ON A CERTAIN COMPLETE, SEPARABLE AND METRIC SPACE.

By

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Introduction. As a characterization of a HILBERT space, J. von NEUMANN considered the next condition in a separable BANACH space R : $(\alpha) |f+g|^2 + |f-g|^2 = 2(|f|^2 + |g|^2)$ for all f, g of R .

In our paper we will discuss the properties of a complete, separable and metric space R with the condition (α) , *except the condition of linearity*. But in our space R , the condition (α) must be changed as follows; For arbitrary two points B, C of R , there exists a point P satisfying the condition:

$$(I) \quad \overline{AB}^2 + \overline{AC}^2 = 2\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 \quad \text{for all } A \text{ of } R.$$

The above condition must be supposed for all A of R . Otherwise, the point P subject to the condition (I), in a EUCLIDEAN space, runs over the circumference of the diameter $\overline{AM}$, where M is the middle point of the segment $[B, C]$, therefore the condition (I) does not imply the condition (α) . If the condition (I) is supposed for all A of R , then the point P satisfying our condition (I) is unique, and as to such a point P , $\overline{BP} = \overline{CP} = \frac{1}{2}\overline{BC}$, as will be shown in §1. Therefore, we can rewrite the condition (I) as follows:

$$\overline{AB}^2 + \overline{AC}^2 = 2(\overline{AP}^2 + \overline{BP}^2).$$

That is, the condition (I) implies the condition (α) .

In this paper we will derive the conclusion that R is isometrically isomorphic with a closed convex in a HILBERT space. If our space R is linear, a closed convex in our conclusion is extended to a linear subspace in a HILBERT space. Therefore the case considered by J. von Neumann is included in our one.

§1. $[B, C]$ and $R_1[B, C]$.

1. Let R denote a complete, separable and metric space, and

suppose that for arbitrary two points B, C of R , there exists a point P satisfying the condition,

$$(I) \quad \overline{AB^2} + \overline{AC^2} = 2\overline{AP^2} + \overline{BP^2} + \overline{CP^2} \text{ for all } A \text{ of } R.$$

Then we have:

Lemma 1.1. *The point P subject to the condition (I) is unique, and as to such a point P , $\overline{BP} = \overline{CP} = \frac{1}{2}\overline{BC}$, and for arbitrary two points B, C*

$$(I') \quad \overline{AB^2} + \overline{AC^2} = 2\overline{AP^2} + \frac{1}{2}\overline{BC^2} \text{ for all } A \text{ of } R.$$

Proof. Suppose that we have the following equation besides (I):

$$\overline{AB^2} + \overline{AC^2} = 2\overline{AQ^2} + \overline{BQ^2} + \overline{CQ^2} \text{ for all } A \text{ of } R.$$

Replace A in the above equation by P , in (I) by Q and add each other. Then we obtain:

$$\overline{BP^2} + \overline{CP^2} + \overline{BQ^2} + \overline{CQ^2} = 4\overline{PQ^2} + \overline{BQ^2} + \overline{CQ^2} + \overline{BP^2} + \overline{CP^2}.$$

From this equation, we get $Q=P$.

Furthermore, we consider the case $A=B$ and $A=C$ in (I), that is, $\overline{BC^2} = 3\overline{BP^2} + \overline{CP^2}$ and $\overline{BC^2} = \overline{BP^2} + 3\overline{CP^2}$. From these two equations, we have: $\overline{BP} = \overline{CP} = \frac{1}{2}\overline{BC}$. Therefore we obtain the equation (I'). Q.E.D.

We denote the set $\{P; \overline{BP} + \overline{CP} = \overline{BC}\}$ by $[B, C]$, and denote the set $\{P; P \in [B, C], P \neq B, C\}$ by (B, C) . Now, Lemma 1.1 shows that, if we denote the points B, C by P_0, P_1 respectively, then there exists a point $P_{1/2}$ such that $\overline{P_0P_{1/2}} = \overline{P_{1/2}P_1} = \frac{1}{2}\overline{P_0P_1}$, and in general, if there exist $2^n + 1$ points $P_0, P_{1/2^n}, \dots, P_{k/2^n}, \dots, P_1$ such that $\overline{P_2^k/2^n} = \overline{P_{k/2^n+1}}$ for $k=0, 1, \dots, 2^n-1$ and $\overline{P_{2^k/2^n}P_{2^k+1/2^n}} = \overline{P_{2^k+1/2^n}P_{2^k+2/2^n}} = \frac{1}{2}\overline{P_{2^k/2^n}P_{2^k+2/2^n}}$ for $k=0, 1, 2, \dots, 2^n-1$, then there exist $2^{n+1} + 1$ points P_l ($l=0, 1, \dots, 2^{n+1}$) such that $\overline{P_{2^k/2^n+1}} = \overline{P_{k/2^n}}$ for $k=0, 1, 2, \dots, 2^n$ and $\overline{P_{2^k/2^n+1}P_{2^k+1/2^n+1}} = \overline{P_{2^k+1/2^n+1}P_{2^k+2/2^n+1}} = \frac{1}{2}\overline{P_{2^k/2^n+1}P_{2^k+2/2^n+1}}$ for $k=0, 1, \dots, 2^n-1$.

For all $P_{k/2^n}$ ($k=0, 1, \dots, 2^n$; $n=1, 2, \dots$), we can show the following lemma by mathematical induction, using the equation (I') in Lemma 1.1.

Lemma 1.2. *For all $k=0, 1, \dots, 2^n$ and $n=1, 2, \dots$,*

$$(a) \quad \overline{BP_{k/2^n}} = \frac{k}{2^n}\overline{BC} \quad \text{and } P_{k/2^n} \in [B, C],$$

(b) *for arbitrary two points B, C of R*

$$(1 - \frac{k}{2^n})\overline{AB^2} + \frac{k}{2^n}\overline{AC^2} = \overline{AP_{k/2^n}^2} + \frac{k}{2^n}(1 - \frac{k}{2^n})\overline{BC^2} \text{ for all } A \text{ of } R.$$

Furthermore from the above equation,

$$(c) \quad \overline{P_{k/2^n} P_{l/2^m}} = (\frac{k}{2^n} \sim \frac{l}{2^m}) \overline{BC} = \overline{BP_{k/2^n}} \sim \overline{BP_{l/2^m}}.$$

As R is complete, and the set $[B, C]$ is closed, observing that for arbitrary real number x , $0 \leq x \leq 1$, there exists a sequence $\{k_v/2^{n_v}\}$ ($0 \leq k_v \leq 2^{n_v}$) such that $x = \lim_{v \rightarrow \infty} k_v/2^{n_v}$, we can show the following lemma.

Lemma 1.3. *For all x , $0 \leq x \leq 1$, there exists a point P such that $\overline{BP} = x\overline{BC}$ and $P \in [B, C]$. We indicate such a point P by P_x . As to a point P_x , for arbitrary two points B, C of R , we have the following equation by (b) in Lemma 1.2:*

$$(II) \quad (1-x)\overline{AB}^2 + x\overline{AC}^2 = \overline{AP_x}^2 + x(1-x)\overline{BC}^2 \text{ for all } A \text{ of } R.$$

Furthermore, from the above equation (II), $\overline{P_x P_y} = (x \sim y)\overline{BC}$. From this equation, the point P_x , subject to the first statement, is unique. And the set $[B, C]$ is composed of all points P_x for all x , $0 \leq x \leq 1$.

2. We denote the set $\{P; \overline{BC} + \overline{CP} = \overline{BP}\}$ by $I'[B, C]$, and the set $[B, C] + I'[B, C] + I'[C, B]$ by $R_1[B, C]$. And then we denote the point P in $R_1[B, C]$, such that $\overline{BP} = x\overline{BC}$, by P_x . If $P_x \in [B, C]$ then $0 \leq x \leq 1$, and if $P_x \in I'[B, C]$, then $x \geq 1$. If $P_x \in I'[C, B]$, then we make a promise that x is non-positive. Then, $\overline{BP_x} = |x|\overline{BC}$. Now, if $P_x \in R_1[B, C]$, then we can show that for arbitrary two points B, C of R ,

$$(II') \quad (1-x)\overline{AB}^2 + x\overline{AC}^2 = \overline{AP_x}^2 + x(1-x)\overline{BC}^2 \text{ for all } A \text{ of } R,$$

and that $\overline{BP_x}^2 = x^2\overline{BC}^2$, $\overline{CP_x}^2 = (1-x)^2\overline{BC}^2$.

From these equations, we have the following lemmas:

Lemma 1.4. *If $P_x, P_y \in R_1[B, C]$, then $\overline{P_x P_y} = (x \sim y)\overline{BC}$.*

Lemma 1.5. *If $P_x, P_y \in R_1[B, C]$, then $[P_x, P_y] \subset R_1[B, C]$.*

When x and y are real numbers such that $x+y=1$, we rewrite P_y by $P_{x,y}$. Then, if $P = P_{x_1, y_1}$ and $Q = P_{x_2, y_2}$, then $\overline{PQ}^2 = -(x_1 - x_2) \times (y_1 - y_2) \times \overline{BC}^2$ by Lemma 1.4. Thus we have:

Theorem 1. *$R_1[B, C]$ is isometrically isomorphic with an extension of the segment $[B^*, C^*]$ in a Hilbert space such that $\overline{BC} = \overline{B^*C^*}$.*

Proof. For all x, y such that $P_{x,y} \in R_1[B, C]$, we consider the set $R_1^*[B^*, C^*]$ of all points $P_{x,y}^* = xB^* + yC^*$. The correspondence $P_{x,y} \rightarrow P_{x,y}^*$ is clearly one-to-one, and if $P = P_{x_1, y_1}$ and $Q = P_{x_2, y_2}$ correspond

to $P^* = P_{x_1, y_1}^*$ and $Q^* = Q_{x_2, y_2}^*$ respectively, then

$$\overline{PQ}^2 = -(x_1 - x_2)(y_1 - y_2)\overline{BC}^2 = -(x_1 - x_2)(y_1 - y_2)\overline{B^*C^*}^2 = \overline{P^*Q^*}^2.$$

Therefore $R_1[B, C]$ is isometrically isomorphic with the set $R_1^*[B^*, C^*]$.

Furthermore, by Lemma 1.5, $R_1^*[B^*, C^*]$ is convex, and clearly closed. Hence, it is an extension of $[B^*, C^*]$.

Corollary. *$[B, C]$ is isometrically isomorphic with the segment $[B^*, C^*]$ in a Hilbert space such that $\overline{BC} = \overline{B^*C^*}$.*

§2. $[A, B, C]$

Suppose that $A \in R_1[B, C]$. Let $[A, B, C]$ and (A, B, C) denote two sets $\{P; P_\varepsilon[A, Q], Q_\varepsilon[B, C]\}$ and $\{P; P_\varepsilon(A, Q), Q_\varepsilon(B, C)\}$ respectively.

Lemma 2.1. *If $Q_\varepsilon(B, C), R_\varepsilon(A, C)$ in arbitrary one set $[A, B, C]$, then there exists a point P such that $(A, Q) \cdot (B, R) = \{P\}$.*

Furthermore, $(A, B, C) = \{P; (A, Q)(B, R) = \{P\}, Q_\varepsilon(B, C), R_\varepsilon(A, C)\}$.

Proof. Let $\overline{BQ} = x\overline{BC}$, $\overline{AR} = y\overline{AC}$. Now we can choose a point P in (B, R) such that $\overline{BP} = [x/\{x+y \cdot (1-x)\}]\overline{BR}$. Then, by the equation (II) in Lemma 1.4., we have five equations for pairs: (A, B, C, Q) , (P, B, C, Q) , (P, A, C, R) , (B, A, C, R) , (A, B, R, P) . From these five equations, we have:

$$\overline{AP} = [y/\{y+x(1-y)\}]\overline{AQ}, \quad \overline{PQ} = [x(1-y)/\{y+x(1-y)\}]\overline{AQ}.$$

Therefore, $P_\varepsilon(A, Q)(B, R)$. The uniqueness of such a point P follows from the fact: $A \in R_1[B, C]$. We can see the last assertion by (III).

Lemma 2.2. *Let $Q_\varepsilon(B, C)$, $R_\varepsilon(A, C)$, and $S_\varepsilon(A, B)$. If $\frac{\overline{BQ}}{\overline{QC}} \cdot \frac{\overline{CR}}{\overline{RA}} \cdot \frac{\overline{AS}}{\overline{SB}} = 1$, then there exists a point P such that $(A, Q) \cdot (B, R) \cdot (C, S) = \{P\}$.*

Proof. We can show this lemma easily by Lemma 2.1 and (III). When three positive numbers x, y, z such that $x+y+z=1$, is given, we can determine three points $Q_\varepsilon(B, C)$, $R_\varepsilon(A, C)$ and $S_\varepsilon(A, B)$ uniquely such that $\overline{BQ}:\overline{QC}=y:x$, $\overline{CR}:\overline{RA}=z:y$ and $\overline{AS}:\overline{SB}=x:z$ respectively. Then we denote the point P , such that $(A, Q) \cdot (B, R) \cdot (C, S) = \{P\}$ as shown in Lemma 2.2, by $P(x, y, z)$. Furthermore, we indicate $Q_{x, y \varepsilon}(B, C)$ ($\overline{BQ}:\overline{QC}=y:x$) $R_{y, z \varepsilon}(A, C)$ ($\overline{CR}:\overline{RA}=z:y$), and $S_{z, x \varepsilon}(A, B)$ ($\overline{AS}:\overline{SB}=x:z$), by $Q(x, y, 0)$, $R(0, y, z)$ and $S(x, 0, z)$ respectively, and let $A=A(0, 0, 1)$, $B=B(1, 0, 0)$ and $C=C(0, 1, 0)$.

Lemma 2.3. *If $P_1 = P_1(x_1, y_1, z_1)$ and $P_2 = P_2(x_2, y_2, z_2)$, then*

$$\overline{P_1 P_2}^2 = -(x_1 - x_2)(y_1 - y_2) \overline{BC}^2 - (y_1 - y_2)(z_1 - z_2) \overline{CA}^2 - (z_1 - z_2)(x_1 - x_2) \overline{AB}^2.$$

Proof. We can prove this lemma by calculation, using (II) and (III). According to (III), $\overline{AP} \cdot \overline{PQ} = (x + y) : z$, where $P = P(x, y, z)$. —

From Lemma 2.3, we can see that, if $x_1 \neq x_2$, $y_1 \neq y_2$ and $z_1 \neq z_2$, then $P_1 \neq P_2$. Therefore the correspondence between the pair (x, y, z) and the point $P(x, y, z)$ is one-to-one. Furthermore we can see easily that $[A, B, C]$ is the set of points $P(x, y, z)$ for all $x, y, z (\geq 0)$ such that $x + y + z = 1$. Thus we have:

Theorem 2. *The set $[A, B, C]$ in R is isometrically isomorphic with the triangle $[A^*, B^*, C^*]$ in a Hilbert space such that*

$$\overline{BC} = \overline{B^* C^*}, \overline{AC} = \overline{A^* C^*}, \text{ and } \overline{AB} = \overline{A^* B^*}.$$

Proof. We can prove this lemma in the same way as in Theorem 1. — We denote three sets $[A, B]$, $[B, C]$, $[C, A]$ in $[A, B, C]$ by Ω_1 , Ω_2 , Ω_0 , and the set $\{P; P \in R_1[Q, R]; Q \in \Omega_q, R \in \Omega_r, q \neq r, q, r = 0, 1, 2\}$ by $R_2[A, B, C]$. The set $\{P; P \in [Q, R]; Q \in \Omega_q, R \in \Omega_r, q \neq r, q, r = 0, 1, 2\}$ is clearly $[A, B, C]$, and the set $\{P; P \in (Q, R); Q \in \Omega_q, R \in \Omega_r, q \neq r, q, r = 0, 1, 2\}$ is (A, B, C) .

§3. $[A_0, A_1, \dots, A_n]$ and $R_n[A_0, A_1, \dots, A_n]$.

1. Let three sets $[A_0, A_1, \dots, A_{n-1}]$, $(A_0, A_1, \dots, A_{n-1})$ and $R_{n-1}[A_0, A_1, \dots, A_{n-1}]$ be defined, and suppose that $[A_0, A_1, \dots, A_{n-1}]$ is isometrically isomorphic with a simplex $[A_0^*, A_1^*, \dots, A_{n-1}^*]$ in a Hilbert space such that

$$\overline{A_i A_j} = \overline{A_i^* A_j^*} \quad (i \neq j, i, j = 0, 1, \dots, n-1).$$

Then we define two sets $[A_0, A_1, \dots, A_n]$, $(A_0, A_1, \dots, A_n)$ as sets $\{P; P \in [A_n, Q], Q \in [A_0, A_1, \dots, A_{n-1}]\}$, $\{P; P \in (A_n, Q), Q \in (A_0, A_1, \dots, A_{n-1})\}$ respectively, where $A_n \in R_{n-1}[A_0, A_1, \dots, A_{n-1}]$.

We denote $[A_0, A_1, \dots, A_{n-1}]$, $[A_0, A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n]$ and $[A_1, A_2, \dots, A_n]$ by Ω_n , Ω_i and Ω_0 respectively ($i = 1, 2, \dots, n-1$). Now let $n+1$ positive numbers $\lambda_0, \lambda_1, \dots, \lambda_n$ such that $\lambda_0 + \lambda_1 + \dots + \lambda_n = 1$ be given. If we choose $n(n-1)/2$ points Q_{ij} ($Q_{ij} = Q_{ji}$) such that $\overline{A_i Q_{ij}} \cdot \overline{Q_{ij} A_j} = \lambda_j : \lambda_i$ ($i, j = 0, 1, \dots, n; i \neq j$), then we can determine $n+1$ points $Q_i = Q_i$ ($\frac{\lambda_0}{\rho}, \frac{\lambda_1}{\rho}, \dots, \frac{\lambda_{i-1}}{\rho}, \frac{\lambda_{i+1}}{\rho}, \dots, \frac{\lambda_n}{\rho}$) in Ω_i by Q_{ij} , where $\rho = \sum_{(k \neq i)}^n \lambda_k$. (Pairs: $(\frac{\lambda_0}{\rho}, \frac{\lambda_1}{\rho}, \dots, \frac{\lambda_{i-1}}{\rho}, \frac{\lambda_{i+1}}{\rho}, \dots, \frac{\lambda_n}{\rho})$; $\rho = \sum_{(k \neq i)}^n \lambda_k$ correspond to the barycentric

coordinate in a Euclidean space.)

Lemma 3.1. *If Q_i , $i=0,1,\dots,n$ are $n+1$ points such that we have described above, then there exists a point P such that*

$$(A_0, Q_0) \cdot (A_1, Q_1) \cdots (A_n, Q_n) = \{P\}, \text{ and } P \in (A_0, A_1, \dots, A_n).$$

Proof. We can prove this lemma by Lemma 2.2. and 2.3, and equation $\overline{P_1 P_2}^2 = - \sum_{i,j=0}^{n-1} (\lambda_i - \lambda'_i)(\lambda_j - \lambda'_j) \overline{A_i A_j}^2$ in $[A_0, A_1, \dots, A_{n-1}]$, where

$$P_1 = P_1(\lambda_0, \lambda_1, \dots, \lambda_{n-1}), \quad P_2 = P_2(\lambda'_0, \lambda'_1, \dots, \lambda'_{n-1}) \quad \text{and} \quad \sum_{k=0}^{n-1} \lambda_k = 1, \quad \sum_{k=0}^{n-1} \lambda'_k = 1.$$

We denote the point P determined by Lemma 3.1 by $P(\lambda_0, \lambda_1, \dots, \lambda_n)$. Furthermore, we indicate $Q_i(\lambda_0, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n)$ in $\mathcal{Q}_i$ by $Q_i(\lambda_0, \lambda_1, \dots, \lambda_{i-1}, 0, \lambda_{i+1}, \dots, \lambda_n)$ ($i=0,1,\dots,n$), where $\sum_{\substack{k=0 \\ (k \neq i)}}^n \lambda_k = 1$, and let $A_i = A_i(\delta_{i,0}, \delta_{i,1}, \dots, \delta_{i,n})$, ($\delta_{i,j}$ is Kronecker's delta.)

Lemma 3.2. *If $P_1 = P_1(\lambda_0, \lambda_1, \dots, \lambda_n)$ and $P_2 = P_2(\lambda'_0, \lambda'_1, \dots, \lambda'_n)$, then*

$$\overline{P_1 P_2}^2 = - \sum_{i,j=0}^n (\lambda_i - \lambda'_i)(\lambda_j - \lambda'_j) \overline{A_i A_j}^2 \quad (i \neq j).$$

Proof. We can prove this lemma by calculation, using (II).

From Lemma 3.2, we can see that, if $\lambda_i \neq \lambda'_i$ ($i=0,1,\dots,n$), then $P \neq P'$. Therefore the correspondence: $(\lambda_0, \lambda_1, \dots, \lambda_n) \rightarrow P(\lambda_0, \lambda_1, \dots, \lambda_n)$ is one-to-one. Furthermore we can see easily that $[A_0, A_1, \dots, A_n]$ is the set of points $P(\lambda_0, \lambda_1, \dots, \lambda_n)$ for all λ_i , $i=0,1,\dots,n$ ($\lambda_i \geq 0$) such that $\sum_{k=0}^n \lambda_k = 1$.

In the same way as in Theorem 1, we can show the following theorem.

Theorem 3. *The set $[A_0, A_1, \dots, A_n]$ in R is isometrically isomorphic with the simplex $[A_0^*, A_1^*, \dots, A_n^*]$ in a Hilbert space such that $\overline{A_i A_j} = \overline{A_i^* A_j^*}$ ($i \neq j$, $i, j=0,1,\dots,n$).*

2. We denote the set $\{P; P \in R_1[Q, R]; Q \in \mathcal{Q}_q, R \in \mathcal{Q}_r, q \neq r, q, r=0,1,\dots,n\}$ by $R_n[A_0, A_1, \dots, A_n]$. The set $\{P; P \in [Q, R], Q \in \mathcal{Q}_q, R \in \mathcal{Q}_r, q \neq r, q, r=0,1,\dots,n\}$ is clearly $[A_0, A_1, \dots, A_n]$, and the set $\{P; P \in (Q, R), Q \in \mathcal{Q}_q, R \in \mathcal{Q}_r, q \neq r, q, r=0,1,\dots,n\}$ is $(A_0, A_1, \dots, A_n)$. Then we can easily show the following lemma.

Lemma 3.3. *If $G = G(1/n, 1/n, \dots, 1/n)$, then $R_n[A_0, A_1, \dots, A_n] = \{P; P \in [G, K] + I[G, K], K \in \sum_{k=0}^n \mathcal{Q}_k\}$.*

We denote the point P in $R_n[A_0, A_1, \dots, A_n]$, such that $K = K(\lambda_0, \lambda_1, \dots, \lambda_n)$ in $\sum_{k=0}^n \mathcal{Q}_k$ and $\overline{GP} = t \overline{GK}$, by $P(\lambda_0, \lambda_1, \dots, \lambda_n; t)$. (At least one λ_i of $\lambda_0, \lambda_1, \dots, \lambda_n$ is constantly zero.) Then we have:

Lemma 3.4. *If $P_1 = P_1(\lambda_0, \lambda_1, \dots, \lambda_n; t)$ and $P_2 = P_2(\lambda_0', \lambda_1', \dots, \lambda_n'; t')$, then*

$$\overline{P_1 P_2^2} = - \sum_{i,j=0}^n \{t(1/n - \lambda_i) - t'(1/n - \lambda_i')\} \{t(1/n - \lambda_j) - t'(1/n - \lambda_j')\} \overline{A_i A_j^2}.$$

Proof. If $K_1 = K_1(\lambda_0, \lambda_1, \dots, \lambda_n)$ and $K_2 = K_2(\lambda_0', \lambda_1', \dots, \lambda_n')$, then by (II')

$$\overline{P_1 P_2^2} = t'(t' - t) \overline{G K_2^2} + t(t - t') \overline{G K_1^2} + t t' \overline{K_1 K_2^2}.$$

Therefore we can show this lemma by the above equation and Lemma 3.2.

Lemma 3.5. *If $P_1, P_2 \in R_n[A_0, A_1, \dots, A_n]$, then $[P_1, P_2] \subset R_n[A_0, A_1, \dots, A_n]$.*

Proof. Let $K_1 \in \Omega_x$, $K_2 \in \Omega_y$, and $P_1 \in [G, K_1] + I'[G, K_1]$, $P_2 \in [G, K_2] + I'[G, K_2]$.

If $P \in [P_1, P_2]$, then there exists a point K such that $(G, P) \cdot (K_1, K_2) = \{K\}$ or $I'[G, P] \cdot (K_1, K_2) = \{K\}$. Now, if $x = y$, then $K \in \Omega_x$. Hence $P \in [G, K] + I'[G, K]$, and $K \in \Omega_x$. If $x \neq y$, then $K \in (A_0, A_1, \dots, A_n)$. Therefore, there exists a point K' in Ω_x such that $I'[G, K] \cdot \Omega_x = \{K'\}$. Therefore $P \in [G, K'] + I'[G, K']$, $K' \in \Omega_x$. That is $P \in R_n[A_0, A_1, \dots, A_n]$. Thus we have Lemma 3.5.

In the same manner as Theorem 1, we can show the following theorem.

Theorem 4. $R_n[A_0, A_1, \dots, A_n]$ is isometrically isomorphic with a closed convex in a Hilbert space, in which a simplex $[A_0^*, A_1^*, \dots, A_n^*]$, such that $\overline{A_i A_j} = \overline{A_i^* A_j^*}$ ($i \neq j$, $i, j = 0, 1, \dots, n$), is included.

Conclusion. Since R is separable, we can obtain the following conclusion.

Theorem 5. Let R be a complete, separable and metric space, and suppose that for arbitrary two points B, C , there exists a point P satisfying the condition:

$$\overline{AB^2} + \overline{AC^2} = 2\overline{AP^2} + \overline{BP^2} + \overline{CP^2} \text{ for all } A \text{ of } R,$$

then R is isometrically isomorphic with a closed convex in a Hilbert space.