ON THE CONVERGENCE OF THE FOURIER SERIES OF $|f(x)|$ AT ONE POINT

By

Noboru MATSUYAMA

(Received October 1, 1951)

1. Let $f(x)$ be an integrable and periodic function with the period 2π , and its Fourier series be

$$(1) \quad f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

One of the most familiar test for the convergence of series (1) is the Dini's test.¹⁾ That is:

(1.1) If for any $\delta > 0$,

$$(2) \quad \int_0^{\delta} |\varphi(x, t)|/t \, dt$$

is finite, then (1) converges to $f(x)$ at x , where

$$\varphi(x, t) = f(x+t) + f(x-t) - 2f(x).$$

Now for any even function $f(x)$ satisfying $f(0) = 0$, condition (2) is replaced at $x = 0$ by

$$(2') \quad \int_0^{\delta} |f(t)|/t \, dt.$$

Consequently for such function, $|f(x)|$ satisfies (2') and the Fourier series of $|f(x)|$ converges to 0 at $x = 0$.

We can show that the same arguments are satisfied for some other tests, for instance, JORDAN's tests

On the other hand we can show by the PALEY's method²⁾ that there exists an even function $f(x)$ such that the Fourier series of $f(x)$ converges at point $x = 0$, although the Fourier series of $|f(x)|$ is not

¹⁾ See [1].

²⁾ See [2].

necessary convergent at that point. Thus we have the following theorems.

Theorem 1. There exists an even function $f(x)$ such that the Fourier series of $f(x)$ converges at point $x = 0$, but the Fourier series of $|f(x)|$ is not convergent at that point.

Theorem 2. There exists an even function $f(x)$ such that its Fourier series is summable (C, 1) at $x = 0$, but the Fourier series of $|f(x)|$ is not summable (C, 1) at that point.

2. Let $\{c_n\}$, $\{t_n\}$, $\{u_n\}$ and $\{v_n\}$ be positive decreasing sequences and $v_n \ll u_n$ for $n = 1, 2, \dots$. If we denote the intervals $(t_n - u_n, t_n - v_n)$ and $(t_n + v_n, t_n + u_n)$ by Δ_n' and Δ_n'' respectively, then our even function $f(x)$ is defined by

$$f(x) = \begin{cases} c_n x/(x - t_n) & \text{for } x \in \Delta_n, \\ 0 & \text{for } x \in \Delta_n^c. \end{cases}$$

and $f(0) = 0$, where $\Delta_n = \Delta_n' + \Delta_n''$.

Now

$$\begin{aligned} \int_{\Delta_n} |f(x)| dx &= c_n \int_{t_n - u_n}^{t_n - v_n} x/(t_n - x) dx + c_n \int_{t_n + v_n}^{t_n + u_n} x/(x - t_n) dx \\ &= 2c_n t_n \log u_n/v_n. \end{aligned}$$

Consequently if

$$(3) \quad \sum_{n=1}^{\infty} c_n t_n \log u_n/v_n < \infty,$$

$f(x)$ is integrable in the Lebesgue sense.

Let S_m and T_m be the m -th partial sums of the Fourier series of $f(x)$ and $|f(x)|$ at $x = 0$ respectively, and if $\sum c_n < \infty$, then

$$\begin{aligned} S_m &= S_m(f; 0) = \frac{2}{\pi} \int_0^\pi f(x) \frac{\sin mx}{x} dx + o(1) \\ &= o(1) + \frac{2}{\pi} \sum_{n=1}^{\infty} c_n \int_{\Delta_n} \frac{\sin mx}{x - t_n} dx \\ &= o(1) + \frac{2}{\pi} \sum_{n=1}^{\infty} c_n \int_{v_n}^{u_n} \frac{\sin m(x - t_n) + \sin m(x + t_n)}{x} dx \\ (4) \quad &= o(1) + \frac{4}{\pi} \sum_{n=1}^{\infty} c_n \int_{v_n}^{u_n} \frac{\cos mt_n \sin mx}{x} dx, \end{aligned}$$

for the last series converges absolutely and uniformly in m .

Hence

$$\begin{aligned}\lim_{m \rightarrow \infty} S_m &= \lim_{m \rightarrow \infty} \frac{4}{\pi} \sum_{n=1}^{\infty} c_n \int_{v_n}^{u_n} \cos mt_n \sin mx/x \, dx \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} c_n \lim_{m \rightarrow \infty} \int_{v_n}^{u_n} \cos mt_n \sin mx/x \, dx \\ &= 0.\end{aligned}$$

Thus if $\sum c_n < \infty$, then the Fourier series of $f(x)$ converges to 0 at $x = 0$.

On the other hand

$$\begin{aligned}T_m &= \frac{2}{\pi} \int_0^{\pi} |f(x)| \frac{\sin mx}{x} \, dx + o(1) \\ &= o(1) + \frac{2}{\pi} \sum_{n=1}^{\infty} c_n \left(\int_{t_n - u_n}^{t_n - v_n} \frac{\sin mx}{t_n - x} \, dx + \int_{t_n + v_n}^{t_n + u_n} \frac{\sin mx}{x - t_n} \, dx \right) \\ &= o(1) + \frac{4}{\pi} \sum_{n=1}^{\infty} c_n \int_{v_n}^{u_n} \frac{\sin mt_n \cos mx}{x} \, dx \\ &= o(1) + \frac{4}{\pi} \left(\sum_{\nu=1}^{n-1} + \sum_{\nu=n}^{\infty} \right) c_{\nu} \sin mt_{\nu} \int_{v_{\nu}}^{u_{\nu}} \frac{\cos mx}{x} \, dx \\ &= o(1) + P + Q,\end{aligned}$$

say. Then

$$|P| \leq \frac{4}{\pi} \sum_{\nu=1}^{n-1} c_{\nu} \int_{v_{\nu}}^{u_{\nu}} x^{-1} \, dx = \frac{4}{\pi} \sum_{\nu=1}^{n-1} c_{\nu} \log \frac{u_{\nu}}{v_{\nu}},$$

and

$$\begin{aligned}Q &= \frac{4}{\pi} \sum_{\nu=n}^{\infty} c_{\nu} \sin mt_{\nu} \int_{v_{\nu}}^{u_{\nu}} \frac{\cos mx}{x} \, dx \\ &= \frac{4}{\pi} \sum_{\nu=n}^{\infty} c_{\nu} \sin mt_{\nu} \int_{v_{\nu}}^{u_{\nu}} \frac{1 - 2 \sin^2 mx/2}{x} \, dx \\ &= \frac{4}{\pi} \sum_{\nu=n}^{\infty} c_{\nu} \sin mt_{\nu} \log \frac{u_{\nu}}{v_{\nu}} - \frac{8}{\pi} \sum_{\nu=n}^{\infty} c_{\nu} \sin mt_{\nu} \int_{v_{\nu}}^{u_{\nu}} \frac{\sin^2 mx/2}{x} \, dx \\ &= Q_1 - Q_2,\end{aligned}$$

say. If we put $m = \left[\frac{\pi}{2} t_n^{-1} \right]$, then

$$\frac{\pi}{4} Q_1 \geq \sum_{\nu=n}^{\infty} c_{\nu} \log \frac{u_{\nu}}{v_{\nu}} \frac{2}{\pi} m t_{\nu} = \frac{2}{\pi} m \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} \log \frac{u_{\nu}}{v_{\nu}}.$$

Hence

$$Q_1 \geq \frac{4}{\pi} \cdot \frac{2}{\pi} \cdot \frac{\pi}{2} \frac{1}{t_n} \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} \log u_{\nu}/v_{\nu} = \frac{4}{\pi} \frac{1}{t_n} \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} \log u_{\nu}/v_{\nu}.$$

By the same consideration,

$$\begin{aligned} |Q_2| &\leq \frac{8}{\pi} \sum_{\nu=n}^{\infty} c_{\nu} m t_{\nu} \int_{v_{\nu} m/2}^{u_{\nu} m/2} \frac{\sin^2 x}{x} dx \\ &\leq \frac{8}{\pi} m \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} \int_{m v_{\nu}/2}^{m u_{\nu}/2} \frac{x}{x^2} dx \\ &= \frac{8}{\pi} m \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} \frac{1}{2} \left(\frac{m}{2} \right)^2 (u_{\nu}^2 - v_{\nu}^2) \\ &\leq \frac{1}{\pi} m^3 \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} u_{\nu}^2 \leq \frac{1}{\pi} \left(\frac{\pi}{2} \frac{1}{t_n} \right)^3 \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} u_{\nu}^2 \\ &= \frac{\pi^2}{8} \frac{1}{t_n^3} \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} u_{\nu}^2. \end{aligned}$$

Thus we have

$$\begin{aligned} (5) \quad T_m &\geq Q_1 - (|Q_2| + |P|) - o(1) \\ &= \frac{4}{\pi} \frac{1}{t_n} \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} \log \frac{u_{\nu}}{v_{\nu}} - \frac{\pi^2}{8} \frac{1}{t_n^3} \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} u_{\nu}^2 \\ &\quad - \frac{4}{\pi} \sum_{\nu=1}^{n-1} c_{\nu} \log \frac{u_{\nu}}{v_{\nu}} - o(1). \end{aligned}$$

Let us put

$$(6) \quad t_n = n^{-3}, u_n = n^{-4}, v_n = e^{-n^3} \text{ and } c_n = n^{-2},$$

then $\sum c_n < \infty$ and the series (3) converges, for

$$\sum_{\nu=1}^{\infty} c_{\nu} t_{\nu} \log \frac{u_{\nu}}{v_{\nu}} = \sum_{\nu=1}^{\infty} \nu^{-2} \nu^{-3} (\nu^3 - 4 \log \nu) < \infty.$$

By (5) and (6)

$$\begin{aligned} T_m &\geq \frac{4}{\pi} n^3 \sum_{\nu=n}^{\infty} \nu^{-2} \nu^{-3} (\nu^3 - 4 \log \nu) \\ &\quad - \frac{\pi^2}{8} n^9 \sum_{\nu=n}^{\infty} \nu^{-2} \nu^{-3} \nu^{-8} - \frac{4}{\pi} \sum_{\nu=1}^{n-1} \nu^{-2} (\nu^3 - 4 \log \nu) - o(1) \\ &\geq \frac{4}{\pi} n^2 - \frac{4}{\pi} \frac{1}{2} n(n-1) - O(1) \rightarrow \infty, \quad (n \rightarrow \infty). \end{aligned}$$

Thus Theorem 1 is proved.

For the proof of Theorem 2, we consider the function of the same type as above. Thus if (4) it satisfied and $\sum c_n < \infty$, then function $f(x)$ is integrable in the Lebesgue sense and its Fourier series is summable (C, 1) to 0 at $x = 0$.

Let $\sigma_m \equiv \sigma_m(0)$ be the (C, 1) mean of the m -th order of the Fourier series of $|f(x)|$ at $x = 0$, then

$$\begin{aligned}\sigma_m &= \sigma_m(|f|, 0) = \frac{1}{\pi} \int_0^\pi |f(x)| \frac{\sin^2 mx}{mx^2} dx + o(1) \\ &= \frac{1}{\pi} \frac{1}{m} \sum_{n=1}^\infty c_n \left(\int_{t_n - u_n}^{t_n - v_n} \frac{\sin^2 mx}{x(t_n - x)} dx + \int_{t_n - v_n}^{t_n + u_n} \frac{\sin^2 mx}{x(x - t_n)} dx \right) + o(1) \\ &= \frac{1}{\pi m} \sum_{n=1}^\infty c_n \left(\int_{v_n}^{u_n} \frac{\sin^2 m(t_n - x)}{x(t_n - x)} dx + \int_{v_n}^{u_n} \frac{\sin^2 m(t_n + x)}{x(t_n + x)} dx \right) + o(1) \\ &\equiv P + Q + o(1).\end{aligned}$$

say. Then

$$\begin{aligned}P &= \frac{1}{\pi m} \left\{ \sum_{\nu=1}^{n-1} c_\nu \int_{v_\nu}^{u_\nu} \frac{\sin^2 m(t_\nu - x)}{x(t_\nu - x)} dx + \sum_{\nu=n}^\infty c_\nu \int_{v_\nu}^{u_\nu} \frac{\sin^2 (t_\nu - x)}{x(t_\nu - x)} dx \right\} \\ &\equiv P_1 + P_2.\end{aligned}$$

say. New

$$\begin{aligned}|P_1| &\leq \frac{1}{\pi m} \sum_{\nu=1}^{n-1} c_\nu \int_{v_\nu}^{u_\nu} \frac{1}{x(t_\nu - x)} dx = \frac{1}{\pi m} \sum_{\nu=1}^{n-1} c_\nu \int_{v_\nu}^{u_\nu} \frac{1}{t_\nu} \left(\frac{1}{x} + \frac{1}{t_\nu - x} \right) dx \\ &\leq \frac{1}{\pi m} \sum_{\nu=1}^{n-1} \frac{c_\nu}{t_\nu} \left(\log \frac{u_\nu}{v_\nu} + \log \frac{t_\nu - v_\nu}{t_\nu - u_\nu} \right) \\ &\leq \frac{2}{\pi m} \sum_{\nu=1}^{n-1} \frac{c_\nu}{t_\nu} \log \frac{u_\nu}{v_\nu},\end{aligned}$$

and

$$\begin{aligned}P_2 &= \frac{1}{\pi m} \sum_{\nu=n}^\infty c_\nu \int_{v_\nu}^{u_\nu} \frac{\sin^2 m(t_\nu - x)}{x(t_\nu - x)} dx \\ &\geq \frac{1}{\pi m} \sum_{\nu=n}^\infty c_\nu \int_{v_\nu}^{u_\nu} \frac{m^2(t_\nu - x)^2}{x(t_\nu - x)} \left(\frac{2}{\pi} \right)^2 dx \\ &= \frac{1}{\pi m} \frac{4}{\pi^2} m^2 \sum_{\nu=n}^\infty c_\nu \int_{v_\nu}^{u_\nu} \frac{t_\nu - x}{x} dx,\end{aligned}$$

where $m = \left\lceil \frac{\pi}{2} t_n^{-1} \right\rceil$. Hence

$$P_2 \geq \frac{4}{\pi^3} m \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} \log \frac{u_{\nu}}{v_{\nu}} - \frac{8}{\pi^3} m \sum_{\nu=n}^{\infty} c_{\nu} u_{\nu}.$$

Thus we have

$$(9) \quad P \geq P_2 - |P_1| \geq \frac{4}{\pi^3} m \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} \log \frac{u_{\nu}}{v_{\nu}} - \frac{8}{\pi^3} m \sum_{\nu=n}^{\infty} c_{\nu} u_{\nu} \\ - \frac{2}{\pi m} \sum_{\nu=1}^{n-1} \frac{c_{\nu}}{t_{\nu}} \log \frac{u_{\nu}}{v_{\nu}}.$$

By the same calculation, we have

$$Q \geq \frac{4}{\pi^3} m \sum_{\nu=n}^{\infty} c_{\nu} t_{\nu} \log \frac{u_{\nu}}{v_{\nu}} - \frac{8}{\pi^3} m \sum_{\nu=n}^{\infty} c_{\nu} u_{\nu} - \frac{2}{\pi m} \sum_{\nu=1}^{n-1} \frac{c_{\nu}}{t_{\nu}} \log \frac{u_{\nu}}{v_{\nu}}.$$

Now if we put

$$t_n = n^{-3}, \quad c_n = n^{-1}(\log n)^{-2}, \quad u_n = n^{-4} \text{ and } v_n = e^{-n^3},$$

then series (3) and (5) converge, and

$$P \geq \frac{4}{\pi^3} \frac{\pi}{2} n^3 \sum_{\nu=n}^{\infty} \frac{1}{\nu (\log \nu)^2} - \frac{1}{\nu^3} (\nu^3 - 4 \log \nu) \\ - \frac{8}{\pi^3} \frac{\pi}{2} n^3 \sum_{\nu=n}^{\infty} \frac{1}{5\nu (\log \nu)^2} - \frac{2}{\pi} \frac{2}{\pi} \frac{1}{n^3} \sum_{\nu=1}^{n-1} \frac{\nu^3}{\nu (\log \nu)^2} (\nu^3 - 4 \log \nu) \\ \geq \frac{2}{\pi^2} n^3 \frac{1}{\log n} - \frac{4}{\pi^2} \frac{1}{n^3} \sum_{\nu=1}^{n-1} \frac{5\nu}{(\log \nu)^2} - o(1) \\ = \frac{2}{n^2} \frac{n^3}{\log n} \left[1 - \frac{1}{\log n} + \frac{1}{n \log n} \right] - o(1) \rightarrow \infty.$$

And by the same way we have $Q \rightarrow \infty (n \rightarrow \infty)$. Thus the (C, 1) mean of the Fourier series of $|f(x)|$ at $x = 0$ diverges to $+\infty$, and thus Theorem 2 is proved.

References

- [1] A. ZYGMUND, *Trigonometrical Series*, Warsaw (1935), § 2, 4.
- [2] R. E. A. C. PALEY, *Proc. Camb. Phy. Soc.*, **26**, (1930).