ON SYMMETRIC POLYNOMIALS

by

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1. It is well known that any symmetric polynomial $P(x_1, x_2, \dots, x_n)$ of n variables $x_1, x_2, \dots, x_n$ is uniquely represented by the polynomial $Q(a_1, a_2, \dots, a_n)$ of n fundamental polynomials $a_1, a_2, \dots, a_n$, where

$$a_k = \sum_{i_1 \dots i_k}^{1 \dots n} x_{i_1} x_{i_2} \dots x_{i_k}, \quad k=1, 2, \dots, n.$$

Here $\sum_{i_1 \dots i_k}^{1 \dots n}$ means the sum for $i_1=1, \dots, n; \dots; i_k=1, \dots, n$. In this paper we shall go more deeply to decide the form $Q(a_1, a_2, \dots, a_n)$ exactly.

2. Putting $s_p = x_{p_1} + x_{p_2} + \dots + x_{p_n}$, we shall consider the relations between $a_k; k=1, 2, \dots, n$ and $s_p; p=1, 2, \dots$.

We begin with the equality

$$(1+x_1 t)(1+x_2 t) \dots (1+x_n t) = 1 + a_1 t + a_2 t^2 + \dots + a_n t^n.$$

Taking the logarithms of both sides for real numbers x_i and sufficiently small $|t|$, we have

$$\sum_i^{1 \dots n} \log(1+x_i t) = \log(1+a_1 t + \dots + a_n t^n),$$

that is,

$$\sum_i^{1 \dots n} \sum_p^{1 \dots \infty} \frac{(-1)^{p-1}}{p} (x_i t)^p = \sum_p^{1 \dots \infty} \frac{(-1)^{p-1}}{p} (a_1 t + \dots + a_n t^n)^p.$$

Comparing the coefficients of t^p of both sides, we have

$$s_p = p \sum_{(\alpha_1 \dots \alpha_n)} \frac{(-1)^{\alpha_1 + \dots + \alpha_n - p} (\alpha_1 + \dots + \alpha_n - p)!}{\alpha_1! \dots \alpha_n!} a_1^{\alpha_1} \dots a_n^{\alpha_n}.$$

Here $\sum_{(\alpha_1 \dots \alpha_n)}^{\alpha_1 + \dots + \alpha_n = p}$ means the sum for all $(\alpha_1 \dots \alpha_n)$ such that $\alpha_1 + 2\alpha_2 + \dots + n\alpha_n = p$.

It is clear that this equality holds algebraically

3. Put

$$(1) \quad P(x_1, x_2, \dots, x_n) = \sum_{p_1 \dots p_n}^{0 \dots \infty} c_{p_1 \dots p_n} x_1^{p_1} \dots x_n^{p_n},$$

where only finite coefficients $c_{p_1 \dots p_n}$ differ from 0. Operating a substitution $\begin{pmatrix} x_1 \dots x_n \\ x_{r_1} \dots x_{r_n} \end{pmatrix}$ in (1), we have

$$P(x_{r_1}, x_{r_2}, \dots, x_{r_n}) = \sum_{p_1 \dots p_n}^{0 \dots \infty} c_{p_1 \dots p_n} x_{r_1}^{p_1} \dots x_{r_n}^{p_n}.$$

There exist $n!$ kinds of $P(x_{r_1}, x_{r_2}, \dots, x_{r_n})$ in those manners and each of them coincides with $P(x_1, x_2, \dots, x_n)$ owing to the symmetricity of $P(x_1, x_2, \dots, x_n)$. Taking the arithmetical mean of them, we have

$$P(x_1, x_2, \dots, x_n) = \sum_{p_1 \dots p_n}^{0 \dots \infty} \frac{c_{p_1 \dots p_n}}{n!} \sum_{(r_1 \dots r_n)}^{1 \dots n} x_{r_1}^{p_1} \dots x_{r_n}^{p_n}.$$

Here $\sum_{(r_1 \dots r_n)}^{1 \dots n}$ means the sum for all permutations $(r_1 \dots r_n)$ of $1, 2, \dots, n$.

4. Then we have only to represent any symmetric polynomial $\sum_{(r_1 \dots r_n)}^{1 \dots n} x_{r_1}^{p_1} \dots x_{r_n}^{p_n}$ by the polynomial of such s_i ; $i=1, 2, \dots$.

Theorem 1. Any symmetric polynomial $\sum_{(r_1 \dots r_n)}^{1 \dots n} x_{r_1}^{p_1} \dots x_{r_n}^{p_n}$ is represented in the form

$$\begin{vmatrix} (s_{p_1})_1^1 & (s_{p_1})_2^1 & \dots & (s_{p_1})_n^1 \\ (s_{p_2})_1^2 & (s_{p_2})_2^2 & \dots & (s_{p_2})_n^2 \\ \dots & \dots & \dots & \dots \\ (s_{p_n})_1^n & (s_{p_n})_2^n & \dots & (s_{p_n})_n^n \end{vmatrix} \\ = \sum_{(\alpha_1 \dots \alpha_n)}^{1 \dots n} \text{sgn} \begin{pmatrix} 1 & 2 & \dots & n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix} (s_{p_1})_{\alpha_1}^1 (s_{p_2})_{\alpha_2}^2 \dots (s_{p_n})_{\alpha_n}^n$$

where $(s_{p_1})_{\alpha_1}^1 (s_{p_2})_{\alpha_2}^2 \dots (s_{p_n})_{\alpha_n}^n$ means

$$s_{p_1 \nu(1,1) + \dots + p_1 \nu(1,k_1)} \times \dots \times s_{p_l \nu(l,1) + \dots + p_l \nu(l,l)},$$

when

$$(\nu(1, 1), \dots, \nu(1, k_1)) \dots (\nu(l, 1), \dots, \nu(l, k_l))$$

is the decomposition of $\begin{pmatrix} 1 & \dots & n \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}$ into cyclic substitutions.

For instance we shall explain about $\sum_{(r_1 \dots r_4)}^{1 \dots 4} x_{r_1}^{p_1} \dots x_{r_4}^{p_4}$.

The decompositions of all substitutions of four natural numbers 1, 2, 3, 4 into cycles are as follows:

$$\begin{array}{rcccccc}
(1)(2)(3)(4) + & (3)(4)(12) & - & (4)(132) & + & (1432) & - \\
(1)(234) & + & (1234) & - & (2)(134) & + & (2)(3)(14) & - \\
(1)(243) & + & (1234) & - & (13)(24) & + & (1423) & - \\
(1)(3)(24) & - & (3)(124) & + & (1324) & - & (14)(23) & + \\
(1)(2)(34) & - & (12)(34) & + & (1342) & - & (3)(142) & + \\
(1)(4)(23) & - & (4)(123) & + & (2)(4)(13) & - & (2)(143) & +,
\end{array}$$

where the symbol $+$ or $-$ means that the substitution is even or odd respectively. According to the notation of expansion in this theorem, we have

[illegible]

The above theorem follows from the next theorem.

Theorem 2. Let

$$V^i = (v_1^i, v_2^i, \dots, v_n^i); \quad i=1, 2, \dots, n,$$

be n vectors. Then it follows that

$$\begin{aligned}
 (2) \quad & \sum_{(\beta^1 \dots \beta^n)}^{1 \dots n} v_{\beta^1 \dots}^1 v_{\beta^n}^n = \begin{bmatrix} (V^1)_1^1 \dots (V^1)_n^1 \\ \vdots \\ (V^n)_1^n \dots (V^n)_n^n \end{bmatrix} \\
 & = \sum_{(\alpha^1 \dots \alpha^n)}^{1 \dots n} sgn \left(\begin{smallmatrix} 1 \dots n \\ \alpha^1 \dots \alpha^n \end{smallmatrix} \right) (V^1)_{\alpha^1}^1 \dots (V^n)_{\alpha^n}^n.
 \end{aligned}$$

Here, if we denote the sets of all elements which belong to each cycle in the decomposition of the substitution $\left(\begin{smallmatrix} 1 \dots n \\ \alpha_1 \dots \alpha_n \end{smallmatrix}\right)$ into cycles by $M_1, M_2, \dots, M_k$, then we define that

$$(3) \quad (V^1)_{\alpha_1}^1 \dots (V^n)_{\alpha_n}^n = \left(\sum_j \prod_i v_j^i \right) \times \dots \times \left(\sum_j \prod_i v_j^i \right).$$

Remark. Putting $v_j^i = x_j^{p_i}$, we can easily obtain Theorem 1 from this theorem.

Proof of theorem 2. It is clear that we are able to reform the right side of (2) into the form

$$\sum_{\beta_1 \dots \beta_n} c_{\beta_1 \dots \beta_n} v_{\beta_1}^1 \dots v_{\beta_n}^n.$$

Then we have only to prove that

$$(4) \quad c_{\beta_1 \dots \beta_n} = \begin{cases} 0; & \text{if } \beta_i = \beta_j \text{ for some } i \neq j. \\ 1; & \text{if } \beta_i \neq \beta_j \text{ for all } i \neq j. \end{cases}$$

Expanding the right side of (3), we have

$$(5) \quad \sum_{j_1 \dots j_k} \left(\prod_i v_{j_1}^i \prod_i v_{j_2}^i \dots \prod_i v_{j_k}^i \right).$$

If a term $v_{\beta_1}^1 v_{\beta_2}^2 \dots v_{\beta_n}^n$ belongs to this formula, we say that the term is derived by the substitution $S = \begin{pmatrix} 1 \dots n \\ \alpha_1 \dots \alpha_n \end{pmatrix}$.

When S runs over all substitutions by which a fixed term $v_{\beta_1}^1 v_{\beta_2}^2 \dots v_{\beta_n}^n$ is derived, then

$$(6) \quad \sum \operatorname{sgn} S = c_{\beta_1 \dots \beta_n},$$

because the coefficient of each term $v_{\beta_1}^1 v_{\beta_2}^2 \dots v_{\beta_n}^n$ in the formula (5) is equal to 1 or 0.

Now we shall consider the condition for a term $v_{\beta_1}^1 \dots v_{\beta_n}^n$ to be derived by a substitution $S = \begin{pmatrix} 1 \dots n \\ \alpha_1 \dots \alpha_n \end{pmatrix}$.

In the decomposition of S into cycles the sets of all elements which belong to each cycle have been denoted by $M_1, M_2, \dots, M_k$. We next classify the set $\{1, 2, \dots, n\}$ by unifying i and j such that $\beta_i = \beta_j$ and denote these classes by $N_1, N_2, \dots, N_l$. If $v_{\beta_1}^1 \dots v_{\beta_n}^n$ is a term of the formula (5), then

(*) Every N_λ ; $\lambda = 1, 2, \dots, l$ is the union of some M_μ ; $\mu = 1, 2, \dots, k$.

Conversely if the condition (*) is satisfied, then $v_{\beta_1}^1 \dots v_{\beta_n}^n$ is a term of the formula (5).

Put $S = C_1 C_2 \dots C_k$, where C_μ is a cycle of M_μ .

If the condition (*) is satisfied, we have

(*) $S = S_1 S_2 \dots S_l$, where S_λ is a substitution of N_λ ,

because the product of some number of C_μ is a substitution of N_λ . Conversely if the condition (*) is satisfied, on decomposing S into cycles, each S_λ is decomposed into cycles. As S_λ is a substitution of N_λ , the condition (*) is satisfied.

Finally we reach to prove the condition (4).

i) If there exists at least one N_λ which contains no less than two elements, by considering the condition (*), the totality of S by which a term $v_{\beta_1}^1 \dots v_{\beta_n}^n$ is derived contains the same number of even and odd substitutions; hence

$$\sum \operatorname{sgn} S = 0,$$

that is, if $\beta_i = \beta_j$ for some $i \neq j$, from (6)

$$c_{\beta_1 \dots \beta_n} = 0.$$

ii) If every N_λ consists of only one element, by considering the condition (*), the substitution S by which a term $v_{\beta_1}^1 \dots v_{\beta_n}^n$ is derived must be the identical substitution $(1)(2) \dots (n)$; hence

$$\sum \operatorname{sgn} S = 1,$$

that is, if $\beta_i \neq \beta_j$ for all $i \neq j$, from (6)

$$c_{\beta_1 \dots \beta_n} = 1.$$

References

- FAA DI BRUNO; *Théorie des formes binaires*.
VANDER WAERDEN; *Moderne Algebra*.