} g_4(x, y)$ so that g_1, g_2, g_3, g_4 are holomorphic at (a, b) , and the restrictions of g_1, g_2, g_3 to $\{D = 0\}$ are linearly independent, and the restriction of g_4 to $\{D = 0\}$ is not (identically) zero. This means, by definition, that $\mathcal{F}_4$ has the exponent $\left(0, 0, 0, \gamma + \gamma' - \alpha - \beta - \frac{1}{2} \right)$ along $\{D = 0\}$.

2.5 Exponent along $\{y = \infty\}$ in $\mathbf{P}^1 \times \mathbf{P}^1$

Assume that the difference of any two members of $\{\alpha, \beta, 1 + \alpha - \gamma, 1 + \beta - \gamma\}$ is not an integer.

As stated in 2.3, $y^{-\alpha} w(x/y, 1/y)$ belongs to $\mathcal{F}_4$ if $w(\xi, \eta)$ is a solution of $(F_4(\alpha, \alpha - \gamma' + 1, \gamma, \alpha - \beta + 1; \xi, \eta))$. In particular

$$z = y^{-\alpha} F_4(\alpha, \alpha - \gamma' + 1, \gamma, \alpha - \beta + 1; x/y, 1/y),$$

$$z = y^{-(1+\alpha-\gamma)} x^{1-\gamma} F_4(\alpha - \gamma + 1, \alpha - \gamma - \gamma' + 2, 2 - \gamma, \alpha - \beta + 1; x/y, 1/y),$$

$$z = y^{-\beta} F_4(\beta, \beta - \gamma' + 1, \gamma, \beta - \alpha + 1; x/y, 1/y),$$

$$z = y^{-(1+\beta-\gamma)} x^{1-\gamma} F_4(\beta - \gamma + 1, \beta - \gamma - \gamma' + 2, 2 - \gamma, \beta - \alpha + 1; x/y, 1/y)$$

are solutions near $\{y = \infty\}$.

Thus $\mathcal{F}_4$ has the exponent $(\alpha, \beta, 1 + \alpha - \gamma, 1 + \beta - \gamma)$ at $\{y = \infty\}$. This means, by definition, $\mathcal{F}_4$ has a basis $y^{-\alpha} g_1, y^{-\beta} g_2, y^{-(1+\alpha-\gamma)} g_3, y^{-(1+\beta-\gamma)} g_4$, where g_k are holomorphic at (a, ∞) ($a \neq 0, \infty$) and the restrictions $g_k(x, \infty)$ are not zero. In our case, $g_1(x, \infty) = g_2(x, \infty) = 1$ and $g_3(x, \infty) = g_4(x, \infty) = x^{1-\gamma}$.

2.6 Exponent of $\mathcal{F}_4$ along $\{x = \infty\}$

Assume that the difference of any two members of $\{\alpha, \beta, 1 + \alpha - \gamma', 1 + \beta - \gamma'\}$ is not an integer. Then $\mathcal{F}_4$ has the exponent $(\alpha, \beta, 1 + \alpha - \gamma', 1 + \beta - \gamma')$ along $\{x = \infty\}$.

3. Riemann's Problem of (F_4) on $\mathbf{P}^2$

We consider (x, y) to be the inhomogeneous coordinate of $\mathbf{P}^2$. Let again $D = (x - y)^2 - 2(x + y) + 1$. Let L_x, L_y and C be subsets of $\mathbf{P}^2$ defined by $x = 0, y = 0$ and $D = 0$ respectively.

Assume that none of $1 - \gamma, 1 - \gamma', \alpha - \beta, \gamma + \gamma' - \alpha - \beta - 1/2$ is zero and put

$$\mathcal{F} = P \left\{ \begin{array}{cccc} x = 0 & y = 0 & (x - y)^2 - 2(x + y) + 1 = 0 & L_\infty \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha \\ 1 - \gamma & 1 - \gamma' & 0 & \beta \\ 1 - \gamma & 1 - \gamma' & \gamma + \gamma' - \alpha - \beta - \frac{1}{2} & \beta \end{array} \right\}.$$

This means that $\mathcal{F}$ is a 4 dimensional vector space consisting of holomorphic functions on the universal covering space of $\mathbf{P}^2 - L_x \cup L_y \cup L_\infty \cup C$ with the following properties.

(EXP_x): $\mathcal{F}$ has the exponent $(0, 0, 1 - \gamma, 1 - \gamma')$ along L_x . This means, by definition, that at each point of $L_x - L_y \cup L_\infty \cup C$ there exist an open neighbourhood U and holomorphic functions g_1, g_2, g_3, g_4 on U so that $g_1, g_2, x^{1-\gamma}g_3, x^{1-\gamma'}g_4$ form a basis of $\mathcal{F}$ and that both of the pairs g_1, g_2 and g_3, g_4 are linearly independent when they are restricted to L_x .

(EXP_y): $\mathcal{F}$ has the exponent $(0, 0, 1 - \gamma', 1 - \gamma')$ along L_y .

(EXP_∞): $\mathcal{F}$ has the exponent $(\alpha, \alpha, \beta, \beta)$ along L_∞ . This means, by definition, that at each point of $L_\infty - L_x \cup L_y \cup C$ there exist an open neighbourhood U in $\mathbf{P}^2$ and holomorphic functions g_1, g_2, g_3, g_4 on U and coordinate functions ξ, η with $L_\infty \cap U = \{\xi = 0\}$ so that $\xi^\alpha g_1, \xi^\alpha g_2, \xi^\beta g_3, \xi^\beta g_4$ form a basis of $\mathcal{F}$ and that both of the pairs g_1, g_2 and g_3, g_4 are linearly independent when they are restricted to L_∞ .

(EXP_C): $\mathcal{F}$ has the exponent $\left(0, 0, 0, \gamma + \gamma' - \alpha - \beta - \frac{1}{2}\right)$ along C . This

means, by definition, that at each point of $C - L_x \cup L_y \cup L_\infty$ there exist an open neighbourhood U and holomorphic functions g_1, g_2, g_3, g_4 on U so that $g_1, g_2, g_3, D^{\gamma+\gamma'-\alpha-\beta-1/2}g_4$ form a basis of $\mathcal{F}$ and that the restrictions of g_1, g_2, g_3 to C are linearly independent and that of g_4 is not (identically) zero.

Under these conditions we shall prove that $\mathcal{F}$ is the solution space $\mathcal{F}_4$ of the Appell's differential equations (F_4).

3.1

Fix a basis f_1, f_2, f_3, f_4 of $\mathcal{F}$.

Let f be any element of $\mathcal{F}$. Then we shall prove that f belongs to $\mathcal{F}_4$, that is, both of

$$Q_1(f) = x(1-x)f_{xx} - y^2f_{yy} - 2xyf_{xy} + (\gamma - (\alpha + \beta + 1)x)f_x - (\alpha + \beta + 1)yf_y - \alpha\beta f$$

$$Q_2(f) = y(1-y)f_{yy} - x^2f_{xx} - 2xyf_{xy} + (\gamma' - (\alpha + \beta + 1)y)f_y - (\alpha + \beta + 1)xf_x - \alpha\beta f$$

are zero.

We define the differential operator D_2 by $D_2(f) = xf_{xx} + yf_{yy} + (x + y - 1)f_{xy}$. Expanding the following determinant (which is identically zero)

$$\begin{vmatrix} f_1 & f_2 & f_3 & f_4 & f \\ f_{1x} & f_{2x} & f_{3x} & f_{4x} & f_x \\ f_{1y} & f_{2y} & f_{3y} & f_{4y} & f_y \\ D_2(f_1) & D_2(f_2) & D_2(f_3) & D_2(f_4) & D_2(f) \\ Q_i(f_1) & Q_i(f_2) & Q_i(f_3) & Q_i(f_4) & Q_i(f) \end{vmatrix} x^{2\gamma} y^{2\gamma'} D^{-\gamma-\gamma'+\alpha+\beta+\frac{3}{2}}$$

by the last column, we obtain the equation

$$(3.1.EQ_i) \quad \Delta_0^i f - \Delta_1^i f_x + \Delta_2^i f_y - \Delta_3^i D_2(f) + \Delta Q_i(f) = 0 \quad i = 1, 2.$$

We will prove that Δ is a nonzero constant and Δ_k^i ($i = 1, 2$ $k = 0, 1, 2, 3$) is identically zero.

3.2 Local behavior of Δ, Δ_k^i

We note that if we exchange the basis f_1, f_2, f_3, f_4 then Δ_k^i and Δ change to $c\Delta_k^i$ and $c\Delta$ for some nonzero constant c .

3.2.1 Behavior along L_x

For each point of $L_x - L_y \cup L_\infty \cup C$, take a neighbourhood U disjoint from $L_y \cup L_\infty \cup C$ and a basis $g_1, g_2, x^{1-\gamma}g_3, x^{1-\gamma}g_4$ of $\mathcal{F}$ satisfying (EXP_x).

Then

$$D_2(x^{1-\gamma}g_k) = x^{-\gamma}((1-\gamma)(-\gamma g_k + (y-1)g_{ky}) + xg'_k)$$

$$Q_i(x^{1-\gamma}g_k) = x^{1-\gamma}g_{ik}$$

for some holomorphic functions g'_k, g_{ik} on U ($k = 3, 4; i = 1, 2$).

$$\begin{aligned} \Delta = & \begin{vmatrix} g_1 & g_2 & xg_3 & xg_4 \\ g_{1x} & g_{2x} & (1-\gamma)g_3 + xg_{3x} & (1-\gamma)g_4 + xg_{4x} \\ g_{1y} & g_{2y} & xg_{3y} & xg_{4y} \\ D_2(g_1) & D_2(g_2) & (1-\gamma)(-\gamma g_3 + (y-1)g_{3y}) & (1-\gamma)(-\gamma g_4 + (y-1)g_{4y}) \\ & & + xg'_3 & + xg'_4 \end{vmatrix} \\ & \times y^{2\gamma'} D^{-\gamma-\gamma'+\alpha+\beta+3/2} \end{aligned}$$

is holomorphic in U , and the restriction $\Delta|_{L_x}$ of Δ to L_x is equal to

$$\begin{aligned} & \begin{vmatrix} g_1 & g_2 & 0 & 0 \\ g_{1x} & g_{2x} & (1-\gamma)g_3 & (1-\gamma)g_4 \\ g_{1y} & g_{2y} & 0 & 0 \\ D_2(g_1) & D_2(g_2) & (1-\gamma)((y-1)g_{3y}-\gamma g_3) & (1-\gamma)((y-1)g_{4y}-\gamma g_4) \end{vmatrix} \\ & \times y^{2\gamma'}(y-1)^{2(-\gamma-\gamma'+\alpha+\beta)+3} \\ & = -(1-\gamma)^2 \begin{vmatrix} g_1 & g_2 \\ g_{1y} & g_{2y} \end{vmatrix} \begin{vmatrix} g_3 & g_4 \\ g_{3y} & g_{4y} \end{vmatrix} y^{2\gamma'}(y-1)^{2(-\gamma-\gamma'+\alpha+\beta+2)} \end{aligned}$$

which is nonzero because both pairs g_1, g_2 and g_3, g_4 are linearly independent on L_x .

By the same reasoning, Δ_k^i are holomorphic on U and Δ_1^i, Δ_3^i vanish on $L_x \cap U$.

3.2.2 Behavior along L_y

For any point of $L_y - L_x \cup L_\infty \cup C$ there exists an open neighbourhood U so that Δ, Δ_k^i are holomorphic on U and that Δ_2^i, Δ_3^i vanish on $L_y \cap U$. Δ is not zero along L_y .

3.2.3 Behavior along L_∞

For any point of $L_\infty - L_x \cup L_y \cup C$ there exists an open neighbourhood U disjoint from $L_x \cup L_y \cup C$ and a basis $\xi^\alpha g_1, \xi^\alpha g_2, \xi^\beta g_3, \xi^\beta g_4$ of $\mathcal{F}$ satisfying

(EXP $_{\infty}$). Then for $f_k = \xi^{\lambda} g_k$ ($k = 1, 2, 3, 4$; $\lambda = \alpha, \beta$)

$$(f_k)_x = \xi^{\lambda+1} f_{k1}, (f_k)_y = \xi^{\lambda+1} f_{k2}, D_2(f_k) = \xi^{\lambda+1} f_{k3}, Q_i(f_k) = \xi^{\lambda+1} f'_{ki}$$

for some holomorphic functions f_{kj}, f'_{ki} . Since $x^{2\gamma} y^{2\gamma'} D^{-\gamma-\gamma'+\alpha+\beta+3/2} = \xi^{-2\alpha-2\beta-3} u$ for some holomorphic function u on U , Δ and Δ_k^i are holomorphic on U and moreover the restrictions $\Delta_0^i|_{L_{\infty}}$ of Δ_0^i to L_{∞} is zero ($i = 1, 2$). Δ is not zero along L_{∞} .

3.2.4 Behavior along C

For any point of $C - L_x \cup L_y \cup L_{\infty}$ there exist an open neighbourhood U disjoint from $L_x \cup L_y \cup L_{\infty}$ and a basis $g_1, g_2, g_3, D^{\gamma+\gamma'-\alpha-\beta-1/2} g_4$ of $\mathcal{F}$ satisfying (EXP $_C$). Put $f_4 = D^{\gamma+\gamma'-\alpha-\beta-1/2} g_4$ then $D_2(f_4)/D^{\gamma+\gamma'-\alpha-\beta-3/2}$ and $Q_i(f_4)/D^{\gamma+\gamma'-\alpha-\beta-3/2}$ are holomorphic on U . Hence Δ and Δ_k^i are all holomorphic on U .

3.2.5 Behavior on $\mathbf{P}^2$

We have proved that Δ and Δ_k^i are holomorphic on $\mathbf{P}^2$ except the six point $L_x \cap L_y, L_x \cap L_{\infty}, L_y \cap L_{\infty}, L_x \cap C, L_y \cap C, L_{\infty} \cap C$. By the Hartogs' extension theorem, Δ and Δ_k^i are holomorphic on $\mathbf{P}^2$. Since Δ_k^i vanish at L_x or L_y or L_{∞} , Δ_k^i are identically zero.

Since Δ is not identically zero, Δ is a non-zero constant.

Thus by (3.1.EQ $_i$), $Q_i(f) = 0$ and $\mathcal{F}$ coincides with the solution space $\mathcal{F}_4$ of the Appell's differential equation (F_4).

4. Riemann's Problem of (F_4) on $\mathbf{P}^1 \times \mathbf{P}^1$

Let

$$\mathcal{F} = P \left\{ \begin{array}{ccccc} x=0 & y=0 & (x-y)^2 - 2(x+y) + 1 = 0 & x=\infty & y=\infty \\ 0 & 0 & 0 & \alpha & \alpha \\ 0 & 0 & 0 & \beta & \beta \\ 1-\gamma & 1-\gamma' & 0 & 1+\alpha-\gamma' & 1+\alpha-\gamma \\ 1-\gamma & 1-\gamma' & \gamma+\gamma'-\alpha-\beta-\frac{1}{2} & 1+\beta-\gamma' & 1+\beta-\gamma \end{array} \right\}$$

We replace the assumption (EXP $_{\infty}$) at the section 3 by the following (EXP $_{x=\infty}$) and (EXP $_{y=\infty}$):

(EXP $_{x=\infty}$): $\mathcal{F}$ has the exponent $(\alpha, \beta, 1 + \alpha - \gamma', 1 + \beta - \gamma')$ along $\{x = \infty\}$.

This means, by definition, that at each point $P = (\infty, b)$ ($b \neq 0, \infty$) there exist an open neighbourhood U and holomorphic functions g_1, g_2, g_3, g_4 on U so that $x^{-\alpha}g_1, x^{-\beta}g_2, x^{-(1+\alpha-\gamma')}g_3, x^{-(1+\beta-\gamma')}g_4$ form a basis of $\mathcal{F}$ and that the restrictions of g_i to $\{x = \infty\}$ are not identically zero.

(EXP $_{y=\infty}$): $\mathcal{F}$ has the exponent $(\alpha, \beta, 1 + \alpha - \gamma, 1 + \beta - \gamma)$ along $\{y = \infty\}$.

Assume that none of $1 - \gamma, 1 - \gamma', \gamma + \gamma' - \alpha - \beta - 1/2$ is zero. Moreover we assume that $\alpha, \beta, 1 + \alpha - \gamma', 1 + \beta - \gamma'$ are all distinct and also that $\alpha, \beta, 1 + \alpha - \gamma, 1 + \beta - \gamma$ are all distinct.

4.1

Fix a basis f_1, f_2, f_3, f_4 of $\mathcal{F}$. Let f be any element of $\mathcal{F}$. Then we shall prove that f belongs to $\mathcal{F}_4$ that is $Q_1(f) = Q_2(f) = 0$, the notations are the same as in 3.1.

4.2 Local behavior of Δ and Δ_k^i on $\mathbf{P}^1 \times \mathbf{P}^1$

4.2.1 Behavior along $\{x = \infty\}$

Assume that g is of order λ at $\{x = \infty\}$ that is $x^\lambda g$ is holomorphic at $\{x = \infty\}$ then $g_y, D_2(g), Q_i(g)$ are of order λ and g_x is of order $\lambda + 1$. Hence Δ, Δ_k^i ($k = 0, 2, 3$) are holomorphic along $\{x = \infty\}$. But Δ_1^i has at most a simple pole along $\{x = \infty\}$.

4.2.2 Behavior along $\{y = \infty\}$

Δ, Δ_k^i ($k = 0, 1, 3$) are holomorphic along $\{y = \infty\}$ and Δ_2^i has at most simple pole along $\{y = \infty\}$.

4.2.3

Δ is holomorphic on $\mathbf{P}^1 \times \mathbf{P}^1$ not identically zero. Hence Δ is a nonzero constant.

Δ_0^i is holomorphic on $\mathbf{P}^1 \times \mathbf{P}^1$. Hence $\Delta_0^i = c^i$ for some constant c^i .

Δ_3^i is holomorphic on $\mathbf{P}^1 \times \mathbf{P}^1$ and vanishes at L_x and L_y . Hence Δ_3^i is zero.

Δ_1^i vanishes at L_x and has at most simple pole along $\{x = \infty\}$ and holomorphic elsewhere. Hence $\Delta_1^i = c_1^i x$ for some constant c_1^i .

$\Delta_2^i = c_2^i y$ for some constant c_2^i .

4.3

The equation (3.1.EQ $_i$) now takes the form

$$(4.3.EQ_i) \quad Q_i(f) - c_1^i x f_x + c_2^i y f_y + c^i f = 0 \quad i = 1, 2.$$

We will show that c_k^i and c^i are all zero. Let $\xi = \frac{1}{x}$ and $f = \xi^\lambda(\varphi_1(y) + \xi\varphi_1(y) + \xi^2\varphi_2(y) + \cdots)$ be a solution of (4.3.EQ_i) ($\lambda = \alpha, \beta, 1 + \alpha - \gamma', 1 + \beta - \gamma'$). Taking the coefficients of ξ^λ in (4.3.EQ_i), we obtain two differential equations $E_i(\varphi_0)$.

$$E_1(\varphi_0): -y^2\varphi_0'' + (2\lambda - \alpha - \beta - 1)y\varphi_0' - (\lambda - \alpha)(\lambda - \beta)\varphi_0 \\ - c_1^1\lambda\varphi_0 + c_2^1y\varphi_0' + c^1\varphi_0 = 0$$

$$E_2(\varphi_0): y(1 - y)\varphi_0'' + (\gamma' + (2\lambda - \alpha - \beta - 1)y)\varphi_0' - (\lambda - \alpha)(\lambda - \beta)\varphi_0 \\ - c_1^2\lambda\varphi_0 + c_2^2y\varphi_0' + c^2\varphi_0 = 0$$

Eliminating φ_0'', φ_0' from $E_1(\varphi_0), E_2(\varphi_0)$ we obtain an equation $E(\lambda) = 0$, where $E(\lambda) = \lambda^4 + \cdots$ is a polynomial of $\lambda, y, \alpha, \beta, \gamma, \gamma', c_j^i, c^i$. The roots of $E(\lambda) = 0$ must be $\alpha, \beta, 1 + \alpha - \gamma', 1 + \beta - \gamma'$. This happens only when $c_1^1 = c_2^1 = c^1 = 0$. By the same consideration at $y = \infty$, we also have $c_1^2 = c_2^2 = c^2 = 0$.

This proves that any f in $\mathcal{F}$ satisfies $Q_i(f) = 0$ $i = 1, 2$, that is, f belongs to $\mathcal{F}_4$.

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