A REPRESENTATION OF A FAMILY OF RANDOM VARIABLES AND THEIR MEANS

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1. The object of this paper is to give a representation of a certain family of random variables by means of a set of operators on certain Banach space, and to investigate the relation between the mathematical expectation, that is the mean of random variable, and the trace of its corresponding operator algebra.

On the other hand, A. KOLMOGOROFF [1] suggested various systems of axioms for probability theory which are different from his own system that depends upon the theory of Lebesgue measure. Y. KAWADA [2] constructed a system of axioms by Vector lattice. In this paper, we also investigate a system of axioms on a complete modular lattice making use of our representation, where a modular function m on the modular lattice L , stands for a distribution, or the trace $\phi(A)$ induced by m on the ring of operators N_L , corresponds to a mean value of A that is a representation of a certain random variable in L .

2. We suppose that a set of "random variables" L to which any definition is not given, is a complete modular lattice with the zero-element θ and the unit element I , and let C be a set of all bounded real function defined on L . Then C is a Banach space with norm $\|\varphi\|$ ($\varphi \in C$) defined by

$$(2.1) \quad \|\varphi\| = \text{l.u.b.}_{x \in L} |\varphi(x)|.$$

Now we introduce two operators ∇_a, Δ_a ($a \in L$) on C , which are defined as follows:

$$(2.2) \quad \nabla_a \varphi(x) = \varphi(a \cup x), \quad \Delta_a \varphi(x) = \varphi(a \cap x).$$

Lemma 1. $\nabla_a(J_a)$ is a linear bounded and idempotent operator with the norm 1, i.e. a projection on C .

Proof: The linearity is evident. The boundedness of ∇_a follows from

$$(2.3) \quad \|\nabla_a \varphi\| = l.u.b._{x \in L} |\varphi(a \cup x)| \leq l.u.b._{x \in L} |\varphi(x)| = \|\varphi\|$$

and hence $\|\nabla_a\| \leq 1(*)$. It is easily seen from a definition that $\nabla_a^2 = \nabla_a$, and $\|\nabla_a\| \neq 0$, hence $\|\nabla_a\| \geq 1(**)$. In combination of (*) and (**), we have $\|\nabla_a\| = 1$. Similarly for J_a . q.e.d.

Let $\nabla_L(J_L)$ be a set of all $\nabla_a(J_a)$ for $a \in L$.

Lemma 2. One to one correspondence T can be defined between $\nabla_L(J_L)$ and L by the mapping which transforms each a to ∇_a .

Proof: It is sufficient to prove that $\nabla_a = \nabla_b$ implies $a = b$.

Let $\nabla_a = \nabla_b$, then $\nabla_a \varphi = \nabla_b \varphi$ for all $\varphi \in L$, that is, $\varphi(a \cup x) = \varphi(b \cup x)$ for all $\varphi \in C$, and all $x \in L$. Here we consider two functions $\varphi_1(x)$, $\varphi_2(x)$ on L such that

$$(2.4) \quad \begin{array}{lll} \varphi_1(a) = 1, & \varphi_1(x) = 0, & x \neq a \\ \varphi_2(b) = 1, & \varphi_2(x) = 0, & x \neq b \end{array}$$

But being evidently elements of C , φ_1, φ_2 should satisfy

$$\varphi_1(a \cup a) = \varphi_1(b \cup a) = 1, \quad \varphi_2(a \cup b) = \varphi_2(b \cup b) = 1$$

and hence $a = a \cup b$, $a \cup b = b$. These two facts leads us $a = b$, as was to be proved. It is similarly for J_a .

We introduce a semi-order in $\nabla_L(J_L)$ as follows,

$$(2.5) \quad \nabla_a \geq \nabla_b (J_a \geq J_b) \text{ when and only when } \nabla_a \nabla_b = \nabla_a (J_a J_b = J_b)$$

Then, each of following conditions (1), (2), (3), ((1)', (2)', (3)') implies other two.

$$(2.6) \quad \begin{array}{lll} (1)^\circ a \geq b, & (2)^\circ \nabla_a \nabla_b = \nabla_a & (3)^\circ \nabla_a \geq \nabla_b \\ (1)' a \geq b, & (2)' J_a J_b = J_b & (3)' J_a \geq J_b \end{array}$$

Consequently, we have

Theorem 1. L can be one to one corresponding and lattice-isomorphic to $\nabla_L(J_L)$.

3. We shall consider an operator ring N_L generated by the set of projection operators J_L with respect to a weak topology, and introduce a semi-order into it, which is defined in the following manner. An operator $A \in N_L$ is said to be positive, when $A\varphi(x) \geq 0$ for all

$\varphi(x) \in L$ provided that $\varphi(x) > 0$, and $x \geq y$ implies $\varphi(x) \geq \varphi(y)$ for all $x, y \in L$. We write $A \geq B$, when $A - B \geq 0$.

It is easily seen that N_L is a semi-ordered additive Abelian group with real multipliers, including the Archimedian Unit $\mathcal{A}'_I$ satisfying following conditions,

- (3.1) i) If $X, Y \in N_L$ and $X \geq 0, Y \geq 0$, then $X + Y \geq 0$.
 ii) If $X \geq 0, \alpha$ (real numer) ≥ 0 , then $\alpha X \geq 0$.
 iii) If $X \geq 0$ and $-X \geq 0$, then $X = \theta$.
 iv) $\inf_{\epsilon > 0} \epsilon \mathcal{A}'_I$ exists and equal to θ .
 v) N_L is complete with respect to the norm

$$\|X\| = \inf_{\alpha} (-\alpha \mathcal{A}'_I \leq X \leq \alpha \mathcal{A}'_I)$$

Then the Theorem 4 of K. Yosida and S. NAKYAMA [3] makes us possible to represent an element A of N_L in the following form in the sense of Riemann-Stieltjes integral in semi-order;

$$(3.2) \quad A = \int \lambda d\mathcal{A}'_I$$

where $\mathcal{A}'_{I_\lambda} \in N_L$, it is verified from the completeness of L is a resolution of the unity $\mathcal{A}'_I$ with the properties

- i) $\mathcal{A}'^2_{I_\lambda} = \mathcal{A}'_{I_\lambda} \leq \mathcal{A}'_{I_\mu} = \mathcal{A}'^2_{I_\mu}$ if $\lambda \leq \mu$,
 ii) if $\lambda_n \downarrow \lambda$ then order-limit $\mathcal{A}'_{I_{\lambda_n}} = \mathcal{A}'_{I_\lambda}$,
 iii) $\mathcal{A}'_{I_\lambda} = \mathcal{A}'_I$ for $\lambda \geq \|A\|$ and $\mathcal{A}'_{I_\lambda} = \mathcal{A}_\theta$ for $\lambda < -\|A\|$

We call a element m of C a *distribution* which satisfies

- (3.3) i) If $a, b \in L$ $m(a \cup b) + m(a \cap b) = m(a) + m(b)$
 ii) If $a, b \in L$ $a \geq b$, then $m(a) \geq m(b)$.

Let Z_L be the set of all distribution functions and define a *distribution* m_ϕ on $\mathcal{A}_L$ with respect to ϕ belonging to Z_L as follows

$$(3.4) \quad m_\phi(\mathcal{A}_a) = \phi(a)$$

Then, from the representation (3.1), we can define the linear functional $\phi(A)$ on the Ring of operators N_L

$$(3.5) \quad \phi(A) = \int \lambda dm_\phi(\mathcal{A}'_{I_\lambda})$$

where the integral denotes the formal sum in the sense of Riemann-Stieltjes. Thus

Theorem 2. For any element A of N_L there is a linear functional $\phi(A)$;

$$\phi(A) = \int \lambda dm_\phi(\mathcal{I}'_{I_\lambda}).$$

4. In this section we shall prove the representation theorem of type of CRAMER [4], MARUYAMA [5] with respect to stationary stochastic process. Let A, B be any two elements of N_L , and consider the set $\mathfrak{R} = \{A + iB; A, B \in N_L\}$ that is, any element $\tilde{A}$ of $\mathfrak{R}$ can be written as

$$(4.1) \quad \tilde{A} = A + iB \quad A, B \in N_L.$$

We write $A - iB \equiv \tilde{A}^*$. Let us consider a positive definite functional on $\mathfrak{R}$ which satisfies the following conditions

$$(4.2) \quad f(\tilde{A} \tilde{A}^*) \geq 0, \text{ for any element } \tilde{A} \text{ of } \mathfrak{R}.$$

$$(4.3) \quad f(\tilde{A}^*) = \overline{f(\tilde{A})}, \text{ where } \overline{f(\tilde{A})} \text{ denotes the conjugate complex of } f(\tilde{A}).$$

$$(4.4) \quad f(\tilde{I}) = 1, \tilde{I} \text{ is the Unit of } \mathfrak{R}, \text{ that is, } \tilde{I} = I.$$

Let $\mathfrak{R}^b$ be the set of all $\tilde{A}$ that $|f(\tilde{A} \tilde{A}^*)| < +\infty$. For any two elements $\tilde{A}, \tilde{B}$ of $\mathfrak{R}^b$, we define the inner product in $\mathfrak{R}^b$ by $f(\tilde{A} \tilde{B}^*) \equiv (\tilde{A}, \tilde{B})$ then it is well known that (e.g. see [7])

$$(4.5) \quad |(\tilde{A}, \tilde{B})| \leq (\tilde{A}, \tilde{A})^{\frac{1}{2}} (\tilde{B}, \tilde{B})^{\frac{1}{2}}.$$

Lemma 3. *We put $\mathfrak{N} = \{\tilde{A}; f(\tilde{A} \tilde{A}^*) = 0\}$, then $\mathfrak{N}$ is left ideal of $\mathfrak{R}^b$.*

Proof: If for any three elements $\tilde{B} \in \mathfrak{R}^b, \tilde{A}_1, \tilde{A}_2 \in \mathfrak{N}$ then by (4.5) $f(\tilde{A}_1 \tilde{A}_1^*) = 0$ implies $f(\tilde{B} \tilde{A}_1^*) = 0$. On the other hand, $|f(\tilde{B} \tilde{A}_1^*)| = 0$ and $|f(\tilde{B} \tilde{A}_2^*)| = 0$ imply $|f\{\tilde{B}(\tilde{A}_1 + \tilde{A}_2)^*\}| = 0$. The homogeneity is evident. As was to be proved. The rest class $\mathfrak{R}^b/\mathfrak{N}$ becomes a incomplete Hilbert space $\mathfrak{H}$ with respect to the inner product which is induced from (4.5). We let $\mathfrak{H}$ complete and rewrite it as $\mathfrak{H}$ again, and an element of $\mathfrak{H}$ is written as $\tilde{\tilde{A}}, \tilde{\tilde{B}}, \dots$. $\tilde{\tilde{I}}$ is the class to which $\tilde{I}$ belongs. We can define a functional f' on $\mathfrak{H}$ induced by the functional f on $\mathfrak{R}^b$ by putting $f'(\tilde{\tilde{A}}) = f(\tilde{A})$ then $f'(\tilde{\tilde{I}}) = 1$ by (4.4).

Theorem 3. *If a one parameter family $\tilde{\tilde{X}}_t$ of $\mathfrak{H}$ ($-\infty < t < +\infty$) satisfies $\|\tilde{\tilde{X}}_t\| = \text{constant}$ irrespective of t ($-\infty < t < +\infty$) then $\tilde{\tilde{X}}_t$ is represented as*

$$(4.6) \quad \tilde{\tilde{X}}_t = \int_{-\infty}^{\infty} e^{i\lambda t} d\tilde{\tilde{E}}_\lambda$$

where $\tilde{\tilde{E}}_\lambda$ is a resolution of the identity on $\mathfrak{H}$ and

$$(4.7) \quad f'(\tilde{X}_t) = \int_{-\infty}^{\infty} e^{it} d(\tilde{E}_\lambda, \tilde{I}) = \int_{-\infty}^{\infty} e^{i\lambda t} df'(\tilde{E}_\lambda).$$

Proof: When we put $\tilde{B}\tilde{A} = H_{\tilde{B}}\tilde{A}$ for any two elements $\tilde{A}, \tilde{B} \in \mathfrak{H}$, $H_{\tilde{B}}$ can be considered as bounded operator in $\mathfrak{H}$. So, one parameter subspace $\tilde{X}_t$ ($-\infty < t < +\infty$) of $\mathfrak{H}$, is, at same time, considered as one parameter family of bounded operators on Hilbert space $\mathfrak{H}$. And so, when $\|\tilde{X}_t\|$ is a constant irrespective of t , whose condition corresponds to stationarity of stochastic process, $\tilde{X}_t$ ($-\infty < t < +\infty$) is considered as one parameter family of unity operators on $\mathfrak{H}$. Thus by well-known Stone's theorem, the representation (4.6) is possible, and (4.7) is easily seen by the definition of the functional f' .

References

- [1] A. KOLMOGOROFF. *Grundbegriffe der Wahrscheinlichkeitsrechnung Berlin*, (1933).
- [2] Y. KAWADA. *Die verbandtheoretische Begründung der Wahrscheinlichkeitsrechnung. Japan. Journ. of Math.* Vol. XIX. No. 1. (1944).
- [3] K. YOSHIDA and T. NAKAYAMA. *On the semi-ordered ring and its application to the spectral theory. II. Proc. Imp. Acad. Tokyo.* **19** (1943).
- [4] H. CRAMER. *On the theory of stationary processes. Annals of Math.* vol. **41**. (1940).
- [5] G. MARUYAMA. *The harmonic analysis of stationary stochastic processes. Memoirs of the Fac. of Sci. Kyusyu University.* vol. **IV**. No. 1. (1949).
- [6] M. ORIHARA. *Rings of operators and their traces. Memo. of Fac. Sci. Kyusyu University.* vol. **V**. No. 2. (1950).
- [7] J. V. NEUMANN. *Allgemeine Eigenwert theorie Hermitescher Funktional operatoren Math. Ann.* **102**. p. 64 (1929).