ON ASYMPTOTIC SOLUTIONS
OF NON-LINEAR DIFFERENCE EQUATIONS
OF THE FIRST ORDER. I.

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§ 1. Introduction.

Studies on difference equations have been done since the 19th century and those celebrated theories were developed by N. E. NÖLRUND, G. D. BIRKHOFF and W. J. TRJITZINSKY, but they treated only linear difference equations. Theories on non-linear difference equations are indeed very few as far as we know.

In order to develop the theory on the difference equation we should begin with the studies on local problems, that is, the studies on the behavior of solutions in the neighbourhood of the singularity as they have been done in the theory of differential equations. Point at infinity is the most important singularity among non-movable singularities for applications and it has quite different properties compared with those of the differential equations. On this standpoint we classified systematically the difference equations. We found a formal solution in the point at infinity and proved the existence of a solution, the asymptotic expansion of which is nothing but the formal solution. This solution is a particular solution. We shall prove analogous theorems on the general solution in other papers.

To develop our theory we established some fundamental theorems, e. g. the existence theorem of the solution and the theorem on its uniqueness. To prove the existence theorem we applied the Fixed-Point Theorem in the functional space, the idea of which we owed to Professor M. HUKUHARA of Tokyo University when he had been the professor of Kyushu University, under whose guidance our studies were begun.

In the preparation of this paper we are much obliged to Professor W. SIBAGAKI who gave us kindly many important criticisms and improvements without whose encouragement this paper would not have been accomplished.

Here we express our hearty thanks to Professor HUKUHARA and Professor SIBAGAKI.

§ 2. Difference equations.

Let x be a real or complex variable and y be a real or complex-valued function of x . Let h be a fixed value, real or complex. If $y(x)$ is defined at x and $x + h$,

$$\frac{y(x+h) - y(x)}{h}$$

is called the *difference* of y and designated by $\Delta_h y(x)$. Without loss of generality we may take $h = 1$ and write simply $\Delta y(x)$ for $\Delta_1 y(x)$. This is the difference of the first order. Higher differences are then defined successively by the relations

$$\Delta^n y(x) = \Delta(\Delta^{n-1} y(x)).$$

A *difference equation* is an equation which involves the difference of the unknown function. Particularly, if the equation involves only the difference of the first order of the unknown function, then it is called the difference equation of the first order, and is written generally by

$$(2.1) \quad F(x, y(x), \Delta y(x)) = 0,$$

which, when we use the relation $\Delta y(x) = y(x+1) - y(x)$, becomes

$$(2.2) \quad G(x, y(x), y(x+1)) = 0.$$

In this paper we shall consider the form (2.1) for the classification of difference equations. When the equation is solved for $\Delta y(x)$, we have

$$\Delta y(x) = f(x, y(x))$$

which is called the *canonical form*. In the following, we shall assume that the equation is given in the canonical form.

§ 3. Fundamental theorems used in the following discussions.

I. Fixed-Point Theorem.

Let E be a point set in the complex plane. Let $\mathfrak{F}$ be a family of sets of m analytic functions defined in E , the element of which we denote simply by $f(x) \equiv \{f_1(x), f_2(x), \dots, f_m(x)\}$.

When both $f(x)$ and $g(x)$ belong to $\mathfrak{F}$, we assume that for any real number λ ($0 < \lambda < 1$) the function $\lambda f(x) + (1 - \lambda)g(x)$ also belongs to $\mathfrak{F}$.

If the sequence $\{f_k(x)\}$ consists of functions that belong to $\mathfrak{F}$ and converges uniformly in the general sense to $f(x)$ on E , then we assume that $f(x)$ also belongs to $\mathfrak{F}$.

Let T be a mapping which maps each function $f(x)$ which belongs to $\mathfrak{F}$ to a function $\bar{f}(x)$ which also belongs to $\mathfrak{F}$.

If the sequence $\{f_k(x)\}$ of functions which belong to $\mathfrak{F}$ converges uniformly in the general sense on E , we assume that the sequence $\{\bar{f}_k(x)\}$ of the mapped functions also converges uniformly in the general sense and the limit function $\bar{f}(x)$ is the function that corresponds to $f(x)$ by T .

Lastly we assume that the totality of functions that correspond to the functions of $\mathfrak{F}$ by T forms a normal family of functions on E .

Then there exists a function $f(x)$ such that $f(x) = \bar{f}(x)$, that is, a function $f(x)$ which remains fixed under the mapping T .

II. Comparison theorem.

We write $x = \sigma + i\tau$, $i = \sqrt{-1}$.

Let $F(\sigma)$ be defined and > 0 in the interval

$$(3.1) \quad a \leq \sigma < \infty$$

and $Y(\sigma)$ be a solution of the linear difference equation

$$(3.2) \quad Y(\sigma + 1) = F(\sigma) Y(\sigma).$$

Let $f(x, y)$ be a function of x and y defined in the region

$$(3.3) \quad a \leq \sigma < \infty, \quad |y| \leq b,$$

such as

$$|f(x, y)| \geq F(\sigma) |y|$$

in (3.3). If $\varphi(x)$ is a solution of the difference equation

$$(3.4) \quad y(x + 1) = f(x, y(x))$$

such that $|\varphi(x)| \leq b$ holds in (3.1) and the inequality

$$(3.5) \quad Y(\sigma) \leq |\varphi(x)|$$

holds in $\sigma = \sigma_0$, then this inequality also holds in $\sigma = \sigma_0 + m$, ($m = 1, 2, \dots$).

Proof. We shall prove this by induction.

Assume that the inequality holds as far as an integer m , where $m \geq 0$, so that

$$(3.6) \quad |\varphi(\sigma_0 + m + i\tau)| \geq Y(\sigma_0 + m).$$

Then for $\varphi(\sigma_0 + m + 1 + i\tau)$ we shall have, when we put $t = \sigma_0 + m$ and $x = t + i\tau$, for simplicity,

$$\begin{aligned} |\varphi(\sigma_0 + m + 1 + i\tau)| &= |\varphi(x + 1)| = |f(x, \varphi(x))| \\ &\geq F(t) |\varphi(x)| \geq F(t) Y(t) \\ &= Y(t + 1) = Y(\sigma_0 + m + 1). \end{aligned}$$

This shows that (3.6) holds also for $m + 1$. Q.E.D.

III. Uniqueness Theorem.

Let $F(\sigma)$ be defined and > 0 in the interval

$$(3.7) \quad a \leq \sigma < \infty$$

and such that every solution of the linear difference equation

$$(3.8) \quad Y(\sigma + 1) = F(\sigma) Y(\sigma)$$

which is initially positive tends to $+\infty$ as $\sigma \rightarrow +\infty$.

Let M be a positive quantity and assume that the function $g(x, z)$ be defined in the region

$$(3.9) \quad a \leq \sigma < \infty, \quad |z| \leq 3M$$

and satisfy the inequality

$$(3.10) \quad |g(x, z_1) - g(x, z_2)| \geq F(\sigma) |z_1 - z_2|$$

when $a \leq \sigma < \infty$, $|z_1| \leq M$, $|z_2| \leq M$.

Then the solution of the difference equation

$$(3.11) \quad z(x + 1) = g(x, z)$$

such as $|z| \leq M$ in the half plane

$$(3.12) \quad a \leq \sigma < \infty$$

is unique, if it exists.

Proof. If there exists two solutions $\varphi_1(x)$, $\varphi_2(x)$ of (3.11) such that $|\varphi_1(x)|$, $|\varphi_2(x)| \leq M$ in the half plane $a \leq \sigma < \infty$, their difference

$$y = \varphi_2(x) - \varphi_1(x)$$

satisfies the difference equation

$$(3.13) \quad y(x + 1) = f(x, y)$$

in (3.7), where

$$(3.14) \quad f(x, y) = g(x, y + \varphi_1(x)) - g(x, \varphi_1(x))$$

and this is defined in the region

$$(3.15) \quad a \leq \sigma < \infty, \quad |y| \leq 2M.$$

Using (3.10) and (3.14), the inequality

$$f(x, y) = |g(x, y + \varphi_1(x)) - g(x, \varphi_1(x))| \geq F(\sigma)|y|$$

holds in (3.15).

If there exists a solution $y_0(x)$ of (3.11) which is not identically zero in (3.12), then there must exist x_0 such as $2M \geq |y_0(x_0)| > 0$ in (3.12). Let $Y_0(\sigma)$ be a solution of (3.8) which has the initial value $Y_0(\sigma_0) = |y(x_0)|$ where $x_0 = \sigma_0 + i\tau_0$. Then the two difference equations (3.8), (3.13) and their solutions $Y_0(\sigma)$, $y_0(x)$ satisfy all the conditions of the Comparison Theorem. Therefore the inequality

$$(3.16) \quad Y_0(\sigma) \leq |y_0(x)|$$

holds for $\sigma = \sigma_0 + m$, ($m = 1, 2, \dots$). On the other hand, since

$$|y_0(x)| \leq 2M$$

in (3.12), using (3.16), we have

$$Y_0(\sigma) \leq 2M$$

in (3.12). This contradicts the property of $Y_0(\sigma)$ tending to positive infinity as $\sigma \rightarrow +\infty$. Therefore (3.13) has only a solution which is identically zero in (3.12). This means that the two solutions of (3.11) are identical in (3.12).
Q. E. D.

§ 4. Classification of the difference equations.

We consider a single difference equation

$$(4.1) \quad \Delta y(x) = f(x, y(x))$$

and assume that $f(x, y)$ is an analytic function of x and y in the region $|x| \geq R$, $|y| \leq r$. We suppose that there exists a value g_0 such as $|g_0| < r$ that satisfies the relation

$$(4.2) \quad f(\infty, g_0) = 0,$$

which we call the *indicial equation* of (4.1).

When we write $f_y^{(m)}(a, b)$ for

$$\left[\frac{\partial^m f}{\partial y^m} \right]_{x=a, y=b},$$

the formal solution of (4.1) is, as we shall show in the following, different according to the following respective cases:

$$\text{Case (1)} \quad f_y'(\infty, g_0) \neq 0,$$

$$\text{Case (2)} \quad f_y^{(k)}(\infty, g_0) = 0, \quad (k = 1, 2, \dots, n-1),$$

$$f_y^{(n)}(\infty, g_0) \neq 0.$$

$$\text{Case (3)} \quad f_y^{(k)}(\infty, g_0) = 0, \quad (k = 1, 2, \dots).$$

In this paper we shall treat only the Case (1).

§ 5. Formal solution in the Case (1).

By hypothesis, in the difference equation

$$(5.1) \quad \Delta y = f(x, y),$$

the function $f(x, y)$ is analytic for x and y in the region $|x| \geq R, |y| \leq r$. Let the expansion be

$$(5.2) \quad f(x, y) = \sum_{j=0}^{\infty} a_j(x) y^j, \\ a_j(x) = a_{j0} + a_{j1} x^{-1} + a_{j2} x^{-2} + \dots, \quad (j = 0, 1, \dots).$$

g_0 is a value such as $|g_0| < r$ and such that the relations

$$(5.3) \quad f(\infty, g_0) = 0, \\ f_y'(\infty, g_0) \neq 0$$

hold.

Now to obtain a formal solution of (5.1) we put

$$(5.4) \quad y = g_0 + z$$

and get, as the difference equation for z ,

$$(5.5) \quad \Delta z = g(x, z),$$

where

$$g(x, z) \equiv f(x, g_0 + z).$$

$g(x, z)$ is an analytic function in $|x| \geq R, |z| \leq \rho (= r - |g_0|)$.

Let

$$(5.6) \quad g(x, z) = \sum_{j=0}^{\infty} b_j(x) z^j, \\ b_j(x) = b_{j0} + b_{j1} x^{-1} + b_{j2} x^{-2} + \dots.$$

Then clearly we have

$$(5.7) \quad b_j(x) = f_y^{(j)}(x, g_0), \quad (j = 0, 1, 2, \dots)$$

for $|x| \geq R$ and the relations

$$(5.8) \quad \begin{aligned} b_0(\infty) &= f(\infty, g_0) = b_{00} = 0, \\ b_1(\infty) &= f_y'(\infty, g_0) = b_{10} \neq 0 \end{aligned}$$

hold.

Put formally

$$(5.9) \quad z = g_1 x^{-1} + g_2 x^{-2} + \dots + g_n x^{-n} + \dots$$

in the equation (5.5) or

$$(5.10) \quad g(x, z) - \Delta z = 0,$$

then we must have

$$\begin{aligned} & b_{01} x^{-1} + b_{02} x^{-2} \dots \\ & + (b_{10} + b_{11} x^{-1} + b_{12} x^{-2} + \dots)(g_1 x^{-1} + g_2 x^{-2} + \dots + g_n x^{-n} + \dots) \\ & + (b_{20} + b_{21} x^{-1} + b_{22} x^{-2} + \dots)(g_1 x^{-1} + g_2 x^{-2} + \dots + g_n x^{-n} + \dots)^2 \\ & \dots \dots \dots \\ & + (b_{n0} + b_{n1} x^{-1} + b_{n2} x^{-2} + \dots)(g_1 x^{-1} + g_2 x^{-2} + \dots + g_n x^{-n} + \dots)^n \\ & \dots \dots \dots \\ & - g_1 x^{-1} [(1 + x^{-1})^{-1} - 1] - g_2 x^{-2} [(1 + x^{-1})^{-2} - 1] - \dots \\ & - g_n x^{-n} [(1 + x^{-1})^{-n} - 1] \dots = 0. \end{aligned}$$

Expanding and rearranging according to the negative powers of x we find readily that the coefficient of x^{-n} is of the form

$$b_{10} g_n + P_n(g_1, g_2, \dots, g_{n-1}; b_{ij}),$$

where P_n is a polynomial of $(g_1, g_2, \dots, g_{n-1})$ and depends only on those b_{ij} such as $i + j \leq n$.

Now z can be a formal solution only if the relations

$$\begin{aligned} b_{10} g_1 + b_{01} &= 0, \\ b_{10} g_2 + b_{02} + b_{11} g_1 + b_{20} g_1^2 &= 0, \\ &\dots \dots \dots, \end{aligned}$$

in general

$$(5.11) \quad b_{10} g_n + P_n(g_1, g_2, \dots, g_{n-1}, b_{ij}) = 0, \quad (n = 1, 2, 3, \dots)$$

are satisfied. By these relations the coefficients $g_1, g_2, \dots$ are successively

determined and thus we get a formal solution (5.9) for the equation (5.5). Thus for the starting equation (5.1) we get a formal solution

$$(5.12) \quad y = g_0 + g_1 x^{-1} + g_2 x^{-2} + \dots.$$

§ 6. Transformation and the further classification.

Let us put

$$(6.1) \quad \begin{aligned} y_n &= g_0 + g_1 x^{-1} + \dots + g_n x^{-n}, \\ z_n &= y - y_n, \end{aligned}$$

and make the equation with respect to z_n by substituting this in (4.1), then we get

$$(6.2) \quad \Delta z_n = f(x, z_n + y_n) - \Delta y_n.$$

Let $\varphi_n(x, z_n)$ and $c_n(x)$ be the functions defined by the relations

$$(6.3) \quad \begin{aligned} \varphi_n(x, z_n) &= f(x, z_n + y_n) - f(x, y_n) \\ c_n(x) &= f(x, y_n) - \Delta y_n, \end{aligned}$$

then (6.2) becomes

$$(6.4) \quad \Delta z_n = \varphi_n(x, z_n) + c_n(x).$$

Replacing respectively z_n , $\varphi_n(x, z_n)$, $c_n(x)$ by z , $\varphi(x, z)$, $c(x)$ in (6.4), for simplicity, we have

$$(6.5) \quad \Delta z = \varphi(x, z) + c(x).$$

Setting $2r_1 = r - |g_0|$, we may find a positive quantity $R_1 > 1$ so large that the inequality

$$|y_n - g_0| \leq r_1$$

holds for

$$(6.6) \quad |x| \geq R_1.$$

Hence

$$|y_n| \leq |g_0| + r_1 < r$$

in (6.6), and the inequality

$$|y_n + z| \leq |g_0| + 2r_1 = r$$

holds for x and z such that

$$(6.7) \quad |x| \geq R_1, \quad |z| \leq r_1.$$

Therefore $f(x, y_n + z)$ is defined in (6.7) and is analytic in (6.7). Hence

$$\varphi(x, z) = f(x, y_n + z) - f(x, y_n)$$

is analytic in (6.7), and the relations

$$\varphi(x, 0) = 0, \varphi_z^{(m)}(x, 0) = f_y^{(m)}(x, y_n), \quad (m = 1, 2, \dots)$$

hold in (6.6). Therefore $\varphi(x, z)$ may be written as

$$\varphi(x, z) = \varphi(x, 0) + \varphi'_x(x, 0)z + \psi(x, z)z^2,$$

that is,

$$(6.8) \quad \varphi(x, z) = f'_y(x, y_n)z + \psi(x, z)z^2$$

where $\psi(x, z)$ is analytic in (6.7).

Since $f(x, y_n)$ is analytic in (6.6), $c(x)$ is also analytic in $|x| \geq R_1$. We can also easily understand, by simple calculation, that the coefficients of x^{-k} ($k = 0, 1, 2, \dots, n$) of $c(x)$ are equal to those of (5.10), which are all zero. Therefore $x^{n+1}c(x)$ is analytic in $|x| \geq R_1$.

Collecting these results, we have:

the equation (6.5) for the unknown $z = z_n$ has the form:

$$(6.9) \quad \Delta z = f'_y(x, y_n)z + \psi(x, z)z^2 + c(x),$$

where $f'_y(x, y_n)$ and $x^{n+1}c(x)$ are analytic in (6.6) and $\psi(x, z)$ is analytic in (6.7).

Furthermore we shall prove that (6.9) has an analytic solution z such that

$$(6.10) \quad |z| \leq M|x|^{-n}$$

in a certain domain. We will determine the domain of existence in the following cases respectively:

$$\text{Case (1.1)} \quad |1 + f'_y(\infty, g_0)| > 1,$$

$$\text{Case (1.2)} \quad |1 + f'_y(\infty, g_0)| = 1,$$

$$\text{Case (1.3)} \quad |1 + f'_y(\infty, g_0)| < 1.$$

§ 7. Existence theorem in the Case (1.1).

We begin with the Case (1.1), so that we assume that

$$(7.1) \quad |1 + f'_y(\infty, g_0)| > 1.$$

Let us put

$$(7.2) \quad z = u x^{-n}$$

and substitute this in (6.9), then we get a difference equation with respect to u , which is

$$(7.3) \quad u(x) = \frac{(1+x^{-1})^{-n}u(x+1) - x^n c(x) - u^2 x^{-n} \phi(x, u x^{-n})}{1 + f'_y(x, y_n)},$$

where $x^{n+1}c(x)$ and $f'_y(x, y_n)$ are analytic in (6.6), and $\phi(x, z)$ is analytic in (6.7). We shall prove that the equation has a bounded and analytic solution in a certain domain I' defined as

$$(7.4) \quad I': \begin{cases} |x| \geq \rho \\ \text{for } \sigma \geq 0, \\ |\tau| \geq \rho \\ \text{for } \sigma \leq 0, \end{cases}$$

where ρ is a sufficiently large quantity that will be determined in the course of the proof. We shall apply the Fixed-Point Theorem to prove this.

Let M be an arbitrary but fixed positive quantity and let the family of functions $u(x)$ that are analytic and

such that $|u(x)| \leq M$ for x in the domain I' be $\mathfrak{F}$. $\mathfrak{F}$ is a convex set, because if $f(x)$ and $g(x)$ belong to $\mathfrak{F}$, for any number λ such as $0 < \lambda < 1$, the inequality

$$|\lambda f(x) + (1-\lambda)g(x)| \leq \lambda |f(x)| + (1-\lambda)|g(x)| \leq M$$

holds in I' . $\mathfrak{F}$ is a closed set, because if $\{u_k(x)\}$ is a sequence of functions, each belonging to $\mathfrak{F}$, and which converges uniformly in the generalized sense, then the limiting function $u(x)$ is analytic and satisfies

$$|u(x)| \leq M$$

in I' , which means that $u(x)$ belongs to $\mathfrak{F}$.

Now consider the mapping T defined as

$$\bar{u}(x) = T(u(x)) = \frac{(1+x^{-1})^{-n}u(x+1) - x^n c(x) - u^2 x^{-n} \phi(x, u x^{-n})}{1 + f'_y(x, y_n)}.$$

Fig. 1.

In this the function $\phi(x, ux^{-n})$ is an analytic function of x and u such as $|x| \geq R_1$, $|ux^{-n}| \leq r_1$, hence is analytic in $|x| > R_1$, $|u| \leq r_1 R_1^n$. We assume that the quantity R_1 was already chosen sufficiently large such that $M \leq r R_1^n$ holds and assume also that the quantity ρ in the definition of the domain I' was chosen such as $\rho \geq R_1$.

We choose two positive numbers ε and α such that

$$|1 + f'_y(\infty, g_0)| < 1 - \alpha, \quad \text{and} \quad 2\varepsilon = \alpha M.$$

Now we assume that the quantity $\rho (\geq R_1)$ was chosen so large that the three inequalities

$$\left| \frac{(1 + x^{-1})^{-n}}{1 + f'_y(x, y_n)} \right| \leq 1 - \alpha, \quad \left| \frac{x^n c(x)}{1 + f'_y(x, y_n)} \right| \leq \varepsilon, \quad \left| \frac{u^2 x^{-n} \phi(x, ux^{-n})}{1 + f'_y(x, y_n)} \right| \leq \varepsilon$$

hold simultaneously for $|x| \geq \rho$. Since the point $x + 1$ belongs to I' if x belongs to I' , we have

$$\begin{aligned} |\bar{u}(x)| &\leq \left| \frac{(1 + x^{-1})^{-n} u(x + 1)}{1 + f'_y(x, y_n)} \right| + \left| \frac{x^n c(x)}{1 + f'_y(x, y_n)} \right| + \left| \frac{u^2 x^{-n} \phi(x, ux^{-n})}{1 + f'_y(x, y_n)} \right| \\ &\leq (1 - \alpha)M + 2\varepsilon = M. \end{aligned}$$

Hence the mapped function $\bar{u}(x)$ which is clearly analytic in I' belongs also to $\mathfrak{F}$.

If $\{u_k(x)\}$ is a sequence of functions that belong to $\mathfrak{F}$ and converges uniformly in the generalized sense, then the sequence $\{\bar{u}_k(x)\}$ of the mapped functions clearly converges uniformly to a function $\bar{u}(x)$ on I' which is the image of the limit function $u(x)$.

Lastly we show that the family of the mapped functions forms a normal family in I' . This is clear from the fact that $\bar{u}_k(x)$ are analytic and equibounded by $|\bar{u}_k(x)| \leq M$ in I' .

Hence all the necessary assumptions of the Fixed-Point Theorem were shown to be satisfied in our case, so that we have the following existence theorem.

Theorem 1. *Let $f(x, y)$ be analytic in the region*

$$|x| \geq R, \quad |y| \leq r$$

and consider the difference equation

$$(7.5) \quad \Delta y(x) = f(x, y).$$

Let a value g_0 exist such as $|g_0| < r$ and

$$f(\infty, g_0) = 0,$$

$$|1 + f'_y(\infty, g_0)| > 1.$$

Then the equation has the the following formal solution

$$g_0 + g_1 x^{-1} + g_2 x^{-2} + \cdots + g_n x^{-n} + \cdots.$$

We consider the solution of the form

$$(S) \quad y = g_0 + g_1 x^{-1} + g_2 x^{-2} + \cdots + g_n x^{-n} + u_n x^{-n}.$$

Then to an arbitrary but fixed positive quantity M there exists a solution (S) such that the inequality

$$|u_n| \leq M$$

holds in a certain region I_n which is defined for a sufficiently large positive quantity $\rho \geq R$ as

$$\begin{cases} |x| \geq \rho & \text{for } \sigma \geq 0, \\ |\tau| \geq \rho & \text{for } \sigma \leq 0, \end{cases}$$

where we put $x = \sigma + i\tau$.

§ 8. Uniqueness theorem in Case (1.1).

We shall prove that such a solution of the equation (7.5) as having the properties of Theorem 1 is unique by means of the Uniqueness Theorem of § 3.

We may choose ρ such that

$$(8.1) \quad |1 + f'_y(\infty, g_0)| > \rho > 1$$

by the assumption (7.1) and then take the difference equation

$$(8.2) \quad Y(\sigma + 1) = \rho Y(\sigma)$$

for (3.8). The general solution of this is

$$Y(\sigma) = \pi(\sigma) \rho^\sigma,$$

where $\pi(\sigma)$ is any periodic function with period 1. Therefore any solution of (8.2) which has the initial condition such that $Y(\sigma_0) > 0$ must not be bounded as $\sigma \rightarrow +\infty$. Hence (8.2) satisfies all the conditions for the equation (3.8).

In the Uniqueness Theorem of § 3, set

$$g(x, u) \equiv (1 + x^{-1})^n \{ (1 + f'_y(x, y_n)) u + u^2 x^{-n} \phi(x, u x^{-n}) + x^n c(x) \}.$$

Then we may choose R_2 sufficiently large so that $g(x, u)$ is analytic for x and u such that $|x| \geq R_2$, $|u| \leq 3M$. Now, for $|u_1| \leq M$, $|u_2| \leq M$,

$$\begin{aligned} & g(x, u_2) - g(x, u_1) \\ &= g(x, v + u_1) - g(x, u_1), \quad (\text{write } v = u_2 - u_1) \\ &= (1 + x^{-1})^n \{ (1 + f'_y(x, y_n)) v + x^{-n} \phi(x, v) \}, \end{aligned}$$

where

$$\phi(x, v) = \{(v + u_1)^2 \phi(x, (v + u_1)x^{-n}) - u_1^2 \phi(x, u_1 x^{-n})\}(1 + x^{-1})^n,$$

which is analytic and bounded in $|x| \geq R_2$, $|v| \leq 2M$. Since $\phi(x_1 0) = 0$, we can write

$$(8.3) \quad g(x, u_2) - g(x, u_1) = \{(1 + f'_y(x, y_n))(1 + x^{-1})^n + x^{-n} \xi(x, v)\}v,$$

where $\xi(x, v)$ is analytic in $|x| \geq R_2$, $|v| \leq 2M$. We may choose $R_3 (\geq R_2)$ such that

$$(8.4) \quad |(1 + f'_y(x, y_n))(1 + x^{-1})^n + x^{-n} \xi(x, v)| > \rho$$

in $|x| \geq R_3$, $|v| \leq 2M$. By using (8.3) and (8.4), the inequality

$$|g(x, u_2) - g(x, u_1)| \geq \rho |v|,$$

that is,

$$|g(x, u_2) - g(x, u_1)| \geq \rho |u_2 - u_1|$$

holds in $|x| \geq R_3$, $|u_1| \leq M$, $|u_2| \leq M$.

Therefore $g(x, u)$ satisfies all the conditions of the Uniqueness Theorem. Hence the solution of (7.3) such as $|u| \leq M$ is unique in the half plane $R_3 \leq \sigma$. Since the solution is analytic, such a solution is unique in I' . Thus we get the following theorem.

Theorem 2. *Under the conditions of Theorem 1 such a solution as stated in that Theorem is unique.*

§ 9. Existence of an asymptotic solution.

We have proved that for the equation (7.5) there exists an analytic solution of the form

$$y = g_0 + g_1 x^{-1} + \dots + g_n x^{-n} + u_n x^{-n}$$

such as $|u_n| \leq M$ in I'_n and that such a solution is unique in I'_n . But the solution might depend on n , so that temporarily let us denote it by $y(n)$. Then

$$y(n) = g_0 + g_1 x^{-1} + \dots + g_n x^{-n} + u_n x^{-n},$$

where $|u_n| \leq M$ in I'_n .

Now this can be rewritten as

$$y(n) = g_0 + g_1 x^{-1} + \dots + g_{n-1} x^{-n+1} + v_{n-1} x^{-n+1},$$

where

$$v_{n-1} \equiv (g_n + u_n) x^{-1},$$

and since $|u_n| \leq M_n$ in I_n , it follows that

$$|v_{n-1}| \leq M'_{n-1} \quad \text{in } I_n,$$

where M'_{n-1} is some positive quantity. On the other hand according to the Uniqueness Theorem the solution such as

$$y(n-1) = g_0 + g_1 x^{-1} + g_2 x^{-2} + \cdots + g_{n-1} x^{-n+1} + u_{n-1} x^{-n+1},$$

where $|u_{n-1}| \leq M_{n-1}$ in I_{n-1} , is unique, so that it must hold

$$y(n) = y(n-1)$$

and

$$u_{n-1} = (g_n + u_n) x^{-1}, \quad \text{in } I_n \cap I_{n-1}.$$

Therefore we can conclude that there exists an analytic solution which does not depend on n in some domain of the form previously denoted by I' , having the property that it holds

$$y(x) = g_0 + g_1 x^{-1} + \cdots + g_n x^{-n} + u_n x^{-n}$$

and

$$\lim_{x \rightarrow \infty} |u_n| = 0, \quad x \in I',$$

i. e.

$$\lim_{x \rightarrow \infty} x_n [y(x) - (g_0 + g_1 x^{-1} + \cdots + g_n x^{-n})] = 0, \quad x \in I',$$

which means that we have asymptotically

$$y(x) \sim g_0 + g_1 x^{-1} + \cdots + g_n x^{-n} + \cdots$$

in the domain I' . In other word we can conclude the following theorem.

Theorem 3. *Under the conditions of Theorem 1 there exists a solution of (7.5) which has the asymptotic expansion*

$$y \sim g_0 + g_1 x^{-1} + \cdots + g_n x^{-n} + \cdots$$

valid in the angular region $|\arg x| \leq \pi - \varepsilon$.

§ 10. Existence theorem for the Case (1.2).

In this case, we consider, instead of the equation (6.4), the difference equation

$$(10.1) \quad \Delta z_{n+1} = \varphi_{n+1}(x, z_{n+1}) + c_{n+1}(x)$$

with respect to z_{n+1} and notice that $x^{n+2}c_{n+1}(x)$ is analytic in $|x| < R_1$. We write, for simplicity,

$$(12.2) \quad \Delta z = \varphi(x, z) + c(x)$$

for (12.1). Taking a positive quantity M we shall prove that there exists an analytic solution of (12.2) such as $|z| \leq M|x|^{-n}$ in a certain domain. The domains of existence shall be determined according to the following cases.

Case (1.2.1) *Existence domain in the right half plane.*

Case (1.2.2) *Existence domain in the left half plane.*

We shall begin with Case (1.2.1).

Let us put

$$z = u x^{-n}$$

and substitute this in (12.2), then we get a difference equation with respect to u , which is

$$(12.3) \quad u(x) = \frac{(1+x^{-1})^{-n} u(x+1) - x^n c(x) - u^2 x^{-n} \phi(x, u x^{-n})}{1 + f'_y(x, y_{n+1})},$$

where $x^{n+2}c(x)$ and $f'_y(x, y_{n+1})$ are analytic in $|x| \geq R_1$, and $\phi(x, z)$ is analytic in $|x| \geq R_1, |z| \leq r_1$. We shall firstly prove that the equation (12.3) has a bounded and analytic solution in a certain domain I^* defined as

$$(12.4) \quad \begin{cases} I^*: |x| > \rho, \\ |\arg x| \leq \vartheta \\ \quad \equiv \pi/2 - \varepsilon, \end{cases}$$

where ε is an arbitrary but fixed quantity such as $0 < \varepsilon < \pi/2$ and ρ is a sufficiently large quantity that will be determined in the course of the proof.

We shall apply the Fixed-Point Theorem to prove this.

fig. 2.

Let M be an arbitrary but fixed positive quantity and let $\mathfrak{F}$ be the family of functions $u(x)$ that are analytic and such as $|u(x)| \leq M$ for x in the domain I^* . Now consider the mapping T^* defined as

$$(12.5) \quad \bar{u}(x) = T^*(u(x)) \\ = \frac{(1+x^{-1})^{-n} u(x+1) - x^n c(x) - u^2 x^{-n} \psi(x, u x^{-n})}{1 + f'_y(x, y_{n+1})}.$$

The most important fact to be proved is that if $u(x)$ belongs to $\mathfrak{F}$ then $\bar{u}(x)$ also belongs to $\mathfrak{F}$. We shall prove this only, since the rests may be proved similarly as in the Case (1.1). In (12.5) the function $\psi(x, u x^{-n})$ is an analytic function of x and u in the region $|x| \geq R_1$, $|u x^{-n}| \leq r$, hence analytic in $|x| \geq R_1$, $|u| \leq r_1 R_1^n$. We assume that the quantity R_1 was already chosen such as $\rho \geq R_1$ in the definition of the domain I^* .

Since $f'_y(x, y_{n+1})$ is analytic in $|x| \geq R_1$ and $|1 + f'_y(\infty, g_0)| = 1$, we have

$$1 + f'_y(x, y_{n+1}) = e^{i\psi} (1 + c_1 x^{-1} + \dots),$$

where c_1 is independent on n , and indicates the terms of the second and higher degrees of x^{-1} . We therefore get

$$(1 + f'_y(x, y_{n+1})) (1 + x^{-1})^n = e^{i\psi} (1 + (c_1 + n) x^{-1} + \dots).$$

Let us put

$$x = r e^{i\theta}, \quad c_1 = c e^{i\varphi},$$

where $r > 0$, $c \geq 0$. It follows then that

$$(1 + f'_y(x, y_{n+1}))^{-1} (1 + x^{-1})^{-n} = e^{-i\psi} (1 - (c e^{i\varphi} + n) r^{-1} e^{-i\theta} + \dots) \\ = e^{-i\psi} (1 - c r^{-1} e^{i(\varphi-\theta)} - n r^{-1} e^{-i\theta} + \dots).$$

Taking the absolute value, we have

$$(12.6) \quad |(1 + f'_y(x, y_{n+1}))^{-1} (1 + x^{-1})^{-n}| \\ = 1 - c r^{-1} \cos(\varphi - \theta) - n r^{-1} \cos \theta + \dots \\ = 1 - (c \cos(\varphi - \theta) + n \cos \theta) r^{-1} + \dots,$$

where shows the terms of the second and higher degrees of r^{-1} . This value becomes smaller than 1 for sufficiently large r , if n satisfies the inequality

$$(12.7) \quad c \cos(\varphi - \theta) + n \cos \theta > 0,$$

i. e., since $\cos \theta > 0$, if

$$n > -c \cos(\varphi - \theta) \sec \theta.$$

Since $|\arg x| = |\theta| \leq \vartheta$ for x in I' , the inequality (12.7) will be surely satisfied if

$$(12.8) \quad n > c \sec \delta, \quad (\text{where } \vartheta = (\pi/2) - \epsilon).$$

We choose two positive numbers δ and α such that

$$c \cos(\varphi - \theta) + n \cos \theta > \alpha, \quad M\alpha = 2\delta.$$

when n satisfies (12.8). Now we assume that the quantity $\rho (\geq R_1)$ was chosen so large that the three inequalities

$$\begin{aligned} \left| \frac{(1+x^{-1})^{-n}}{1+f'_y(x, y_{n+1})} \right| &\leq 1 - \alpha/r, \\ \left| \frac{x^n c(x)}{1+f'_y(x, y_{n+1})} \right| &\leq \delta/r, \\ \left| \frac{u^2 x^{-n} \phi(x, u x^{-n})}{1+f'_y(x, y_{n+1})} \right| &\leq \delta/r, \end{aligned}$$

hold simultaneously for x in the region

$$|x| \geq \rho, \quad |\arg x| \leq \vartheta.$$

Since $x+1$ belongs to I^* if x belongs to I^* we have

$$\begin{aligned} |\bar{u}(x)| &\leq \left| \frac{(1+x^{-1})^{-n} u(x+1)}{1+f'_y(x, y_{n+1})} \right| + \left| \frac{x^n c(x)}{1+f'_y(x, y_{n+1})} \right| + \left| \frac{u^2 x^{-n} \phi(x, u x^{-n})}{1+f'_y(x, y_{n+1})} \right| \\ &\leq (1 - \alpha/r) M + 2\delta/r \leq M \end{aligned}$$

in I^* . Hence the mapped function $\bar{u}(x)$ which is clearly analytic in I^* belongs also to $\mathfrak{F}$.

We shall denote the solution y of the equation $\Delta y(x) = f(x, y)$ which shall be defined by the above obtained solution $u(x)$ by S^* .

Next we shall prove that the equation (12.3) has another bounded and analytic solution $u(x)$ in a domain I^{**} defined as

$$(12.9) \quad \begin{cases} I^{**}: |x| > \rho, \\ |\arg x - \pi| < \vartheta \equiv \pi/2 - \epsilon, \end{cases}$$

where ρ and ϵ have the same meaning as above. For that purpose we must rewrite the equation (12.3) in the following form:

fig. 3.

$$u(x+1) = (1+x^{-1})^n \{ [1 + f'_y(x, y_{n+1})] u(x) + x^n c(x) + u^2 x^{-n} \phi(x, u x^{-n}) \}$$

or

$$(12.10) \quad u(x) = \left(1 + \frac{1}{x-1}\right)^n \{ [1 + f'_y(x-1, y_{n+1}(x-1))] u(x-1) \\ + (x-1)^n c(x-1) \\ + u^2 (x-1) (x-1)^{-n} \phi(x-1, u(x-1)(x-1)^{-n}) \}.$$

To this equation we can apply the Fixed Point Theorem in almost a similar manner as above, so that we shall omit the details. We shall denote the solution y of the equation $\Delta y(x) = f(x, y)$ which shall be defined by the just obtained solution $u(x)$ by S^{**} .

Theorem 4. *Let $f(x, y)$ be analytic in the region*

$$|x| \geq R, \quad |y| \leq r,$$

and consider the difference equation

$$(12.11) \quad \Delta y(x) = f(x, y).$$

Let a value g_0 exist such that $|g_0| < r$ and satisfy

$$f(\infty, g_0) = 0, \\ |1 + f'_y(\infty, g_0)| = 1.$$

Then the equation has the following formal solution

$$g_0 + g_1 x^{-1} + g_2 x^{-2} + \cdots + g_n x^{-n} + \cdots.$$

Let

$$y_{n+1} = g_0 + g_1 x^{-1} + \cdots + g_{n+1} x^{-(n+1)},$$

and let c_1 be the coefficient of x^{-1} in the expansion of $1 + f'_y(x, y_{n+1})$. Then there exist two solutions S^ and S^{**} , each of which has the form*

$$y = g_0 + g_1 x^{-1} + \cdots + g_{n+1} x^{-(n+1)} + u_{n+1} x^{-n}$$

and defined as follows:

S^* *has the property that to an arbitrary but fixed positive quantity M and an arbitrary small positive quantity $\varepsilon < \pi/2$ the inequality*

$$|u_{n+1}| \leq M$$

holds for those indices n such as $n > |c_1| \sec \vartheta$, where $\vartheta = \pi/2 - \varepsilon$, in a certain region I_{n+1}^ which is defined for a sufficiently large positive quantity $\rho \geq R$ as*

$$|x| \geq \rho, \quad |\arg x| \leq \vartheta = \pi/2 - \varepsilon,$$

whereas S^{**} has the analogous property in the region I_{n+1}^{**} which is defined for a sufficiently large positive quantity $\rho \geq R$ as

$$|x| \geq \rho, \quad |\arg x - \pi| \leq \vartheta = \pi/2 - \varepsilon.$$

§ 11. Uniqueness theorem in the Case (1.2).

We shall prove that the solution of the equation (4.1) that satisfies the conditions of the Theorem 4 is unique by means of the Uniqueness Theorem of § 3.

We consider the difference equation (12.3)

$$(13.1) \quad u(x+1) = g(x, u),$$

where $g(x, u)$ is

$$\{(1 + f_y'(x, y_{n+1})u + x^n c(x) + u^2 x^{-n} \phi(x, u x^{-n}))\}(1 + x^{-1})^n.$$

We may choose R_2 sufficiently large so that $g(x, u)$ is analytic for x and u such that $|x| \geq R_2$, $|u| \leq 3M$. Now, for $|u_1| \leq M$, $|u_2| \leq M$,

$$\begin{aligned} g(x, u_2) - g(x, u_1) &= g(x, v + u_1) - g(x_1, u) \quad (\text{write } v = u_2 - u_1) \\ &= (1 + x^{-1})^n (1 + f_y'(x, y_{n+1}))v + x^{-n} \phi(x, v), \end{aligned}$$

where

$$\phi(x, v) = ((v + u_1)^2 \phi(x, (v + u_1)x^{-n}) - u_1^2 \phi(x, u_1 x^{-n}))(1 + x^{-1})^n,$$

and $\phi(x, v)$ is analytic and bounded in $|x| \geq R_2$, $|v| \leq 2M$. Since $\phi(x, 0) = 0$, we can write

$$(13.2) \quad \begin{aligned} g(x, u_2) - g(x, u_1) &= \{(1 + f_y'(x, y_{n+1}))(1 + x^{-1})^n + x^{-n} \xi(x, v)\}v, \end{aligned}$$

where $\xi(x, v)$ is analytic in $|x| \geq R_2$, $|v| \leq 2M$. The index n being taken as in (12.7) we choose two positive quantities α, β such that the relations

$$\begin{aligned} n \cos \theta + c \cos(\varphi - \theta) &> \alpha > 0, \\ \alpha \cos \vartheta = \beta > 0, \quad (\vartheta = \pi/2 - \varepsilon) \end{aligned}$$

hold. For such α, β and n , we may choose $R_3 (\geq R_2)$ such that the relations

$$(13.3) \quad \begin{aligned} |(1 + f_y'(x, y_{n+1}))(1 + x^{-1})^n + x^{-n} \xi(x, v)| \\ = 1 + c r^{-1} \cos(\varphi - \theta) + n r^{-1} \cos \theta + \dots > 1 + \alpha r^{-1} \geq 1 + \beta \sigma^{-1} \end{aligned}$$

hold for x and v such that $|x| \geq R_3$, $|v| \leq 2M$.

Therefore using (13.2) and (13.3), we get

$$|g(x, u_2) - g(x, u_1)| \geq (1 + \beta/\sigma) |u_2 - u_1|,$$

for x and u such that $|x| \geq R_3$, $|\arg x| \leq \vartheta$ and $|u_1| \leq M$, $|u_2| \leq M$.

Let $F(\sigma) = 1 + \beta/\sigma$, then $F(\sigma)$ is defined and > 0 in the interval $\delta \leq \sigma < \infty$ (δ is any positive quantity). Hence we have the inequality

$$|g(x, u_2) - g(x, u_1)| \geq F(\sigma) |u_2 - u_1|,$$

for x and u such that $|x| \geq R_3$, $|\arg x| \leq \vartheta$, and $|u_1| \leq M$, $|u_2| \leq M$.

We take

$$Y(\sigma + 1) = (1 + \beta/\sigma) Y(\sigma)$$

for (3.8) of the Uniqueness Theorem of § 3. The general solution is

$$Y(\sigma) = \pi(\sigma) \frac{\Gamma(\sigma + \beta)}{\Gamma(\sigma)},$$

where $\pi(\sigma)$ is any periodic function with period 1, and $\Gamma(\sigma)$ is the Gamma function of σ . Therefore any solution of (13.4) which has the initial value $Y(\sigma_0) > 0$ must not be bounded as $\sigma \rightarrow +\infty$. Thus (13.1) and (13.4) satisfy all the conditions of the Uniqueness Theorem of § 3. Hence the solution of (13.1) such as $|u| \leq M$ is unique in $|x| \geq R_3$, $|\arg x| \leq \vartheta$ for n such that $n \geq c \sec \vartheta$. Since the solution is analytic, such a solution is unique in I^* .

In a similar manner we may treat the problem for the solution S^{**} and get the following theorem.

Theorem 5. *Under the conditions of the Theorem 4 such a solution S^* or S^{**} stated in that Theorem is unique respectively.*

§ 12. Existence of an asymptotic solution.

We have proved that for the equation (12.9) there exists an analytic solution of the form

$$y = g_0 + g_1 x^{-1} + \cdots + g_{n+1} x^{-(n+1)} + u_{n+1} x^{-n}$$

such as $|u_{n+1}| \leq M$ in I_{n+1}^* . But the solution might depend on n , so that temporarily let us denote it by $y(n)$. Then

$$y(n) = g_0 + g_1 x^{-1} + \cdots + g_{n+1} x^{-(n+1)} + u_{n+1} x^{-n},$$

where $|u_{n+1}| \leq M$ in I_{n+1}^* .

Now this can be written as

$$y(n) = g_0 + g_1 x^{-1} + \dots + g_{n+1} x^{-(n+1)} + v_n x^{-(n+1)},$$

where

$$v_n = g_{n+1} x^{-1} + u_{n+1} x^{-1},$$

and since $|u_{n+1}| \leq M$ in I'_{n+1} , it follows that

$$|v_n| \leq M'_n \text{ in } I'_{n+1},$$

where M'_n is some positive quantity. On the other hand, according to Theorem 5, the solution such as

$$y(n-1) = g_0 + g_1 x^{-1} + \dots + g_n x^{-n} + u_n x^{-(n+1)},$$

where $|u_n| \leq M_n$ in I'_n , is unique, so that it must hold

$$y(n) = y(n-1) \text{ and } u_n = g_{n+1} x^{-2} + u_{n+1} x^{-1} \text{ in } I'_{n+1} \cap I'_n.$$

Therefore we can conclude that there exists an asymptotic solution which does not depend on n in some domain of the form previously denoted by I'^* , having the property that it holds

$$y(x) = g_0 + g_1 x^{-1} + \dots + g_{n+1} x^{-(n+1)} + u_{n+1} x^{-n}$$

and

$$\lim_{n \rightarrow \infty} |u_{n+1}| = 0, \quad x \in I'^*,$$

i. e.

$$\lim x^n (y - y_n(x)) = 0, \quad x \in I'^*.$$

Similar discussions hold for the solution S^{**} , thus we get the following theorem.

Theorem 6. *Under the conditions of the Theorem 4, there exist two solutions of (12.11), which we denote by S^* and S^{**} , and have the asymptotic expansion*

$$y \sim g_0 + g_1 x^{-1} + \dots + g_n x^{-n} + \dots$$

respectively in the region I'^ and I'^{**} .*