ON THE REGULARITY OF DISTRIBUTIONS DEFINED BY SOME DIFFERENTIAL EQUATIONS

By

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Introduction

KRYLOFF discussed the property of functions whose derivatives as functional belong to some L^p ($p \geq 1$), by dint of SOBOLEFF's lemma. And L. SCHWARTZ extended his theorem to distributions on the same idea. Furthermore, SCHWARTZ pointed out that if $\Delta T \in L^p$ ($p \geq 1$) for all compact set, then $T \in L^q$ ($q = q(p)$), in his celebrated *Théorie des Distributions*.⁽¹⁾

We shall prove analogous results for some other differential operators, the part about the iterated wave and Laplacian cases being due to TASHIRO and the one about the parabolic cases to ONO.

Here we express our hearty thanks to Prof. W. SIBAGAKI for his leading and his criticism throughout the preparation of this memoir.

§ 1. On the LAPLACE'S equation $\Delta T = \frac{\partial^2 T}{\partial x_1^2} + \cdots + \frac{\partial^2 T}{\partial x_n^2} = f$.

We treat the cases $n > 2$ in this paragraph. For the LAPLACE'S equation $\Delta T = f$, where f belongs to L^p ($p \geq 1$), L. SCHWARTZ proved the next theorem in his book.

Theorem I. *If the Laplacian of T , $f = \Delta T$, belongs to L^p ($p \geq 1$) on any compact set,⁽²⁾ T belongs to L^q on any compact set, where $1/q \geq \sup(1/q_1, 0)$, $1/q_1 = 1/p - 2/n$, but when $p = 1$ or $1/p - 2/n = 0$, we must take $>$ instead of $\geq$.*

Now we have the following theorem for the LIPSCHITZ'S condition of T .

(1) See [I] Tome II, pp. 36-46.

(2) From now on we will assume that " $f \in L^p$ " means " $f \in L^p$ on any compact set."

Theorem II. *If $f(x)$ satisfies the condition of the Theorem I, T is a function that satisfies the LIPSCHITZ's condition of order s ($0 \leq s \leq 1$) in L^q on any compact set, where $1/q \geq \sup(1/q_1, 0)$, $1/q_1 = 1/p - 2/n + s/n$, with similar exceptional cases.*

In proving this theorem we use the following lemmas.

Lemma 1.1.⁽³⁾ *For the elementary solution of LAPLACE's equation, we have*

$$\Delta\left(-\frac{1}{N} \frac{1}{r^{n-2}}\right) = \delta,$$

where

$$N = \frac{\Gamma(n/2)}{(n-2) 2(\sqrt{\pi})^n}.$$

Lemma 1.2.⁽⁴⁾ *If f has a compact carrier and belongs to L^p ($p \geq 1$) and if $0 \leq \lambda < n$, the composition product $g = (1/r)^\lambda * f$ belongs to L^q on any compact set, as long as $1/q \geq \sup(1/q_1, 0)$, $1/q_1 = 1/p + \lambda/n - 1$, with some exceptional cases.*

Proof of the Theorem. At first we notice the next majoration for $0 \leq s \leq 1$ and for sufficiently large constant K .

$$\left| \frac{1}{\|x+h\|^m} - \frac{1}{\|x\|^m} \right| \leq K \|h\|^s \left[\frac{1}{\|x+h\|^{m+s}} + \frac{1}{\|x\|^{m+s}} \right].$$

This inequality can be verified easily, when we distinguish the next two cases, i.e. the one in which $\|x\|/\|h\| \leq 1$ and the other in which $\|x\|/\|h\| > 1$. In the first case, let

$$F = \left| \frac{1}{\|x+h\|^m} - \frac{1}{\|x\|^m} \right| / \|h\|^s \left[\frac{1}{\|x+h\|^{m+s}} + \frac{1}{\|x\|^{m+s}} \right].$$

Then

$$\begin{aligned} F &\leq \frac{\|x+h\|^s}{\|h\|^s} + \frac{\|x\|^s}{\|h\|^s} \\ &\leq \frac{\{2\|h\|\}^s}{\|h\|^s} + 1 \leq K_1. \end{aligned}$$

In the second case,

$$F \leq \frac{\left| \frac{\|x\|^m}{\|x+h\|^m \|x\|^m} - \frac{\|x+h\|^m}{\|x+h\|^m \|x\|^m} \right|}{\left| \frac{\|x\|^m}{\|x+h\|^m \|x\|^m} - \frac{\|x+h\|^m}{\|x+h\|^m \|x\|^m} \right|} \frac{\|h\|^s}{\|x+h\|^{m+s}}$$

(3) See [1] Tome I p. 46.

(4) See [1] Tome II p. 37

$$\begin{aligned}
&= \frac{\|x\| - \|x+h\| \{ \|x+h\|^{m-1} + \|x+h\|^{m-2} \|x\| + \dots + \|x\|^{m-1} \}}{\frac{\|x\|^m \|h\|^s}{\|x+h\|^s}} \\
&\leq \frac{\|h\|^{1-s} \{2\|x\|\}^s}{\|x\|^m} \left[\{2\|x\|\}^{m-1} + \{2\|x\|\}^{m-2} \|x\| + \dots + \|x\|^{m-1} \right] \\
&\leq K'_2 \frac{\|h\|^{1-s}}{\|x\|^{1-s}} \leq K_2.
\end{aligned}$$

Thus the above inequality has been verified.

If T has a compact carrier, from

$$\begin{aligned}
\tau_{-h} T - T &= (\partial_{-h} - \partial) * T, \\
-\frac{1}{N} \frac{1}{r^{n-2}} * f &= -\frac{1}{N} \frac{1}{r^{n-2}} * \Delta * T = T
\end{aligned}$$

and the above inequality, we have

$$|\tau_{-h} T - T| \leq K \|h\|^s \left[\frac{1}{r^{n-2+s}} * |f| + \partial_{-h} * \frac{1}{r^{n-2+s}} * |f| \right].$$

Hence, by the Lemma 1.2, we have

$$\|\tau_{-h} T - T\|_q \leq K \|h\|^s,$$

where $\frac{1}{q} \geq \sup\left(\frac{1}{q_1}, 0\right), \quad \frac{1}{q_1} = \frac{1}{p} + \frac{s-2}{n},$

with the exceptional cases.

When T has a non-compact carrier, we take *parametrix*⁽⁵⁾ instead of *elementary solution*.

§ 2. On the iterated wave equation $\varGamma^k T = f$,⁽⁶⁾

$$\text{where } \varGamma = \frac{\partial^2}{\partial \mathbf{x}_n^2} - \frac{\partial^2}{\partial \mathbf{x}_1^2} - \dots - \frac{\partial^2}{\partial \mathbf{x}_{n-1}^2}.$$

For the wave equation $\varGamma T = f$, even if f be a function, T is not always a function, as in the case of the equation $\varGamma T = 0$. But the equation $\varGamma^k T = f$, where $k \geq n/2$ (n an even number) and $k \geq (n-1)/2$ (n an odd number) and $f \in L^p$ ($p \geq 1$), has at least one function as the solution. In these cases we shall discuss properties

(5) See [1] Tome II, p. 39.

(6) We treat the case that f has a compact carrier.

analogous to the LAPLACIAN. For the cases $k > n/2$ (n an even number) or $k > (n-1)/2$ (n an odd number), we notice that the proofs are analogous to the case $k = (n+1)/2$ (n an odd number). Therefore we shall treat the cases $k = n/2$, $k = (n-1)/2$, $(n+1)/2$. Before we proceed to mention the theorem, we shall notice the next lemma.

Lemma 2.1.⁽⁷⁾ *The equation $\nabla^l Z_l = \delta$ is satisfied for a positive integer l , when we define s and Z_l as follows.*

$$s = (x_n^2 - x_1^2 - \dots - x_{n-1}^2)^{1/2} \quad \text{for } \begin{cases} x_n \geq 0 \text{ and} \\ x_n^2 - x_1^2 - \dots - x_{n-1}^2 \geq 0, \end{cases}$$

$$= 0 \quad \text{for the other case,}$$

$$\text{and } Z_l = \frac{1}{\pi^{(n-2)/2} 2^{l-1} \Gamma\left(\frac{l}{2}\right) \Gamma\left(\frac{l+2-n}{2}\right)} \text{pf. } s^{l-n} = \frac{1}{M_l} \text{pf. } s^{l-n}.$$

(The notation *pf.* is the one of HADAMARD's finite part, and in the cases $l-n > -2$, the notation *pf.* is unnecessary.)

Theorem III. *When n is an even number, if $\nabla^{n/2} T = f$ is a function, T is a continuous function,⁽⁸⁾ and if f is a measure, T is a bounded function. When n is an odd number, if $\nabla^{(n-1)/2} T = f$ belongs to L^p , T belongs to L^q , where $1/q > 1/p - 1/2$ (for $1/p - 1/2 \geq 0$), but if $1/p - 1/2 < 0$, T is a continuous function. For $\nabla^{(n+1)/2} T = f$, if f is a measure, T is a continuous function.*

For the first case, take Z_n in the Lemma 2.1, then we have

$$Z_n * f = Z_n * \nabla^{n/2} * T = T.$$

Hence

$$Z_n * f = T.$$

Now

$$Z_n = \frac{1}{M_n} s^0 \quad \begin{cases} = \frac{1}{M_n} & s > 0 \\ = 0 & s = 0, \end{cases}$$

(7) See [1] Tome I pp. 50-51.

(8) Now we treat the solution which is a function.

hence Z_n belongs to L^∞ . According to the HARDY-LITTLEWOOD's theorem,⁽⁹⁾ we have $T \in L^r$ where $1/r = 1 + 0 - 1 = 0$, thus the theorem is established.

For the second part of the theorem, we have similarly

$$Z_{n-1} * f = T,$$

hence, since f belongs to L^p , T belongs to L^q , where $1/q > 1/p - 1/2$, for $1/p - 1/2 \geq 0$, and T is a continuous function for $1/p - 1/2 < 0$. The proof of the last part of the theorem is similar.

Similarly we have the next theorem for LIPSCHITZ's condition.

Theorem IV.

i) When n is an even number, if $\nabla^{n/2} T = f$ is a measure, T satisfies the LIPSCHITZ's condition of order s ($0 < s \leq 1$) in L^q , where $1/q \geq s$.

ii) When n is an odd number, if $\nabla^{(n-1)/2} T = f$ belongs to L^p , T satisfies the LIPSCHITZ's condition of order s ($0 \leq s < 1/2$) in L^q , where $1/q > 1/p + s - 1/2$.

iii) When n is an odd number, if $\nabla^{(n+1)/2} T = f$ is a measure, T satisfies the ordinary LIPSCHITZ's condition of order $1/2$.

At first we shall prove the case (i). According to the HARDY-LITTLEWOOD's theorem, we have

$$\left\{ \iint \cdots \int |\tau_{-h} T - T|^q dx \right\}^{1/q} \leq \left\{ \iint \cdots \int |\tau_{-h} Z_n - Z_n|^q dx \right\}^{1/q} \\ \cdot \left\{ \iint \cdots \int |f|^q dx \right\}^{1/q}, \quad (1 \leq q < \infty),$$

which is, by the definition of Z_n

$$\leq K_1 \left\{ \iint \cdots \int |\tau_{-h} s^0 - s^0|^q dx \right\}^{1/q},$$

and since the integrand is 1 or 0, this is

$$\leq K_2 \left\{ \iint \cdots \int dx \quad + \quad \iint \cdots \int dx \right. \\ \left. \begin{array}{l} x_n^2 - x_1^2 - \cdots - x_{n-1}^2 \geq 0 \quad (x_n \geq 0) \\ (x_n + h_n)^2 - (x_1 + h_1)^2 - \cdots - (x_{n-1} + h_{n-1})^2 < 0 \end{array} \quad \begin{array}{l} x_n^2 - x_1^2 - \cdots - x_{n-1}^2 \geq 0 \quad (x_n \geq 0) \\ (x_n + h_n)^2 < 0 \end{array} \right\}$$

(9) See for example [II] pp. 294-299.

$$\begin{aligned}
& + \iint \cdots \int_{\substack{x_n^2 - x_1^2 - \cdots - x_{n-1}^2 < 0 \\ (x_n + h_n)^2 - (x_1 + h_1)^2 - \cdots - (x_{n-1} + h_{n-1})^2 \geq 0 \\ (x_n + h_n \geq 0)}} dx \Big\}^{1/q} \\
& + \iint \cdots \int_{\substack{x_n < 0 \\ (x_n + h_n)^2 - (x_1 + h_1)^2 - \cdots - (x_{n-1} + h_{n-1})^2 \geq 0 \\ (x_n + h_n \geq 0)}} dx \Big\}^{1/q} \\
& \equiv K_2 \{ (I) + (II) + (III) + (IV) \}^{1/q}.
\end{aligned}$$

Observing the domains of integration, we have for the integral (I)

$$\begin{aligned}
& x_n^2 - x_1^2 - \cdots - x_{n-1}^2 \geq 0 \quad (x_n \geq 0) \\
& x_n \geq (x_1^2 + x_2^2 + \cdots + x_{n-1}^2)^{1/2} \\
& (x_n + h_n)^2 - (x_1 + h_1)^2 - \cdots - (x_{n-1} + h_{n-1})^2 \leq 0 \quad (x_n + h_n \geq 0) \\
& |x_n| \leq |h_n| + \{ (x_1 + h_1)^2 + \cdots + (x_{n-1} + h_{n-1})^2 \}^{1/2} \\
& \leq |h_n| + \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} + \left(\sum_{i=1}^{n-1} h_i^2 \right)^{1/2} \\
& \leq 2 \|h\| + (x_1^2 + \cdots + x_{n-1}^2)^{1/2}.
\end{aligned}$$

Hence

$$(x_1^2 + \cdots + x_{n-1}^2)^{1/2} + 2 \|h\| \geq x_n \geq (x_1^2 + \cdots + x_{n-1}^2)^{1/2}.$$

Similarly we have for (II)

$$\begin{aligned}
& -h_n \geq x_n \geq 0, \\
& |h_n| \geq x_n \geq (x_1^2 + \cdots + x_{n-1}^2)^{1/2},
\end{aligned}$$

for (III)

$$(x_1^2 + \cdots + x_{n-1}^2)^{1/2} \geq x_n \geq (x_1^2 + \cdots + x_{n-1}^2)^{1/2} - 2 \|h\|,$$

for (IV)

$$0 \geq x_n \geq (x_1^2 + \cdots + x_{n-1}^2)^{1/2} - 2 \|h\|.$$

Thus we have for (I)

$$\iint \cdots \int \left\{ \int_{\sqrt{x_1^2 + \cdots + x_{n-1}^2}}^{\sqrt{x_1^2 + \cdots + x_{n-1}^2} + 2 \|h\|} dx_n \right\} dx_1 \cdots dx_{n-1} \leq K_1 \|h\|,$$

for (II)

$$\begin{aligned}
& \iint \cdots \int \left\{ \int_{\sqrt{x_1^2 + \cdots + x_{n-1}^2}}^{\|h\|} dx_n \right\} dx_1 \cdots dx_{n-1} \leq \int \cdots \int \left\{ \int_{\sqrt{x_1^2 + \cdots + x_{n-1}^2}}^{\sqrt{x_1^2 + \cdots + x_{n-1}^2} + \|h\|} dx_n \right\} dx_1 \cdots dx_{n-1} \\
& \leq K_1 \|h\|.
\end{aligned}$$

Similarly

$$\int \cdots \int \left\{ \int_{\sqrt{x_1^2 + \cdots + x_{n-1}^2} - 2\|h\|}^{\sqrt{x_1^2 + \cdots + x_{n-1}^2}} dx_n \right\} dx_1 \cdots dx_{n-1} \leq K_2 \|h\|,$$

$$\int \cdots \int \left\{ \int_0^{\sqrt{x_1^2 + \cdots + x_{n-1}^2} - 2\|h\|} dx_n \right\} dx_1 \cdots dx_{n-1} \leq K_2 \|h\|.$$

Hence for $q < \infty$

$$\left\{ \int \cdots \int |\tau_{-h} T - T|^q dx \right\}^{1/q} \leq K' \|h\|^{1/q}.$$

Now, take s ($0 < s \leq 1$), then

$$\begin{aligned} \|\tau_{-h} T - T\|_q &\leq K' \|h\|^s \|h\|^{1/q-s} \\ &\leq K \|h\|^s, \end{aligned}$$

if q satisfies the condition $1/q \geq s$.

Before we prove the case (ii), we shall prove (iii). By the HARDY-LITTLEWOOD's theorem we have

$$\begin{aligned} |\tau_{-h} T - T| &\leq |\tau_{-h} Z_{n+1} - Z_{n+1}| \iint \cdots \int |f| dx \\ &\leq K_1 |\tau_{-h} s - s|. \end{aligned}$$

Now we consider that s and $\tau_{-h} s$ are not both zero, since when they are both zero, the theorem is evident. We divide the proof into two cases: the case $\|h\|^{1/2}/s \leq 1$ and $\|h\|^{1/2}/s > 1$.

Now we consider $\tau_{-h} s \neq 0$, then for the first case we have

$$\begin{aligned} |\tau_{-h} s - s| &= \frac{\tau_{-h} s^2 - s^2}{\tau_{-h} s + s} \leq \frac{2 \sum_{i=1}^n |x_i| |h_i| + \sum_{i=1}^n h_i^2}{\tau_{-h} s + s} \leq \frac{2(\sum x_i^2)^{1/2} (\sum h_i^2)^{1/2} + \|h\|^2}{\tau_{-h} s + s} \\ &\leq K_2' \frac{\|h\|^{1/2}}{s} \leq K_2. \end{aligned}$$

For the other case, we assume $\tau_{-h} s \neq 0$, then

$$\begin{aligned} \tau_{-h} s &= \{(x_n + h_n)^2 - (x_1 + h_1)^2 - \cdots - (x_{n-1} + h_{n-1})^2\}^{1/2} \\ &\leq \{(|x_n| + |h_n|)^2 - (|x_1| - |h_1|)^2 - \cdots - (|x_{n-1}| - |h_{n-1}|)^2\}^{1/2} \\ &\leq s + \|h\| + \{2(\sum_{i=1}^n x_i^2)^{1/2} (\sum_{i=1}^n h_i^2)^{1/2}\}^{1/2} \\ &\leq s + K_3 \|h\|^{1/2}. \end{aligned}$$

From the assumption, we have $\mathbf{s} < \|h\|^{1/2}$, hence

$$\begin{aligned}\tau_{-h}\mathbf{s} &\leq K_4 \|h\|^{1/2}, \\ |\tau_{-h}\mathbf{s} - \mathbf{s}| &\leq K_5 \|h\|^{1/2}.\end{aligned}$$

Therefore from

$$|\tau_{-h}T - T| \leq K_1 |\tau_{-h}\mathbf{s} - \mathbf{s}|,$$

we have

$$|\tau_{-h}T - T| \leq K \|h\|^{1/2}.$$

For the case (ii), at first we shall prove the following inequality for $0 \leq s \leq 1/2$

$$|\mathbf{s}(\mathbf{x} + h)^{-1} - \mathbf{s}^{-1}| \leq K \|h\|^s \left[\frac{1}{\mathbf{s}^{1+2s}} + \frac{1}{\mathbf{s}(\mathbf{x} + h)^{1+2s}} \right],$$

where

$$\mathbf{s}(\mathbf{x} + h) = \tau_{-h}\mathbf{s}.$$

Take

$$F = \frac{|\mathbf{s}(\mathbf{x} + h)^{-1} - \mathbf{s}^{-1}|}{\|h\|^s |\mathbf{s}^{-1-2s} + \mathbf{s}(\mathbf{x} + h)^{-1-2s}|},$$

with the exception of the cases of $\mathbf{s}(\mathbf{x} + h) = 0$ or $\mathbf{s} = 0$.

Now we distinguish the following two cases, $\mathbf{s}^2/\|h\| \leq 1$ and $\mathbf{s}^2/\|h\| > 1$.

In case $\mathbf{s}^2/\|h\| \leq 1$, we have

$$\begin{aligned}F &\leq \frac{1}{\|h\|^s} [\mathbf{s}^{2s} + \mathbf{s}(\mathbf{x} + h)^{2s}] \\ &\leq \left(\frac{\mathbf{s}^2}{\|h\|} \right)^s + \left(\frac{\mathbf{s}(\mathbf{x} + h)^2}{\|h\|} \right)^s \quad (\tau_{-h}\mathbf{s} \neq 0).\end{aligned}$$

Using the above result, we have

$$\begin{aligned}F &\leq 1 + \left\{ \frac{(\mathbf{s} + K_3 \|h\|^{1/2})^2}{\|h\|} \right\}^s \\ &\leq 1 + (K_3 + 1)^{2s} \leq K' .\end{aligned}$$

In case $\mathbf{s}^2/\|h\| > 1$, we have

$$\begin{aligned}F &= \frac{|\mathbf{s} - \mathbf{s}(\mathbf{x} + h)|}{\|h\|^s \mathbf{s} \cdot \mathbf{s}(\mathbf{x} + h)} \bigg/ (\mathbf{s}(\mathbf{x} + h)^{-1-2s} + \mathbf{s}^{-1-2s}) \\ &\leq \frac{K_3 \|h\|^{1/2}}{\|h\|^s} \mathbf{s}(\mathbf{x} + h)^{2s} / \mathbf{s} \\ &\leq K'' \left\{ \left(\frac{\|h\|}{\mathbf{s}^2} \right)^{1/2-s} + 1 \right\} \leq K .\end{aligned}$$

For $s(x+h) = 0$, we have

$$s \leq K_1 \|h\|^{1/2}.$$

Hence

$$\frac{s^{2s}}{\|h\|^s} \leq K$$

$$\left| \frac{1}{s} \right| \leq \|h\|^s \left| \frac{1}{s^{1+2s}} \right|.$$

By means of the above inequality, the theorem is verified easily. Now we suppose that T has a compact carrier as well as the theorem II. From

$$\tau_{-h}T - T = (\delta_{-h} - \delta) * Z_{n-1} * f,$$

we have for $s < 1/2$

$$|\tau_{-h}T - T| \leq K \|h\|^s \left[\frac{1}{s^{1+2s}} * |f| + \delta_{-h} * \frac{1}{s^{1+2s}} * |f| \right].$$

Since s^{1+2s} belongs to L^r ($r \geq 1$), where $(1+2s)r < 2$ for $s < 1/2$, according to HARDY-LITTLEWOOD's theorem we have

$$\|\tau_{-h}T - T\|_q \leq K \|h\|^s,$$

where

$$\frac{1}{q} > \frac{1}{p} + s - \frac{1}{2}.$$

§ 3. On the parabolic equation $DT = \left(\frac{\partial}{\partial x_n} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_{n-1}^2} \right) T = f$.

We mention some lemmas.

Lemma 3.1.⁽¹⁰⁾ *The elementary solution $e(t)$ of the parabolic equation*

$$\left(\frac{\partial}{\partial x_n} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_{n-1}^2} \right) T = 0$$

is given by

$$e(t) = \left(\frac{1}{2\sqrt{\pi} t_n} \right)^{n-1} \exp \left[-\frac{t_1^2 + \dots + t_{n-1}^2}{4t_n} \right] \quad t_n > 0$$

$$= 0 \quad t_n \leq 0.$$

(10) See for example [III] pp. 70-110.

Lemma 3.2. $e(t)$ belongs to L^k for all compact set for any $k < 1 + 2/(n-1)$.

Proof. $|e(t)|^k = \left(\frac{1}{2\sqrt{\pi}}\right)^{(n-1)k} \left(\frac{1}{t_n}\right)^{(n-1)k/2} \prod_{i=1}^{n-1} \exp\left[-k \frac{t_i^2}{4t_n}\right]$.

Therefore

$$\int \cdots \int |e(t)|^k dt_1 \cdots dt_n \leq K \int_{0 < t_n < M} \cdots \int t_n^{-(n-1)(k-1)/2} \prod_{i=1}^{n-1} \exp(-t_i^2) dt_1 \cdots dt_n.$$

Therefore, if $(n-1)(k-1)/2 < 1$, i.e. $k < 1 + 2/(n-1)$, then $e(t) \in L^k$ on any compact set.

By means of these lemmas we shall prove the

Theorem V. If for the equation $DT = f, f \in L^p$ ($p \geq 1$), then T is a function and belongs to L^q as long as $1/q > 1/p - 2/(n+1)$. If $1/p - 2/(n+1) < 0$, then T is a continuous function.

Proof. At first, we assume that the carrier of f is compact. Then

$$e * f = e * DT = De * T = \delta * T = T,$$

i.e.

$$T = e * f.$$

Other T which satisfy $DT = f$ satisfy $D(T - e * f) = 0$, so that $T - e * f \in C^\infty$ according to the SCHWARTZ's fundamental theorem.⁽¹¹⁾ Therefore it is sufficient to consider the function $e * f$.

Now $f \in L^p$ and $e \in L^k$, where $k < 1 + 2/(n-1)$ by Lemma 3.2., so that we have by the HARDY-LITTLEWOOD's theorem $e * f \in L^q$, where

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{k} - 1 > \frac{1}{p} + \frac{n-1}{n+1} - 1 = \frac{1}{p} - \frac{2}{n+1}.$$

Therefore

$$\frac{1}{q} > \frac{1}{p} - \frac{2}{n+1}.$$

If the carrier of f is not compact, then let us take a γ which is equal to 1 in a neighbourhood of the origin and belongs to $(\mathfrak{D})$.

(11) See [I] pp. 136-143.

Then

$$D(re) = \zeta + \delta, \quad \text{where } \zeta \in (\mathfrak{D}),$$

because e has only singularity at the origin.

Therefore

$$re * f = re * DT = D(re) * T = (\zeta + \delta) * T = \zeta * T + T.$$

Therefore

$$re * f = \zeta * T + T.$$

Now

$$re \in L^k, f \in L^p \quad \text{and} \quad \zeta * T \in (\mathfrak{E}).$$

Hence

$$T \in L^q, \quad \text{where } \frac{1}{q} > \frac{1}{p} - \frac{2}{n+1}. \quad \text{q.e.d.}$$

Next we prove the following important inequality.

Lemma 3.3. *If $t_n > h_n > 0$, then*

$$\begin{aligned} & \left| \left(\frac{1}{2\sqrt{\pi}(t_n - h_n)} \right)^{n-1} \exp \left[-\frac{(t_1 - h_1)^2 + \dots + (t_{n-1} - h_{n-1})^2}{4(t_n - h_n)} \right] \right. \\ & \quad \left. - \left(\frac{1}{2\sqrt{\pi}t_n} \right)^{n-1} \exp \left[-\frac{t_1^2 + \dots + t_{n-1}^2}{4t_n} \right] \right| \\ & \leq K(h_1^2 + \dots + h_n^2)^{s/2} \left\{ \left(\frac{1}{2\sqrt{\pi}(t_n - h_n)} \right)^{n-1+2s} \right. \\ & \quad \cdot \exp \left[-\theta \frac{(t_1 - h_1)^2 + \dots + (t_{n-1} - h_{n-1})^2}{4(t_n - h_n)} \right] \\ & \quad \left. + \left(\frac{1}{2\sqrt{\pi}t_n} \right)^{n-1+2s} \exp \left[-\theta \frac{t_1^2 + \dots + t_{n-1}^2}{4t_n} \right] \right\}, \end{aligned}$$

where $0 < \theta < 1$ and K is a sufficiently large number that depends on θ only.

Proof. At first we assume that the point t belongs to a fixed neighbourhood $U(0)$, and divide the proof into the following cases.

(i) $t_n - h_n \leq c_1 h_n$, where c_1 is a fixed positive number.

In this case we may majorate the first and second term of the first member by the corresponding term of the second member.

(ii) $t_n - h_n \geq c_1 h_n$, in which case

the first member

$$\begin{aligned}
 &= \left(\frac{1}{\sqrt{t_n - h_n}} \right)^{n-1} \left\{ \exp \left[- \frac{(t_1 - h_1)^2 + \cdots + (t_{n-1} - h_{n-1})^2}{4(t_n - h_n)} \right] \right. \\
 &\quad \left. - \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4t_n} \right] \right\} + \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4t_n} \right] \left\{ \left(\frac{1}{\sqrt{t_n - h_n}} \right)^{n-1} \right. \\
 &\quad \left. - \left(\frac{1}{\sqrt{t_n}} \right)^{n-1} \right\}.
 \end{aligned}$$

In this firstly

$$\begin{aligned}
 \text{the second term} &= \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4t_n} \right] \\
 &\cdot \frac{(\sqrt{t_n} - \sqrt{t_n - h_n})(\sqrt{t_n})^{n-2} + \cdots + (\sqrt{t_n - h_n})^{n-2}}{(\sqrt{t_n})^{n-1} (\sqrt{t_n - h_n})^{n-1}} \leq A \frac{h_n (\sqrt{t_n})^{n-2}}{(\sqrt{t_n})^{2n-1}} \\
 &\cdot \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4t_n} \right] = A \frac{h_n}{(\sqrt{t_n})^{n+1}} \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4t_n} \right],
 \end{aligned}$$

which is easily majorated by the second term of the second member.
Next as for the first term, i.e.

$$\begin{aligned}
 &\left(\frac{1}{\sqrt{t_n - h_n}} \right)^{n-1} \left\{ \exp \left[- \frac{(t_1 - h_1)^2 + \cdots + (t_{n-1} - h_{n-1})^2}{4(t_n - h_n)} \right] \right. \\
 &\quad \left. - \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4t_n} \right] \right\},
 \end{aligned}$$

we further divide the cases:

(ii)₁ $t_n \leq c_2(h_1^2 + \cdots + h_{n-1}^2)^{1/2}$, where c_2 is a fixed positive number.
In this case the majoration is trivial.

(ii)₂ $t_n \geq c_2(h_1^2 + \cdots + h_{n-1}^2)^{1/2}$

$$\begin{aligned}
 &\exp \left[- \frac{(\sqrt{t_1^2 + \cdots + t_{n-1}^2} \pm \sqrt{h_1^2 + \cdots + h_{n-1}^2})^2}{4(t_n - h_n)} \right] \\
 &= \exp \left[- \frac{(t_1^2 + \cdots + t_{n-1}^2) \pm 2|t||h| + |h|^2}{4(t_n - h_n)} \right] \\
 &= \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4(t_n - h_n)} \right] \exp \left[\mp \frac{2|t| + |h|}{4(t_n - h_n)} |h| \right] \\
 &\sim \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4(t_n - h_n)} \right] \left\{ 1 \mp \frac{2|t| + |h|}{4(t_n - h_n)} |h| \right\} \\
 &= \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4(t_n - h_n)} \right] \mp \frac{2|t| + |h|}{4(t_n - h_n)} |h| \exp \left[- \frac{t_1^2 + \cdots + t_{n-1}^2}{4(t_n - h_n)} \right],
 \end{aligned}$$

the second term of which is majorated by the first term of the second member. So that it remains to consider the term

$$\left(\frac{1}{\sqrt{t_n - h_n}}\right)^{n-1} \left\{ \exp\left[-\frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n}\right] - \exp\left[-\frac{t_1^2 + \dots + t_{n-1}^2}{4(t_n - h_n)}\right] \right\},$$

which can be written as

$$\begin{aligned} & \left(\frac{1}{\sqrt{t_n - h_n}}\right)^{n-1} \exp\left[-\frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n}\right] \\ & \cdot \left\{ 1 - \exp\left[-h_n \frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n(t_n - h_n)}\right] \right\}. \quad (H) \end{aligned}$$

Now let us take a fixed positive number c_3 and distinguish the two cases :

$$(\alpha) \quad \frac{h_n(t_1^2 + \dots + t_{n-1}^2)}{t_n(t_n - h_n)} \leq c_3,$$

then

$$|(H)| \leq B \left(\frac{1}{\sqrt{t_n - h_n}}\right)^{n-1} \exp\left[-\frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n}\right] \frac{h_n(t_1^2 + \dots + t_{n-1}^2)}{t_n(t_n - h_n)},$$

the quotient of which to the second term of the second member of our lemma is as follows :

$$\begin{aligned} & \sim \frac{(\sqrt{t_n})^{n-1+2s}}{(h_1^2 + \dots + h_n^2)^{s/2}} \left(\frac{1}{\sqrt{t_n - h_n}}\right)^{n-1} \\ & \cdot \exp\left[-(1-\theta) \frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n}\right] \frac{h_n(t_1^2 + \dots + t_{n-1}^2)}{t_n(t_n - h_n)} \\ & \sim \frac{|h|^{1-s}}{(\sqrt{t_n})^{2-2s}} \frac{t_1^2 + \dots + t_{n-1}^2}{t_n} \exp\left[-(1-\theta) \frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n}\right] \leq K'. \end{aligned}$$

$$(\beta) \quad \frac{h_n(t_1^2 + \dots + t_{n-1}^2)}{t_n(t_n - h_n)} \geq c_3,$$

then

$$\begin{aligned} |(H)| & \leq \left(\frac{1}{\sqrt{t_n - h_n}}\right)^{n-1} \exp\left[-\frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n}\right] \{1 - e^{-c_3}\} \\ & \leq \left(\frac{1}{\sqrt{t_n - h_n}}\right)^{n-1} \exp\left[-\frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n}\right], \end{aligned}$$

the quotient of which to the second term is as follows:

$$\begin{aligned} & \sim \frac{(\sqrt{t_n})^{n-1+2s}}{(h_1^2 + \dots + h_n^2)^{s/2}} \left(\frac{1}{\sqrt{t_n} - h_n} \right)^{n-1} \exp \left[-(1-\theta) \frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n} \right] \\ & \sim \frac{|t_n|^s}{|h|^s} \exp \left[-(1-\theta) \frac{t_1^2 + \dots + t_{n-1}^2}{4 t_n} \right]. \end{aligned}$$

On the other hand, $\frac{t_1^2 + \dots + t_{n-1}^2}{t_n} \geq c_3 \frac{t_n - h_n}{h_n}.$

Therefore

$$\leq \frac{|t_n|^s}{|h|^s} \exp \left[-(1-\theta) c_3 \frac{t_n - h_n}{h_n} \right] = K'' \frac{|t_n|^s}{|h|^s} \exp \left[-(1-\theta) c_3 \frac{t_n}{h_n} \right],$$

where

$$K'' = \exp[(1-\theta) c_3].$$

Therefore in this case the majoration is possible.

Thus we proved the inequality for the case when t belongs to a fixed $U(0)$. When t does not belong to a fixed neighborhood $U(0)$, then the given function is infinitely differentiable, so that the proof is trivial. Thus we have a complete proof of our lemma for any compact set.

In other cases which will be necessary in the proof of the next theorem, for instance, when either $e(t+h)$ or $e(t)$ is 0, we can give analogous inequalities.

Theorem VI. *T in Theorem V satisfies the Lipschitz condition of order s ($0 < s < 1$) in $L^{q'}$, where $\frac{1}{q'} > \frac{1}{p} - \frac{2}{n+1} + \frac{2s}{n+1}.$*

If $\frac{1}{p} - \frac{2}{n+1} + \frac{2s}{n+1} < 0$, then T satisfies the Lipschitz condition in L^∞ , and so the Lipschitz condition in the usual sense.

Proof. We can prove that the second member of the inequality of Lemma 3.3. belongs to $L^{k'}$, where $k' < \frac{n+1}{n-1+2s}$, in the same way as in Lemma 3.2. Therefore

$$\begin{aligned}
 |\tau_h T - T| &= |\tau_h(e * f) - e * f| = |(\tau_h e - e) * f| \\
 &\leq K(h_1^2 + \dots + h_n^2)^{s/2} \left\{ \left(\frac{1}{\sqrt{t_n - h_n}} \right)^{n-1+2s} \right. \\
 &\quad \cdot \exp \left[-\theta \frac{(t_1 - h_1)^2 + \dots + (t_{n-1} - h_{n-1})^2}{4(t_n - h_n)} \right] * |f| \\
 &\quad \left. + \left(\frac{1}{\sqrt{t_n}} \right)^{n-1+2s} \exp \left[-\theta \frac{t_1^2 + \dots + t_{n-1}^2}{4t_n} \right] * |f| \right\}.
 \end{aligned}$$

Then by means of the theorem of HARDY-LITTLEWOOD the second member belongs to $L^{q'}$, where $\frac{1}{q'} = \frac{1}{p} + \frac{1}{k'} - 1 > \frac{1}{p} + \frac{n-1+2s}{n+1} - 1$
 $= \frac{1}{p} - \frac{2}{n+1} + \frac{2s}{n+1}.$

Therefore

$$\|\tau_h T - T\|_{q'} \leq c(h_1^2 + \dots + h_n^2)^{s/2} \quad (0 < s < 1).$$

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