ON ERGODIC THEOREM IN REFLEXIVE SPACES

By

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Let T be a linear operator on a BANACH space B into itself, and $\|T\| \leq 1$. We shall call an element $x \in B$ "ergodic" if and only if the averages x_n of the $T^k(x)$, defined as $x^n = \sum_{k=0}^{n-1} T^k(x)/n$, converge metrically to a fix point.

In case B is locally weakly compact, every element is ergodic⁽¹⁾. This and the fact "a reflexive space is locally weakly compact"⁽²⁾ lead us to the

Theorem. *Every element of a reflexive space is ergodic.*

In the present note we attempt to give a direct proof of this theorem.

We set

$$T_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k,$$

R ; the range of the operator $T-1$,

Z ; the set of such elements x as $T(x)=x$, viz. fix points of T .

1. *Every element of $\bar{R}$ is ergodic and its limit is θ ⁽³⁾.*

Suppose $T_n(y)$ does not converge to θ ,

$$\lim_{n \rightarrow \infty} \|T_n(y)\| = \alpha > 0.$$

From the condition $\|T_n\| \leq 1$, we have for arbitrary element $(T-1)(z)$ of R

$$\begin{aligned} \|y - (T-1)(z)\| &\geq \|T_n(y - (T-1)(z))\| = \|T_n(y) - \frac{1}{n}(T^n - 1)(z)\| \\ &\geq \|T_n(y)\| - \frac{2}{n}\|z\|, \end{aligned}$$

whence $\|y - (T-1)(z)\| \geq \alpha$.

(1) G. Birkhoff, Lattice theory. p. 268.

(2) V. Gantmacher and V. Šmulian: c.k. URSS. 17 (1937), 91.

(3) θ is the zero element of B .

This proves that y does not belong to $\overline{R}$.

2. Z and $\overline{R}$ have only the zero element in common.

If $x \in Z$, we have $x = T_n(x)$.

Assuming that $x \in \overline{R}$, and letting $n \rightarrow \infty$, we see that $x = \theta$.

3. $Z + \overline{R}$ is closed.

If $x \in Z$, $y \in \overline{R}$, then $\|x + y\| \geq \|T_n(x + y)\| \geq \|x\| - \|T_n(y)\|$, from which we get, letting $n \rightarrow \infty$, $\|x + y\| \geq \|x\|$.

This lemma makes way for our proof.

Let us consider a sequence $\{z_n\}$ which converges to z , together with

$$z_n = x_n + y_n, \quad x_n \in Z, \quad y_n \in \overline{R}, \quad (n=1,2,3,\dots).$$

By the above proposition, we have

$$\|z_m - z_n\| = \|(x_m - x_n) + (y_m - y_n)\| \geq \|x_m - x_n\|.$$

Since $\{z_n\}$ is convergent, $\{x_n\}$ also converges to an element x , which is contained in Z because of its closedness. Accordingly $\{y_n\}$ converges and its limit y belongs to $\overline{R}$, moreover

$$z = x + y \in Z + \overline{R},$$

which is to be proved.

4. Concerning T^* , we obtain similar results, since $\|T^*\| = \|T\| \leq 1$. Among others, we have the proposition 2 with respect to T^* .

Z^* and $\overline{R}^*$ have only the zero functional in common, where R^* is the range of the operator $T^* - 1$ and Z^* is the set of such functionals x^* as $T^*(x^*) = x^*$.

5. $B = Z + \overline{R}$.

If it is not the case, we can find such a non-zero linear functional z_0^* as vanishes on $Z + \overline{R}$, since $Z + \overline{R}$ is a closed and linear set. Then we have, for all elements $z \in B$,

$$(T^* - 1)z_0^*[z] = z_0^*[(T - 1)(z)] = 0,$$

since $(T - 1)(z) \in R$.

Therefore $(T^* - 1)z_0^* = \theta^*$, or $T^*z_0^* = z_0^*$.

Thus we get in the first place $z_0^* \in Z^*$.

In the second place we shall verify that

$$z_0^* \in \overline{R}^*.$$

Suppose $z_0^* \notin \overline{R}^*$. We can find that element $x \in B$, owing to the

reflexibility of the space $\mathbf{B}$, which satisfies the conditions;

$$\begin{aligned} y^*[x] &= 0, \text{ for all } y^* \in \overline{R}^*, \\ z_0^*[x] &= 1. \end{aligned}$$

But they contradict with each other, since the former implies

$$\begin{aligned} (T^* - 1)(z^*)[x] &= 0 \text{ for all } z^* \in \mathbf{B}^* \\ z^*[(T - 1)(x)] &= 0. \end{aligned}$$

This means $(T - 1)(x) = \theta$, $T(x) = x$,
hence $x \in Z$, and so $z_0^*[x] = 0$.

Thus it follows from $\mathbf{B} \neq Z + \overline{R}$ that there exists a non-zero functional z_0^* that is contained simultaneously in Z^* and $\overline{R}^*$, which is in conflict with 4.

6. Here we have made all necessary preparations for the proof of the theorem.

5 and 2 shows that $\mathbf{B} = Z \oplus \overline{R}$, namely, every element z of $\mathbf{B}$ determines uniquely such x, y as

$$z = x + y, \quad x \in Z, \quad y \in \overline{R}.$$

$T_n(x) = x$, $T_n(y) \rightarrow \theta$ as $n \rightarrow \infty$, accordingly $T_n(z) \rightarrow x$ as $n \rightarrow \infty$. This completes the proof.