ON THE MEAN-VALUE THEOREM OF HARMONIC FUNCTIONS

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§ 1. Introduction.

Let D be a domain in the space of n real variables $x \equiv (x_1, x_2, \dots, x_n)$, whose volume element we denote by $dv_n \equiv dx_1 dx_2 \dots dx_n$. Assume that the functions $f(x)$ and $\varphi(x)$ are continuously differentiable up to the second order in D , and that the carrier S of φ is compact and is contained in D . Let

$$(1.1) \quad \Delta \equiv \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

be the Laplacian operator, which is self-adjoint. Then we have the relation

$$(1.2) \quad \int_D (\Delta f) \cdot \varphi dv_x = \int_D f \cdot (\Delta \varphi) dv_x.$$

Now in the following the functions f and φ will play very distinct roles. The function f will be a fixed function defined in D , at first of differentiability class C'' . However φ will be a variable function, sometimes called a testing function in D . It is a continuous and infinitely differentiable function in D , whose carrier S , i.e. the closure of the set of points $x \in D$ such that $\varphi(x) \neq 0$ is a compact contained in D . Following L. SCHWARTZ we call $(\mathfrak{D})$ the space of all such real functions φ in D .

Now, in particular, a function f is a harmonic function, i.e. a strict solution of the LAPLACE's equation

$$(1.3) \quad \Delta f = 0,$$

if and only if for all testing functions we have

$$(1.4) \quad \int_D (\Delta f) \cdot \varphi dv_x = 0.$$

By (1.2) this is entirely equivalent with

$$(1.5) \quad \int_D f \cdot (\Delta \varphi) dv_x = 0.$$

By dint of this S. BOCHNER [1] generalized the concept of a solution of (1.3). Following him we only require that $f(x)$ shall be defined and measurable in D except for a set of measure zero and that in every compact contained in D $f(x)$ shall be Lebesgue integrable. We call $f(x)$ a generalized solution of (1.3), if corresponding to every point P in D there exists a neighbourhood U_0 of P such that for subneighbourhood U of U_0 and for any testing function $\varphi(x)$ vanishing outside of U we shall have the relation (1.5). Clearly if a generalized solution happens to belong to class C'' , it is a strict solution of (1.3).

For the harmonic function we know on the one hand the classical GAUSS' Theorem of the Arithmetic Mean. If $f(x)$ is a strict solution of (1.3), that is, a harmonic function in D , the value of $f(x)$ at each point $x \in D$ is the arithmetic mean of its values on the $(n-1)$ -dimensional boundary of any sphere with center at x contained entirely in D . We write it in the form

$$(1.6) \quad f(x) = \int f(x + \rho \xi) d\omega_\xi \equiv M(f),$$

where ξ is a point of the unit sphere, $d\omega_\xi$ is the $(n-1)$ -dimensional area-element divided by the total area and ρ is an arbitrary sufficiently small positive number.

On the other hand we know as the converse of GAUSS' theorem the celebrated Theorem of KOEBE. (See e.g. [2], p. 224). If $f(x)$ is continuous in D , and at every point of D has as value the arithmetic mean of its values on any sphere with center at that point and lying in D , then $f(x)$ is harmonic in D .

Now about the generalized solution of (1.3) it seems natural for us to raise the question, "Do the following two statements hold as the generalization of the above theorems?"

Theorem I. *If $f(x)$ is a generalized solution of (1.3) in D , the value of $f(x)$ at each point $x \in D$ except for a set of measure zero is the arithmetic mean of its values on the $(n-1)$ -dimensional boundary of any sphere with center at x contained entirely in D .*

Theorem II. *Let $f(x)$ be defined and measurable in D except for a set of measure zero and be Lebesgue integrable in every compact contained*

in D . If at each point $x \in D$ except for a set of measure zero $f(x)$ has as value the arithmetic mean of its values on any sphere with center at that point and lying in D , then $f(x)$ is a generalized solution of (1.3) in D .

In fact these two theorems are true as we shall show in the following. In the proof we shall conveniently use the concepts of the SCHWARTZ's theory of Distribution.

§ 2. Some Lemmas.

Lemma 1. Let λ be a real parameter varying on the interval $[\alpha, \beta]$. For each value of λ let $\varphi(x; \lambda)$ belong to $(\mathfrak{D})$ and have its carrier S_λ contained in a fixed compact $K \subset D$, and let each derivative $D^p \varphi(x; \lambda)$ be continuous with respect to $x \in K$, $\lambda \in [\alpha, \beta]$. Let T be a fixed distribution on D , then $I(\lambda) = T_x(\varphi(x; \lambda))$ is a continuous function of λ in $[\alpha, \beta]$ and its integral may be obtained by integration under the "integral-sign":

$$(2.1) \quad \int_{\alpha}^{\beta} T(\varphi(x; \lambda)) d\lambda = T\left(\int_{\alpha}^{\beta} \varphi(x; \lambda) d\lambda\right).$$

Proof. By the definition of the integral

$$\int_{\alpha}^{\beta} T(\varphi(x; \lambda)) d\lambda = \lim_{n \rightarrow \infty} \sum_{i=1}^n T(\varphi(x; \bar{\lambda}_i)) \Delta \lambda_i,$$

where $\bar{\lambda}_i$ is some point on the interval $\Delta \lambda_i$. Now

$$(2.2) \quad \sum_{i=1}^n T(\varphi(x; \bar{\lambda}_i)) \Delta \lambda_i - T\left(\int_{\alpha}^{\beta} \varphi(x; \lambda) d\lambda\right)$$

$$= T\left(\sum_{i=1}^n \varphi(x; \bar{\lambda}_i) \Delta \lambda_i - \int_{\alpha}^{\beta} \varphi(x; \lambda) d\lambda\right)$$

$$(2.3) \quad = T\left(\sum_{i=1}^n [\varphi(x; \bar{\lambda}_i) - \varphi(x; \lambda'_i)] \Delta \lambda_i\right),$$

where λ'_i is some point on the interval $\Delta \lambda_i$. By dint of the uniform continuity of $\varphi(x; \lambda)$, corresponding to an arbitrary $\varepsilon > 0$ we can find a sufficiently fine division $|\Delta \lambda_i| < \delta_0$ ($i = 1, 2, \dots, n$), such that

$$|\varphi(x; \bar{\lambda}_i) - \varphi(x; \lambda'_i)| < \frac{\varepsilon}{\beta - \alpha} \quad \text{for } x \in K.$$

Therefore

$$\left| \sum_{i=1}^n \varphi(x; \bar{\lambda}_i) \Delta \lambda_i - \int_{\alpha}^{\beta} \varphi(x; \lambda) d\lambda \right| < \varepsilon,$$

that is, the function of x on the left hand side tends to 0 uniformly with respect to $x \in K$ as $\text{Max} |\Delta \lambda_i| \rightarrow 0$. Similarly the derivatives D^p of the same function tend to 0 uniformly with respect to $x \in K$ as $\text{Max} |\Delta \lambda_i| \rightarrow 0$, by reason of the continuity of $D^p \varphi(x; \lambda)$. As the function has its carrier contained in the fixed compact K , it converges to 0 in $(\mathfrak{D})$ pseudo-topologically as $\text{Max} |\Delta \lambda_i| \rightarrow 0$. Therefore by the continuity of the linear form T the quantity (2.3) tends to 0 as $\text{Max} |\Delta \lambda_i| \rightarrow 0$, which proves the lemma.

Lemma 2. *Let K be an arbitrary compact contained in D . Then every generalized solution $f(x)$ of (1.3) is a limit in the sense of L_K^1 -norm of some sequence of strict solutions of (1.3), that is, if $f_1(x), f_2(x), \dots$ is the approximating sequence of strict solutions, then*

$$(2.4) \quad \lim_{s \rightarrow \infty} \int_K |f(x) - f_s(x)| dv_x = 0.$$

Proof. Take a testing function $\varphi(x)$ in D . It vanishes outside some U . Denoting the distance between the boundary of U and D by $\delta > 0$. Then for any vector t whose length does not exceed $\delta/2$ the translated function $\varphi_t(x) \equiv \varphi(x - t)$ is likewise a testing function, albeit in $U_t = U + (t)$. Obviously

$$\Delta \varphi_t(x) = (\Delta \varphi(\xi))_{\xi=x-t}.$$

Now we obtain from (1.5)

$$(2.5) \quad \int_D f(x) \cdot \Delta \varphi_t(x) dv_x = 0$$

for $\|t\| < \delta/2$. In this we can shift the translation from φ onto f , thus obtaining

$$(2.6) \quad \int_D f(x+t) \Delta \varphi(x) dv_x = 0$$

for $\|t\| < \delta/2$. Now if D_ρ denotes any subdomain of D whose boundary has a distance $\geq \rho$ from the boundary of D itself, then we can form the function

$$(2.7) \quad f_h(x) = \frac{1}{(2h)^n} \int_{-h}^h \cdots \int_{-h}^h f(x_1 + t_1, \dots, x_n + t_n) dt_1 \cdots dt_n$$

for x in D_{2h} . Relation (2.6) implies

$$(2.8) \quad \int_{D_{2h}} f(x+t) \Delta \varphi(x) dv_x = 0$$

for $\varphi(x)$ a testing function in D_{2h} , and if we integrate (2.8) with respect to t and interchange integrations with respect to t and x , we obtain

$$(2.9) \quad \int_{D_{2h}} f_h(x) \Delta \varphi(x) dv_x = 0.$$

Thus in D_{2h} , $f_h(x)$ is again a generalized solution of (1.3). But, for fixed h , $f_h(x)$ is a continuous function in x by a fundamental theorem of the Lebesgue theory. Furthermore from

$$|f(x) - f_h(x)| \leq \frac{1}{(2h)^n} \int_{-h}^h \cdots \int_{-h}^h |f(x) - f(x+t)| dv_t,$$

we conclude by FUBINI's theorem that

$$(2.10) \quad \int_K |f(x) - f_h(x)| dv_x \leq \frac{1}{(2h)^n} \int_{-h}^h \cdots \int_{-h}^h \psi(t_1, \dots, t_n) dv_t,$$

where

$$\psi(t_1, \dots, t_n) \equiv \int_K |f(x) - f(x+t)| dv_x.$$

By a fundamental theorem of the Lebesgue theory $\psi(t) \rightarrow 0$ as $\|t\| \rightarrow 0$, and therefore for any sequence $\{h_s\} \rightarrow 0$ we have

$$(2.11) \quad \lim_{s \rightarrow \infty} \int_K |f(x) - f_{h_s}(x)| dv_x = 0.$$

Now starting from the generalized continuous solutions $f_h(x)$ we can iterate the process of averaging, thus defining

$$(2.12) \quad f_{h,2}(x) = \frac{1}{(2h)^n} \int_{-h}^h \cdots \int_{-h}^h f_h(x_1 + t_1, \dots, x_n + t_n) dv_t.$$

These functions are of class C' , and similarly the 3rd h -average, $f_{h,3}(x)$, gives functions of class C'' , which are generalized solutions of (1.3) and thus strict solutions. Therefore there are sequences $\{f_s(x)\}$ among them for which (2.4) holds.

Lemma 3.¹⁾ *Let $f(x)$ be a fixed function, defined and measurable in D , and Lebesgue integrable in every compact contained in D . Let $\varphi(x)$ be*

1) A generalization of the Lemma 1.1. stated in [4], p. 67.

a variable function $\in (\mathfrak{D})$, that is a function, continuous and infinitely differentiable in D , whose carrier is a compact contained in D . If the relation

$$(2.13) \quad \int_D f(x) \varphi(x) dv_x = 0$$

holds for all such testing functions $\varphi(x) \in (\mathfrak{D})$ in D , then $f(x) = 0$ almost everywhere in D .

Proof. First we suppose the function $\varphi(x)$ to be a bounded and measurable function in D , whose carrier is a compact contained in D . We suppose that the relation (2.13) holds for all such testing functions $\varphi(x)$, which is a very severer requirement than in the lemma. Under this condition we shall show that $f(x) = 0$ almost everywhere.

Assume that $|f(x)|$ might be positive on a compact set $E \subset D$ of positive measure $mE > 0$. For this function we can define a testing function as follows:

$$(2.14) \quad \begin{cases} \varphi(x) = 0 & \text{for } x \notin E, \\ \varphi(x) = \frac{f(x)}{|f(x)|} & \text{for } x \in E. \end{cases}$$

Then

$$(2.15) \quad \int_D f(x) \varphi(x) dv_x = \int_E |f(x)| dv_x.$$

Take an arbitrary sequence $\varepsilon_n \downarrow 0$ and put

$$E_n \equiv \{x / |f(x)| > \varepsilon_n, x \in D\}.$$

Then $E = \lim_{n \rightarrow \infty} E_n$, $mE = \lim_{n \rightarrow \infty} m(E_n)$.

Choosing a sufficiently large index N , we have

$$m(E_n) \geq \frac{1}{2} m(E), \quad n \geq N,$$

and from (2.15)

$$(2.16) \quad \begin{aligned} \int f(x) \varphi(x) dv_x &\geq \int_{E_n} |f(x)| dv_x \\ &> \varepsilon_n m(E_n) \\ &\geq \varepsilon_n \cdot \frac{1}{2} m(E) \quad \text{for any } n \geq N. \end{aligned}$$

Hence
$$\int f(x) \varphi(x) dv_x > 0,$$

which is a contradiction.

Now we relax the requirement and suppose the testing functions $\psi(x)$ to be $\epsilon(\mathfrak{D})$. Assume that under this condition $|f(x)|$ might be positive on a compact set $E \subset D$ of positive measure mE .

Let $\{f(x)\}_\nu$ denote $f(x)$ at points where $f(x) \leq \nu$, but ν where $f(x) > \nu$. We write, inserting the function $\varphi(x)$ defined in (2.14),

$$\begin{aligned} (2.17) \quad \int_D f(x) \psi(x) dv_x &= \int_D f(x) \varphi(x) dv_x \\ &+ \int_D [f(x) - \{f(x)\}_\nu] [\psi(x) - \varphi(x)] dv_x \\ &+ \int_D \{f(x)\}_\nu [\psi(x) - \varphi(x)] dv_x. \end{aligned}$$

By (2.16)

$$(2.18) \quad \int_D f(x) \varphi(x) dv_x \geq \epsilon_N \cdot \frac{1}{2} m(E) = A, \text{ say.}$$

We consider only function $\psi(x) \in (\mathfrak{D})$ such as

$$(2.19) \quad |\psi(x) - \varphi(x)| < 1 \quad (x \in D).$$

Then

$$\begin{aligned} (2.20) \quad & \left| \int_D [f(x) - \{f(x)\}_\nu] [\psi(x) - \varphi(x)] dv_x \right| \\ & \leq \int_D |f(x) - \{f(x)\}_\nu| dv_x \\ & < \frac{1}{4} A \quad \text{for } \nu \geq M. \end{aligned}$$

Now under the condition (2.19) we can choose the approximating function $\psi(x)$ so closely as

$$(2.21) \quad \int_D |\psi(x) - \varphi(x)| dv_x < \frac{1}{4M} A^{2\gamma}.$$

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(2) Notice that we need only require that this approximation may be attained for a compact E_1 such as $E \subset E_1 \subset D$.

Then by (2.17), (2.18), (2.20), (2.21)

$$\left| \int_D f(x) \psi(x) dv_x \right| \geq A - \frac{1}{4}A - \frac{1}{4}A = \frac{1}{2}A,$$

which is a contradiction.

§ 3. Proof of Theorem I.

$f(x)$ being a generalized solution of (1.3) in D , consider the function defined by

$$(3.1) \quad F(x) \equiv M(f) = \int f(x + \rho \xi) d\omega_\xi.$$

(See (1.6)). Let $\{f_s(x)\}$ be the sequence of strict solutions of (1.3) introduced in Lemma 2. Then by GAUSS' Theorem we have

$$(3.2) \quad f_s(x) = M(f_s).$$

Consider now the distribution

$$\begin{aligned} F(\varphi) &\equiv \int_D F(x) \varphi(x) dv_x = \int_D \left[\int f(x + \rho \xi) d\omega_\xi \right] \varphi(x) dv_x \\ &= \int_D \left[\int f(x') \varphi(x' - \rho \xi) d\omega_\xi \right] dv_{x'} \\ &= \int \left[\int_D f(x') \varphi(x' - \rho \xi) dv_{x'} \right] d\omega_\xi. \end{aligned}$$

By Lemma 1 this integral can be written as the distribution $f\left(\int \varphi(x' - \rho \xi) d\omega_\xi\right)$. So we have

$$(3.3) \quad F(\varphi) = f\left(\int \varphi(x' - \rho \xi) d\omega_\xi\right).$$

Similarly for the function $f_s(x)$ we obtain by (3.2)

$$(3.4) \quad f_s(\varphi) = f_s\left(\int \varphi(x' - \rho \xi) d\omega_\xi\right).$$

Now by a fundamental theorem of the SCHWARTZ's theory, the convergence of $f_s(x)$ to $f(x)$ in the sense of L_K^1 -norm, where K is an arbitrary compact contained in D , implies that the sequence of distributions $\{f_s\}$ converges to the distribution f in $(\mathfrak{D})'$. On the other hand the relation (3.3) and (3.4)

give us the fact that the same sequence $\{f_s\}$ converges to the distribution F in $(\mathfrak{D})$. Therefore the equality

$$(3.5) \quad F(\varphi) = f(\varphi)$$

holds for any testing function $\varphi \in (\mathfrak{D})$. The equation (3.5) means that

$$\int_D \left[\int f(x + \rho \xi) d\omega_\xi - f(x) \right] \varphi(x) dv_x = 0$$

for any testing function $\varphi \in (\mathfrak{D})$ in D , from which by the Lemma 3 it follows that the required relation

$$\int f(x + \rho \xi) d\omega_\xi = f(x)$$

must hold except for a set of measure zero.

§ 4. Proof of Theorem II.

We assume that $f(x)$ satisfies the relation

$$(4.1) \quad f(x) = \int f(x + \rho \xi) d\omega_\xi$$

at each point x of D except for a set of measure zero. As in § 2 let us form the continuous function (2.7) for x in D_{2h} . Now applying the process on both sides of (4.1), we can conclude by FUBINI's theorem that the relation

$$(4.2) \quad f_h(x) = \int f_h(x + \rho \xi) d\omega_\xi$$

holds, so that by KOEBE's Theorem the function $f_h(x)$ is harmonic in D_{2h} .

Now if we take any compact set $K \subset D_{2h}$,

$$\begin{aligned} \int_K |f_h(x) - f(x)| dv_x &= \frac{1}{(2h)^n} \int_K \left| \int_{-h}^h \cdots \int_{-h}^h [f(x+t) - f(x)] dv_t \right| dv_x \\ &\leq \frac{1}{(2h)^n} \int_{-h}^h \cdots \int_{-h}^h \psi(t_1, \dots, t_n) dv_t, \end{aligned}$$

where

$$\psi(t_1, \dots, t_n) \equiv \int_K |f(x+t) - f(x)| dv_x,$$

from which we conclude, as in § 2, that the sequence $\{f_{h_s}(x)\}$ converges to $f(x)$ in the sense of L_K^1 -norm.

Let us consider now the distribution

$$(4.3) \quad f_h(\varphi) = \int f_h(x) \varphi(x) dv_x.$$

It is a solution of the LAPLACE's equation (1.3) in the sense of the SCHWARTZ's theory of distribution, because

$$(4.4) \quad \begin{aligned} \Delta f_h(\varphi) &= \int_{v_{2h}} f_h(x) \Delta \varphi(x) dv_x \\ &= \int_{v_{2h}} \Delta f_h(x) \varphi(x) dv_x = 0. \end{aligned}$$

Now let $\{h_s\}$ be any sequence of positive numbers that converges to 0 as $s \rightarrow \infty$. Then the sequence of distributions $\{f_{h_s}\}$ converges to the distribution f in $(\mathfrak{D}')$. On the other hand by a fundamental theorem of the SCHWARTZ's theory, the derivation of the distribution is a linear continuous operation, so that the sequence of distributions $\{\Delta f_{h_s}\}$ converges to the distribution Δf in $(\mathfrak{D}')$. Therefore we have

$$(4.5) \quad \Delta f(\varphi) = \int_D f(x) \Delta \varphi(x) dv_x = 0,$$

which shows that the function $f(x)$ is a generalized solution of the equation (1.3).

Note. According to the BOCHNER-SCHWARTZ's theory of elliptic partial differential equation, a generalized solution of (1.3) is automatically a strict solution, so that under the same hypothesis about the function $f(x)$ we can in fact conclude that it is harmonic in D .

References

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