ON A KIND OF REPRESENTATION OF TOPOLOGICAL GROUPS

By

Shin-ichi MATSUSHITA

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1. The purpose of this note is to give a kind of representations of topological groups, not by *matrices* but by *families of matrices*. Our method of discussions is based on the use of the famous theorem of A. MARKOFF¹⁾ and S. KAKUTANI²⁾ concerning on free topological groups. That theorem is written as follows:

For any *completely regular* (c. r.) topological space R , there exists a free topological group F with the following properties;

- i) $R \subset [F]$,
- ii) R generates F algebraically,
- iii) Any continuous mapping φ of R into any topological group G is extended up to the homomorphic mapping ϕ of F into G such that

$$\phi(x) = \varphi(x) \text{ on } R,$$

where $[F]$ is the set of all elements of F with the same topology as F .

Now we shall begin with some notations. Let $\mathfrak{S}$ be an algebra (group, ring, etc.) with some topology or not, and $[\mathfrak{S}]$ be the topological or discrete space of all elements of a topological or discrete algebra $\mathfrak{S}$ respectively.

Further let $\chi = \{x_\lambda\}_{\lambda \in \Lambda}$ be an ordered finite or transfinite family of the elements x_λ of $\mathfrak{S}$ with a set of suffixes Λ , where $\mathfrak{S}$ is a topological or discrete ring. Then we define:

$$D_1) \quad \chi + \chi' = \{x_\lambda + x'_\lambda\}_{\lambda \in \Lambda},$$

where $\chi = \{x_\lambda\}_{\lambda \in \Lambda}$ and $\chi' = \{x'_\lambda\}_{\lambda \in \Lambda}$.

$$D_2) \quad \chi = \chi' \text{ implies that } x_\lambda = x'_\lambda \text{ for all } \lambda \in \Lambda.$$

$$D_3) \quad \chi = \mathfrak{M}, \text{ where } \mathfrak{M} \text{ is a set, implies that } \chi \text{ coincides } \mathfrak{M}$$

in respect to their elements; that is, $x_\lambda \in \mathfrak{M}$ for all $\lambda \in A$ and all m , $m \in \mathfrak{M}$ belong to γ . In this case we are able to put $\mathfrak{M}$ in order according to the corresponding elements of γ . After these procedures, this equality has the same meaning as in D_2).

If $\mathfrak{S}$ has a operator domain A , then we define that

$$D_4) \quad \alpha \cdot \gamma = \{\alpha \cdot x_\lambda\}_{\lambda \in A}, \quad \alpha \in A.$$

And moreover we consider the product set of γ and γ' such that

$$D_5) \quad \gamma * \gamma' \text{ is a set of all } x_\lambda \cdot x'_\mu; \quad \lambda, \mu \in A, \text{ where } x_\lambda \in \gamma \text{ and } x'_\mu \in \gamma'.$$

2. We now assume that G is a *c. r.* topological group: so is the topological space $[G]$. We shall next consider a continuous mapping ϕ of $[G]$ into G , such that $\phi(x)=x$, *i. e.*

$$(2.1) \quad \phi([G]) = G,$$

then we can obtain, from the above theorem, a free topological group F , such that

$$(2.2) \quad \phi(F) \subset G,$$

and

$$(2.3) \quad [G] \subset [F] \text{ besides } [G] \text{ generates } F \text{ algebraically.}$$

Combining (2.1) with (2.2) we have

$$G = \phi([G]) \subset \phi(F) = G,$$

that is

$$\phi(F) = G.$$

Since F is a free topological group, F is *maximally almost-periodic* (*max. a. p.*).³⁾ Then we conclude:

Lemma 1. For any c. r. topological group G , there exists at least one max. a. p. group $\mathfrak{G}$ which has the following properties:

$$A_1) \quad \text{There exists a topological homomorphism } \phi \text{ of } \mathfrak{G} \text{ onto } G,$$

$$\phi(\mathfrak{G}) = G.$$

$$A_2) \quad [G] \subset \mathfrak{G} \text{ and } [G] \text{ is the group-generator of } \mathfrak{G}.$$

$$A_3) \quad \phi \text{ is invariant on } [G], \text{ i. e.}$$

$$\phi(x) = x, \quad x \in [G].$$

For a given *c. r.* topological group G , suppose G^* *max. a. p.* and that G^* has the properties $A_1)$ – $A_3)$ as above. Let $H_c \subset G^*$ be the kernel of the homomorphism ϕ , $\phi(G^*) = G$, *i. e.*

$$\phi(H_c) = e,$$

where e is the unit of G , and H_x be a restclass containing x , while $H_{(x)}$ a restclass which corresponds to x , with respect to the factor-group G/H . Then we have immediately that

$$(2.4) \quad H_x = H_{(x)}.$$

We shall distinguish the group-operation of G^* from that of G , writing the former by $x^* \cdot y^*$ in G^* and the latter by xy in G . Then we have

$$(2.5) \quad H_x \cdot H_y = H_{x \cdot y} = H_{xy} (= H(xy)).$$

Let $\tilde{H} = \{e_\lambda\}_{\lambda \in \Lambda}$ be an ordered family of elements of H , and $\tilde{H}_\lambda = \{x_\lambda\}_{\lambda \in \Lambda}$, $x_\lambda = x \cdot e_\lambda$. Then we have

$$(2.4)' \quad [\tilde{H}_x] = [H_x] = [H(x)],$$

where $[H_x]$ means the set of all elements of H_x , and

$$(2.5)' \quad [\tilde{H}_x \cdot \tilde{H}_y] = [H_x \cdot H_y]$$

therefore we have, from (2.4)' and (2.5)', that

$$(2.6) \quad [\tilde{H}_x \cdot \tilde{H}_y] = [H_{xy}].$$

3. Let $D(x^*)$, $x^* \in G^*$ be an irreducible unitary representation of G by a finite dimensional matrix, and put

$$\mathfrak{D}(x) = \{D(x_\lambda)\}_{\lambda \in \Lambda}$$

for $\tilde{H}_x = \{x_\lambda\}_{\lambda \in \Lambda}$.

Lemma 2. The totality of $\mathfrak{D}(x)$, $x \in G$, forms a group Γ which is homomorphic to G .

Proof. We shall show that

$$(3.1) \quad \mathfrak{D}(x) * \mathfrak{D}(y) = \mathfrak{D}(xy).$$

It follows, from the definition $D_{\tilde{H}}$, that

$$\mathfrak{D}(x) * \mathfrak{D}(y) = [D(x_\lambda) \cdot D(y_\mu)]_{\lambda, \mu \in \Lambda}$$

and from (2.6)

$$x_\lambda \cdot y_\mu \in [\tilde{H}_{xy}].$$

Hence we have

$$D(x_\lambda) \cdot D(y_\mu) = D(x_\lambda \cdot y_\mu) \in [\mathfrak{D}(xy)],$$

that is

$$[\mathfrak{D}(x) * \mathfrak{D}(y)] \subset \mathfrak{D}(xy).$$

Conversely, from (2.5)', we can find two suffixes $\mu, \nu \in \Lambda$ such that

$$(xy)_\lambda = x_\mu \cdot y_\nu$$

and then

$$D((xy)_\lambda) = D(x_\mu \cdot y_\nu) = D(x_\mu) \cdot D(y_\nu) \\ \in [\mathfrak{D}(x) * \mathfrak{D}(y)],$$

that is

$$[\mathfrak{D}(xy)] \subset [\mathfrak{D}(x) \cdot \mathfrak{D}(y)].$$

Finally we can conclude (3.1). It is clear that Γ forms a group, providing $\mathfrak{D}(e)$ to be its unit. Moreover we have

$$\mathfrak{D}(x^{-1}) = \mathfrak{D}(x)^{-1}.$$

We call Γ a representation of G by families of matrices. Γ being dependent on the group G^* , we shall precisely write

$$\Gamma = \Gamma_{G^*}.$$

If there exists a sub-group K of G^* and

$$K \cong G^{*'},$$

$G^{*'}$ being max. a. p. and having the properties $A_1) - A_3)$, then we say that Γ_{G^*} is *reducible* and denote

$$\Gamma_{G^{*'}} < \Gamma_{G^*}.$$

If, for any $G^{*'}$ such that $\Gamma_{G^{*'}} < \Gamma_{G^*}$, it must be that $G^{*'} = G^*$, then we call Γ_{G^*} *irreducible*.

4. We now construct a ring by using $D_1)$ and $D_4)$ from the group Γ_{G^*} , and denote it by R_{G^*} . Next we intend to define a topology on R_{G^*} . For $\mathfrak{A} = \{A_\lambda\}_{\lambda \in \Lambda} \in R_{G^*}$, putting

$$(4.1) \quad \|\mathfrak{A}\| = \sup_{\lambda \in \Lambda} \|A_\lambda\|,$$

where $\|A_\lambda\|$ is the usual norm with respect to matrices, i.e.

$\|A_\lambda\| = \sqrt{\sum_{i,j} |a_{ij}^\lambda|^2}$, we call $\|\mathfrak{A}\|$ the norm of $\mathfrak{A}$. Then we can introduce a topology on R_{G^*} by such norms (4.1).

Lemma 3. If $x \rightarrow y$ on G , then $\mathfrak{D}(x) \rightarrow \mathfrak{D}(y)$.

Proof. $x \rightarrow y$ (on G) implies $x \rightarrow y$ (on G^*) and so we have $D(x) \rightarrow D(y)$. As $D(x^*)$, $x^* \in G$ is bounded, we have

$$\begin{aligned} \|\mathfrak{D}(x) - \mathfrak{D}(y)\| &= \sup_{\lambda \in \Lambda} \|D(x_\lambda) - D(y_\lambda)\| \\ &= \sup_{\lambda \in \Lambda} \|D(x \cdot e_\lambda) - D(y \cdot e_\lambda)\| \\ &= \sup_{\lambda \in \Lambda} \|(D(x) - D(y)) \cdot D(e_\lambda)\| \\ &= \sup_{\lambda \in \Lambda} \|D(x) - D(y)\| \cdot \|D(e_\lambda)\| \leq M \cdot \|D(x) - D(y)\|, \end{aligned}$$

that is

$$\mathfrak{D}(x) \rightarrow \mathfrak{D}(y).$$

Hence Γ_G^* becomes a topological representation of G .

Theorem 1. Every c. r. topological group is topologically represented by the families of finite dimensional unitary matrices.

We next denote the set of all suffices λ such that

$$(4.2) \quad x_n^{-1} \cdot x_{n-1}^{-1} \dots x_1^{-1} \cdot x_1 x_2 \dots x_n = e_\lambda$$

for $n=1, 2, \dots$ and $x_k \in G$, by $A' (\subset A)$. Here x^{-1} means the inverse of x in G^* . Then we have;

Lemma 4. The two conditions $B_1), B_2)$ are equivalent one another for $D \in \Gamma_G^$, which are respectively written as follows;*

$$(4.3) \quad \begin{aligned} B_1) \quad D(e_\lambda) &= 1 \text{ for all } \lambda \in A. \\ B_2) \quad D(e_\lambda) &= 1 \text{ for all } \lambda \in A'. \end{aligned}$$

Proof. $B_1) \rightarrow B_2)$ is clear. $B_2) \rightarrow B_1)$: We now show that the totality of such $D(x)$, $x \in G$ that has the property $B_2)$ forms a representation of G . For such $D(x^*)$, $x^* \in G^*$, we have

$$\begin{aligned} D(y^{-1} \cdot x^{-1} \cdot xy) &= D(y^{-1}) D(x^{-1}) \cdot D(xy) \\ &= D(y)^{-1} \cdot D(x)^{-1} \cdot D(xy) = 1, \end{aligned}$$

that is,

$$(4.4) \quad D(xy) = D(x) \cdot D(y),$$

for every $x, y \in G$, and so $D(e) = 1$.

Next we see that each e_μ , $\mu \in A$ has the representation as

$$e_\mu = x_1 \cdot x_2 \dots x_n, \quad x_k \in G$$

and

$$x_1 x_2 \dots x_n = e.$$

Then we can find a suitable $\lambda \in A'$ such that

$$e_\mu = (x_1 x_2 \dots x_n) \cdot e_\lambda^{-1}$$

and so

$$D(e_\mu) = D(x_1 x_2 \dots x_n) \cdot D(e_\lambda^{-1}) = D(e) \cdot D(e_\lambda)^{-1} = 1.$$

Thus $B_2) \rightarrow B_1)$ is proved.

Theorem 2. A necessary and sufficient condition for a c. r. topological group G to have a finite dimensional unitary representation by a family of matrices $\mathfrak{D}(x)$, $x \in G$, is

$$D(e_\lambda) = 1, \text{ for all } \lambda \in A,$$

where $\mathfrak{D}(x) = \{D(x_\lambda)\}_{\lambda \in A}$.

Proof. If $\mathfrak{D}(x)$ is a finite dimensional unitary representation of G , then $D^*(x^*)$, $x^* \in G$, defining as

$$D^*(x_\lambda) = D(x) \text{ for all } \lambda \in A,$$

is that of G^* . The necessity is obtained from the fact (4. 4).

5. Now we intend to consider more deeply the characters of $\mathfrak{D}(x)$. Let $\mathfrak{D}_{pq}(x)$ be a function on G defined as

$$\mathfrak{D}_{pq}(x) = \{D_{pq}(x_\lambda)\}_{\lambda \in \Lambda},$$

where $D(x^*) = (D_{pq}(x^*))$, then the range of $\mathfrak{D}_{pq}(x)$ is contained in $\hat{K} = \bigotimes_{\lambda \in \Lambda} K_{(\lambda)}$, where each $K_{(\lambda)}$ is the topological space coinciding with the field of complex-numbers K .

$D \in I_{G^*}$, Putting $|\mathfrak{D}_{pq}(x)| = \sup_{\lambda \in \Lambda} |D_{pq}(x_\lambda)|$ and $\|\mathfrak{D}_{pq}\| = \sup_{x \in G} |\mathfrak{D}_{pq}(x)|$, it is easily seen that $\|\mathfrak{D}_{pq}\| = \|D_{pq}\|$, because we have

$$\begin{aligned} \|\mathfrak{D}_{pq}\| &= \sup_{x \in G} \sup_{\lambda \in \Lambda} |D_{pq}(x_\lambda)| \\ &= \sup_{x^* \in G^*} |D_{pq}(x^*)| = \|D_{pq}\|. \end{aligned}$$

Then the almost periodicity of $D_{pq}(x^*)$ suggests that of $\mathfrak{D}_{pq}$, and we claim:

Theorem 3. $\mathfrak{D}_{pq}(x)$ is a a. p. function in the sense of S. BOCHNER and J. VON NEUMANN,⁴⁾ and G is max. a. p. in the above sense.

The max. a. periodicity of G in the above sense is due to the fact that, from that of G^* , we can find a finite dimensional irreducible unitary representation $D(x^*)$ of G^* such that

$$D(x_\lambda) \neq D(y_\lambda)$$

for arbitrary two x, y of G and $\lambda \in \Lambda$. Then we see immediately that

$$\mathfrak{D}(x) \neq \mathfrak{D}(y),$$

that is to say, G is max. a. p.

6. Finally we shall concern the group-ring L_G in the sense of I. E. SEGEL⁵⁾ of a locally bicomact group G with respect to the right-invariant Haar's measure on G ; the multiplication and the norm are defined as follows:

$$(6. 1) \quad f \times g = \int_G f(xy^{-1})g(y)dy,$$

$$(6. 2) \quad \|f\| = \int_G |f(x)| dx,$$

where $f, g \in L_G$. To begin with, putting

$$(6. 3) \quad e(x_1, x_2, \dots, x_n) = x_n^{-1} \cdot x_{n-1}^{-1} \dots x_1^{-1} \cdot x_1 x_2 \dots x_n,$$

and for a finite dimensional irreducible unitary representation $D(x^*)$ of G^* ,

$$(6.4) \quad D(e(x_1, x_2, \dots, x_n)) = \phi(x_1, x_2, \dots, x_n),$$

we define that

$$(6.5) \quad D(f_1) \times D(f_2) \times \dots \times D(f_n) \\ = \int \int \dots \int_{\underbrace{G \times G \times \dots \times G}_n} D(f_1, x) \cdot D(f_2, x) \dots D(f_n, x) \cdot \phi(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n,$$

where $D(f_k, x_k) = f(x_k) D(x_k)$ for $f \in L_G$.

When $n=1$, we see immediately that $e(x)$, $x \in G$, coincides with the unit e^* of G^* and so $\phi(x)=1$ for all $x \in G$; then we have

$$(6.6) \quad D(f) = \int_G f(x) D(x) dx$$

and

$$(6.7) \quad D(\alpha f + \beta g) = \alpha D(f) + \beta D(g), \quad \alpha, \beta \in K,$$

$$(6.8) \quad \|D(f)\| \leq d(D) \cdot M \cdot \|f\|,$$

where $d(D)$ is the degree of $D(x^*)$ and $\|D_{pq}(x^*)\| \leq M$, for $p, q=1, 2, \dots, d(D)$, $x^* \in G^*$, because we get

$$\|D(f)\| = \left(\sum_{p,q} \left| \int_G f(x) D(x) dx \right|^2 \right)^{1/2} \\ \leq [d(D)^2 \cdot M^2 \left(\int_G |f(x)|^2 dx \right)]^{1/2} = d(D) \cdot M \cdot \|f\|.$$

When $n=2$, we have

$$D(f) \times D(g) = \int \int_{G \times G} f(x) D(x) \cdot g(y) \cdot D(y) \phi(x, y) dx dy \\ = \int \int_{G \times G} f(xy^{-1}) g(y) D(xy^{-1}, y) \phi(xy^{-1}, y) dx dy \\ = \int \int_{G \times G} f(xy^{-1}) g(y) dy \cdot D(x) dx \\ = \int_G f \times g(x) \cdot D(x) dx = D(f \times g).$$

And in this manner, for every n we get likewise

$$D(f_1) \times D(f_2) \times \dots \times D(f_n) = D(f_1 \times f_2 \times \dots \times f_n).$$

Theorem 4. The normed ring $R(D)$, which is generated from $D(f)$, $f \in L_G$, by the usual addition of matrices and by the multiplication (6.5), is topologically homomorphic to L_n .

The statement is easily obtained from (6.7), (6.8) and the above gotten equality. To obtain $D(f_1) \times D(f_2) \times \dots \times D(f_n) = D(f_1) \cdot D(f_2) \dots \cdot D(f_n)$, it is necessary that $\phi(x_1, x_2, \dots, x_n) = 1$ almost everywhere on $G_{(1)} \times G_{(2)} \times \dots \times G_{(n)}$. $\phi(x_1, x_2, \dots, x_n)$ being continuous, it must be always equal to 1. Referring *Theorem 3*, we establish:

Theorem 5. The following two conditions are mutually equivalent:

1) The normed ring $R_n(D)$ which is generated from $D(f)$, $f \in L_n$, by the usual addition and multiplication of matrices is topologically homomorphic to L_n .

2) For $R_n(D)$, $D(x)$ is a topological representation of G .
This statement is already obtained in another way by K. IWASAWA.⁶⁾⁷⁾

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