RETRACTS IN THE LOCALLY COMPACT HAUSDORFF SPACES

By

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1. **Introduction.** K. BORSUK defined and studied retracts, absolute retracts and absolute neighborhood retracts in the separable, metrizable spaces. However, many of his results can be extended to more general spaces with slight modifications. We shall consider this problem in the locally compact Hausdorff spaces, and in the following we call these spaces only "spaces."

There are two distinguished facts in our case, i.e., (1) local contractibility can not characterize the property of being absolute neighborhood retracts, and (2) the Sum Theorem does not hold.

2. In this section we shall prove some topological preliminaries.

Lemma 1. *Let R be a space and C_1, C_2 be two disjoint compact sets. Then there exists a continuous function $\varphi(x)$ defined all over R satisfying:*

$$\begin{aligned} 0 \leq \varphi(x) \leq 1 & \quad \text{for all } x \in R, \\ \varphi(x) = \begin{cases} 0 & \text{for } x \in C_1, \\ 1 & \text{for } x \in C_2. \end{cases} \end{aligned}$$

Proof. Since a locally compact Hausdorff space is completely regular, for every $x \in C_1$, there exists a continuous function $f_x(y)$ satisfying

$$0 \leq f_x(y) \leq 1 \quad \text{for all } y \in R,$$

and

$$f_x(y) = \begin{cases} 0 & \text{for } y = x, \\ 1 & \text{for } y \in C_2. \end{cases}$$

Since R is locally compact, for sufficiently small $\varepsilon > 0$

$$V_\varepsilon(x) = \{y; f_x(y) < \varepsilon\}$$

has the compact closure.

Put

$$V_1(x) = V_\delta(x), \text{ where } \delta < \min. \{\varepsilon, 1/3\}$$

$$V_2(C_2, x) = \{y; f_x(y) > 2/3\}.$$

Then V_1 and V_2 are open and mutually disjoint. Clearly

$$\bigcup_{x \in C_1} V_1(x) \supset C_1,$$

and since C_1 is compact, there exist finite elements of C_1 , $x_1, x_2, \dots, x_n$; and it holds $\bigcup_{i=1}^n V_1(x_i) \supset C_1$.

Put $U_1 = \bigcup_{i=1}^n V_1(x_i)$ and $U_2' = \bigcap_{i=1}^n V_2(C_2, x_i)$, then $\bar{U}_1$ is compact and $U_1 \cap U_2' = \phi$. On account of local compactness of R , we have an open set U_2 containing C_2 with compact closure, which is contained in U_2' .

Put $\mathbf{C}U_1 = C_1'$, $\mathbf{C}U_2 = C_2'$ and $C_1' \cap C_2' \equiv C_3$.

Since $C_1' \cap C_1 = \phi$, by the same way as above, we have open sets $U_{11} \supset C_1$ and $U_{13} \supset C_1'$ satisfying $U_{11} \cap U_{12} = \phi$, $\bar{U}_{11}$ is compact. From the fact that $\mathbf{C}U_{13} \subset \bar{U}_1$, $\mathbf{C}U_{13}$ is compact. Therefore we can apply the above method to the pairs of sets $(C_1, \mathbf{C}U_{11})$ and $(\mathbf{C}U_{13}, C_1')$, and so on.

Concerning the pair (C_2, C_2') , we apply the same procedure, too. Then we have an infinite sequence of mutually disjoint closed sets $\{C_1, C_2, C_3, \dots\}$, and can use the Urysohn's metrization method. Thus we have the desired continuous function $\varphi(x)$.

Theorem 1 (Tietze's Extension Theorem). *Let R be a space, and $f(x)$ be a real valued bounded continuous function defined on a compact set M . Then there exists an extension $F(x)$ of f defined over whole R , and its values are bounded by $k = \text{Max}_{x \in M} |f(x)|$.*

Proof. Put $C_1 = \{x; f(x) \geq k/3\}$, $C_2 = \{x; f(x) \leq -k/3\}$. Then C_1 and C_2 are closed subsets of M , therefore compact. Using the continuous function $\varphi(x)$ defined in Lemma 1, we define a new function $F_1(x)$ on R as follows:

$$F_1(x) = \frac{k}{3} (2\varphi(x) - 1).$$

$F_1(x)$ is obviously continuous, and it holds

$$F_1(x) \begin{cases} = k/3 & \text{for } x \in C_1, \\ < k/3, > -k/3 & \text{for } x \in R - (C_1 \cup C_2), \\ = -k/3 & \text{for } x \in C_2. \end{cases}$$

Therefore it satisfies the following conditions :

$$\begin{aligned} |F_1(x)| &\leq k/3 && \text{for all } x \in R, \\ |F_1(x) - f(x)| &\leq 2k/3 && \text{for } x \in M. \end{aligned}$$

Let $F' = F|M$ (in general $f|M$ means a contraction of f to the set M). In place of $f(x)$ and k in the above procedure, we use $f_1(x) = f(x) - F_1(x)$ and $2k/3$, respectively. Then we have a continuous function $F_2(x)$ defined all over R , and satisfying the following conditions :

$$\begin{aligned} |F_2(x)| &\leq 2k/3^2 && \text{for all } x \in R, \\ |f(x) - F_1(x) - F_2(x)| &\leq 2^2 k/3^3 && \text{for all } x \in M. \end{aligned}$$

By the same way, we have a sequence of continuous functions $\{F_n(x)\}$ defined over R , satisfying

$$(1) \quad |F_n(x)| \leq 2^{n-1} k/3^n \quad \text{for all } n = 1, 2, \dots, \text{ and } x \in R,$$

and

$$(2) \quad |f(x) - \sum_{i=1}^n F_i(x)| \leq 2^n k/3^{n+1} \quad \text{for all } n, \text{ and } x \in M.$$

Since $\sum_{i=1}^{\infty} F_i(x)$ converges uniformly on R , $F(x) = \sum_{i=1}^{\infty} F_i(x)$ is a continuous function on R and satisfies, by virtue of (1) and (2)

$$(3) \quad |F(x)| \leq k \quad \text{for all } x \in R,$$

and

$$(4) \quad F(x) = f(x) \quad \text{for all } x \in M.$$

This completes the proof.

3. In this section we shall say some definitions and easy consequences.

Definition 1. Let B be a subset of a space A . We call B a retract of A , provided that there exists a continuous function $f(x)$ defined on A , which satisfies the following conditions :

$$(1) \quad f(x) \in B \quad \text{for all } x \in A,$$

$$(2) \quad f(x) = x \quad \text{for all } x \in B.$$

Then f is called a retraction of A on B .

Proposition 1. A necessary and sufficient condition that B be a retract of A is that any continuous function defined on B can be extended to A without change of its range.

Proof. Suppose that B is a retract of A , $\varphi: A \rightarrow B$ its retraction. Let $f(x)$ be any continuous function defined on A . Define $F(x)$ as follows:

$$F(x) = f\varphi(x) \quad \text{for all } x \in A.$$

It is clear that $F(x)$ is continuous and $F(A) \subset f(B)$. For $x \in B$

$$F(x) = f\varphi(x) = f(x).$$

Hence $F(x)$ is the desired extension of $f(x)$.

Conversely, assuming that any continuous function defined on B can be extended to A consider the function $\varphi(x)$ such as

$$\varphi(x) = x \quad \text{for all } x \in B.$$

Then its extension $\psi(x)$ is a retraction of A on B . Q.E.D.

Proposition 2. *Let B be a retract of A , then B is closed in A .*

Proposition 3. *Let B be a retract of A , and if A has the fixed point property (abbreviation F.P.P.), then B has also the F.P.P.*

Proof. Let $f(x)$ be any intrinsic function on B , and $\varphi(x)$ be a retraction of A on B . Then the function $f\varphi(x)$ is an intrinsic function on A . Since A has the F.P.P., there exists a point $x_0 \in A$, and it holds $f\varphi(x_0) = x_0$. Since $f\varphi(A) \subset B$, $x_0 \in B$, hence $\varphi(x_0) = x_0$. Then we have $f(x_0) = x_0$.

Proposition 4. *Let $f: P \rightarrow A$ be any continuous function, $A \supset B$, B be a retract of A , $P_1 \supset P$. If $F: P \rightarrow A$ is an extension of f , then there exists an extension F' of F on P_1 and $F'(P_1) \subset B$.*

Proof. Let φ be a retraction of A on B . Put $F'(x) = \varphi F(x)$. Obviously F' is a continuous function defined on P_1 and its range is contained in B . If $x \in B$, since F is an extension of f , we have: $F(x) = f(x) \in B$. Therefore $\varphi F(x) = \varphi f(x) = f(x)$. Hence F' is a required extension of f .

Proposition 5. *Let $f: B \rightarrow A$ a continuous function. A necessary and sufficient condition that f be a retraction is that for every $x \in B$ it holds $ff(x) = f(x)$ and $f(B) = A$.*

Proof. Assume $f: B \rightarrow A$ is a retraction. Then the assertions are obvious from the fact:

$$f(x) = x \quad \text{for every } x \in B.$$

Conversely suppose that $f: B \rightarrow A$ satisfies the above stated two conditions. Take a point $x \in A$, and $y \in f^{-1}(x)$. Then $ff(y) = f(y) = f(x) = x$.

4. Definition 2. We call a space A an absolute retract (A.R.) provided that for every space $A \subset B$, A is an retract of B .

Definition 3. We say that a space A is a closed neighborhood retract of B if it holds

- (i) A is a closed subset of B ,
- (ii) there exists an open set U containing A in B , and A is a retract of U .

Furthermore, we call A an absolute neighborhood retract (A.N.R.) provided that (i) A is compact, and (ii) for every $B \supset A$, A is a closed neighborhood retract of B .

Theorem 2. A necessary and sufficient condition that a space A be an A.R. is that there exist a Tychonoff cube T and its subset A_1 which is homeomorphic to A , and A_1 is a retract of T .

Proof. Assume that A is an absolute retract. Since A is completely regular there exist a Tychonoff cube T and its subspace A_1 which is homeomorphic to A_2 by virtue of Tychonoff's theorem. Of course T is a locally compact Hausdorff space, therefore, since A is an absolute retract, A_1 is a retract of T .

Conversely, we assume that there exist a Tychonoff cube T and its subspace A_1 which is homeomorphic to A , and A_1 is a retract of T .

Let h be the homeomorphism, and $\varphi: T \rightarrow A_1$ be the retraction, and M be an arbitrary space containing A . Since T is compact, as a retract of T , A_1 is compact, accordingly A_1 is compact. Then there is an extension H of h defined over whole M by Theorem 1. (We can denote $T = \prod I_x$; $I_x = [0, 1]$) $H: M \rightarrow T$. Put $\psi = h^{-1}\varphi H$. Then $\psi(M) = h^{-1}\varphi H(M) = A$. And it holds:

$\psi(x) = h^{-1}\varphi H(x) = h^{-1}\varphi h(x) = h^{-1}h(x) = x$ for every $x \in A$. Hence $\psi: M \rightarrow A$ is a retraction.

Corollary. Every absolute retract is compact.

Theorem 3. A necessary and sufficient condition that a space A be an absolute neighborhood retract is that there are a Tychonoff cube T and its compact subset A_1 which is homeomorphic to A , and A_1 is a closed neighborhood retract of T .

Proof. We can prove this theorem by the similar way in the above.

Proposition 6. Let A be an absolute retract. If B is a retract of A , then B is also an absolute retract.

Theorem 4. *A necessary and sufficient condition that a space A be an absolute retract is that it satisfies the following conditions:*

- (i) *A is compact Hausdorff space,*
- (ii) *Let f be an arbitrary continuous function defined on a compact space P contained in a space P_1 , and $f(P) \subset A$. Then there exists an extension F of f over P_1 , and $F(P_1) \subset A$.*

Proof. *Necessity.* By Theorem 2, there exist a Tychonoff cube T , its subset A_1 homeomorphic to A , and A_1 is a retract of T . Let h be the homeomorphism, φ be the retraction. Consider the function hf . Then $hf: P \rightarrow T$. Then there exists an extension $g: P \rightarrow T$ of hf . $F = h^{-1}\varphi g$. Then $F: P_1 \rightarrow A$ is obviously continuous, and when $x \in P$, $F(x) = h^{-1}\varphi g(x) = h^{-1}\varphi hf(x) = h^{-1}hf(x) = f(x)$. That is, F is the required extension.

Sufficiency. Suppose that the above stated conditions are satisfied. Let M be a space containing A . Consider the function $f: A \rightarrow A$; $f(x) = x$ for all $x \in A$. Then there exists an extension $F: M \rightarrow A$ of f , this F is a retraction of M on A .

Theorem 5. *A necessary and sufficient condition that a space A be an absolute neighborhood retract is that:*

- (i) *A is a compact Hausdorff space,*
- (ii) *Let f be an arbitrary continuous function defined on a compact subset P of a space P_1 , and $f(P) \subset A$. Then there exist an open set U , $P_1 \supset U \supset P$, and an extension $F: U \rightarrow A$ of f .*

Proposition 7. *Let A be an absolute neighborhood retract. If B is a closed neighborhood retract of A , then B is also an absolute neighborhood retract.*

Proof. Since B is closed in A , it is compact. Let V be an open set containing B in A , and $\varphi: V \rightarrow B$ be the retraction.

Let P be a compact subset of a space P_1 , and $f: P \rightarrow B$ an arbitrary continuous function. Since A is an absolute neighborhood retract, we have an open set U , $P_1 \supset U \supset P$, and an extension $g: U \rightarrow A$ of f . Then $W = g^{-1}(g(U) \cap V)$ is open in U , therefore open in P_1 , and $W \supset P$. Put $g' = g|_W$, and $F = \varphi g'$, then $F: W \rightarrow B$ is the extension of f . Our proof is completed by Theorem 5.

Theorem 6. *Let $\{A_\alpha\}$ be a collection of absolute retract. Then ΠA_α is also an absolute retract.*

Proof. Let P be a compact subset of a space P_1 , and $f: P \rightarrow \prod A_\alpha$ be a continuous function. Then we can consider $f(x) = \{f_\alpha(x)\}$, $f_\alpha: P_1 \rightarrow A_\alpha$.

By the above theorem, we have an extension $F_\alpha: P_1 \rightarrow A_\alpha$ of f_α . Put $F = \{F_\alpha\}$, then $F: P \rightarrow \prod A_\alpha$ is an extension of f . This completes the proof.

Theorem 7. Let $A_1, A_2, \dots, A_n$ be a finite number of absolute neighborhood retracts, then $\prod_{i=1}^n A_i$ is also an absolute neighborhood retract.

Proof. Let P be a compact subset of a space P and $f: P \rightarrow \prod A_i$ be a continuous function. We shall show that f is extensible to a open set V containing P . We can consider $f = \{f_i\}$, $f_i: P \rightarrow A_i$. Since A_i is an absolute neighborhood retract, there exists an open set V_i , $P_1 \supset V_i \supset P$, and f_i can be extensible to V_i . Put this extension F_i' , and $\bigcap_{i=1}^n V_i = V$, then V is an open set containing P , and put $F_i' \upharpoonright V = F_i$, $F = \{F_i\}$. Then F is a required one.

5. We shall show in this section some theorems concerning absolute retracts and absolute neighborhood retracts.

Theorem 8. Let $A \cup B$, $A \cap B$ be absolute retracts, then A and B are both absolute retracts.

Theorem 9. If A , B and $A \cap B$ are weakened absolute retracts, then $A \cup B$ is also a weakened absolute retract.

In general, this sum theorem does not hold without the restriction "weakened". We call a space A a weakened absolute retract, if spaces and retraction are restricted in monotonous locally compact Hausdorff spaces¹⁾. In this case, we have the so-called "Tychonoff's separation theorem", i.e., these spaces are completely normal²⁾. Therefore we can prove this theorem by the analogous way as BORSUK.

Definition 4. A continuous function $f: A \rightarrow A$ is called deformable provided that there exists a function $\varphi(x, t): A \times [0, 1] \rightarrow A$ satisfying:

(i) $\varphi(x, t)$ is continuous with respect to both x and t , $x \in A$, $t \in [0, 1]$,

¹⁾ See [1]

²⁾ See [2]

$$(ii) \quad \varphi(x, t) \begin{cases} = f(x) & \text{for all } x \in A \text{ and } t = 0, \\ \in A & \text{for all } x \in A \text{ and } t \in [0, 1], \\ = x & \text{for all } x \in A \text{ and } t = 1. \end{cases}$$

We call space B a *deformation retract* of A provided that the retraction $\varphi: A \rightarrow B$ is deformable in A .

Definition 5. A space A is called *contractible to a point* $a \in A$ if the function $\varphi: A \rightarrow a$ is deformable in A .

Definition 6. A space A is called *locally contractible at* $a \in A$ provided that for every neighborhood U of a there exists a neighborhood V of a $V \subset U$, and V is contractible to a in U .

Theorem 10. Every absolute retract is contractible.

Proof. Let A be an absolute retract and $a \in A$. Define the function $g(x, t): A \times [0] \cup \{a\} \times [0, 1] \cup A \times [1] \rightarrow A$ as follows:

$$g(x, t) \begin{cases} = a & \text{for all } x \in A \text{ and } t = 0, \\ = a & \text{for } x = a \text{ and } t \in [0, 1], \\ = x & \text{for all } x \in A \text{ and } t = 1. \end{cases}$$

Then g is clearly continuous. Since $A \times [0] \cup \{a\} \times [0, 1] \cup A \times [1]$ is compact in $A \times [0, 1]$ and A is an absolute retract, therefore applying Theorem 4, we have an extension $f(x, t): A \times [0, 1] \rightarrow A$ of g , and this $f(x, t)$ is the required deformation.

Proposition 8. Let A, B ($A \supset B$) be absolute retracts. Then B is a deformation retract of A , furthermore in this deformation every point of B is fixed left.

Proof. By a similar way as in Theorem 10.

Lemma 2. Let X be a Hausdorff space and C its closed subset, A be a compact Hausdorff space. If there is an open set U , $X \times A \supset U \supset C \times A$, there exists an open set V , $X \supset V \supset C$, and $V \times A \subset U$.

Proof. Using the compactness of $\bar{U}$, proof is easy.

Theorem 11. Let X be a compact Hausdorff space, and C its closed subset. If A, B are absolute retracts and $A \supset B$. Then for any continuous function $f: (X \times B) \cup (C \times A) \rightarrow N$, where N is an absolute neighborhood retract, there exists an extension $F: X \times A \rightarrow N$ of f .

Proof. Let $\varphi: A \rightarrow B$ the retraction. Since C is closed in X , C is compact, too. By Proposition 6, B is also an absolute retract,

therefore compact. Then $X \times B \cup C \times A$ is compact in $X \times A$. Using Theorem 5, we have an open set U , $X \times A \supset U \supset (X \times B \cup C \times A)$, and an extension $g: U \rightarrow N$ of f . By Lemma 2, there exists an open set V , $X \supset V \supset C$, such that $V \times A \subset U$. Then C and $C^c V$ is disjoint closed sets in X , then by Lemma 1, there exists a function $\rho(x)$ on X satisfying: $0 \leq \rho(x) \leq 1$, $\rho(x) = 1$ for $x \in C$ and $\rho(x) = 0$ for $x \in C^c V$. By Proposition 8, there exists a function $\phi(a, t): A \times [0, 1] \rightarrow A$, $\phi(a, 0) = \varphi(a)$ and $\phi(a, 1) = a$ for $a \in A$ and $\phi(a, t) = a$ for $a \in B$.

Define $F: X \times A \rightarrow N$ as follows:

$$F(x, a) = g(x, \phi(a, \rho(x))) \quad \text{for } x \in X, a \in A$$

For $(x, a) \in V \times A$, since $\phi(a, \rho(x)) \in A$, $F(x, a)$ is defined. For $(x, a) \in C^c V \times A$, $\phi(a, \rho(x)) = \phi(a, 0) = \varphi(a) \in B$ therefore $F(x, a)$ is also defined. And for $(x, a) \in X \times B \cup C \times A$, $F(x, a) = f(x, a)$, thus F is the required extension.

Proposition 9. *Let B be a retract of A , and $x \in B$. If A is locally contractible at x , then B is locally contractible at x .*

Using this proposition and local contractibility of the Tychonoff cube the following theorem can be proved, but we give here another proof.

Theorem 12. *Every absolute neighborhood retract is locally contractible.*

Proof. Let A be an absolute neighborhood retract, $a \in A$, $U(a)$ be an arbitrary open neighborhood of a . Without loss of generality we may assume that $\bar{U}$ is compact.

Define $g(x, t): \bar{U} \times [0] \cup \{a\} \times [0, 1] \cup \bar{U} \times [1] \rightarrow A$ as follows:

$$\begin{aligned} g(x, 0) &= a && \text{for all } x \in \bar{U} \\ g(a, t) &= a && \text{for all } t \in [0, 1], \\ g(x, 1) &= x && \text{for all } x \in \bar{U}. \end{aligned}$$

Put $P = \bar{U} \times [0] \cup \{a\} \times [0, 1] \cup \bar{U} \times [1]$, then because of compactness of P in $A \times [0, 1]$, and since A is an absolute neighborhood retract, applying Theorem 5, we have an open set W , $A \times [0, 1] \supset W \supset P$, and an extension $f: \rightarrow A$. Then $f^{-1}(U \cap f(W))$ is open in W , therefore open in $A \times [0, 1]$, and contains P , hence contains $\{a\} \times [0, 1]$. By Lemma 2, there exists an open set $V \ni a$ such that $V \times [0, 1] \subset f^{-1}(U \cap f(W))$ and $V \subset U$.

Put $F = f|_{V \times [0, 1]}$, then $F(x, t)$ is the required. In fact,

$$\begin{aligned}
F(x, 0) &= g(x, 0) = a && \text{for } x \in V, \\
F(x, t) &\in U && \text{for } x \in V \text{ and } t \in [0, 1], \\
F(x, 1) &= g(x, 1) = x && \text{for } x \in V.
\end{aligned}$$

This completes our proof.

Proposition 10. *Every absolute retract has the F.P.P.*

Proof. By Tychonoff's theorem³⁾ every Tychonoff cube has the F.P.P., then our proposition is obvious from Proposition 3.

Theorem 14. *Every absolute neighborhood retract which is contractible at a point is an absolute retract.*

Proof. Let A be absolute neighborhood retract which is contractible at $a \in A$. By Theorem 5 we can suppose that A is in a Tychonoff cube T .

Since A is contractible, there exists a continuous function $g: A \times [0, 1] \rightarrow A$ satisfying:

$$\begin{aligned}
g(x, 0) &= a && \text{for } x \in A, \\
g(x, t) &\in A && \text{for } x \in A \text{ and } t \in [0, 1], \\
g(x, 1) &= x && \text{for } x \in A.
\end{aligned}$$

Define $f: T \times [0] \cup A \times [0, 1] \rightarrow A$ as follows:

$$\begin{aligned}
f(x, 0) &= a && \text{for } x \in T \\
f(x, t) &= g(x, t) && \text{for } x \in A, t \in [0, 1].
\end{aligned}$$

Since $[0, 1]$ and $[0]$ are absolute retracts, using Theorem 11 we have an extension $F: T \times [0, 1] \rightarrow A$ of f . Put $\varphi(x) = F(x, 1)$, then $\varphi: T \rightarrow A$ is a retraction of T on A .

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³⁾ See [3].