ON SOME SUMMATIONS OF DOUBLE SERIES

By

Hiroshi WATANABE

(Received Sep. 17, 1954)

1. Recently I. M. SHEFFER [1] and A. WILANSKY [2] discussed general summations of a double series

$$(1) \quad \sum_{i,j=0}^{\infty} a_{i,j},$$

where $\{a_{i,j}\}$ are real numbers. Here we concern ourselves with some special summations which are used frequently. Our notations are as follows:

$$\begin{aligned} S_{m,n} &= \sum_{i,j=0}^{m,n} a_{i,j}, & c_n &= \sum_{i+j=n} a_{i,j}, & S_n &= \sum_{i=0}^n c_i, \\ S_{m,n}^{(\alpha,\beta)} &= \sum_{i,j=0}^{m,n} A_{m-i}^{\alpha-1} A_{n-j}^{\beta-1} S_{i,j}, & S_n^{(\gamma)} &= \sum_{i=0}^n A_{n-i}^{\gamma-1} S_i, \\ \sigma_{m,n}^{(\alpha,\beta)} &= S_{m,n}^{(\alpha,\beta)} / A_m^{\alpha} A_n^{\beta}, & \sigma_n^{(\gamma)} &= S_n^{(\gamma)} / A_n^{\gamma}, \end{aligned}$$

where α, β and γ are real numbers greater than -1 , and $A_n^{\alpha} = \binom{n+\alpha}{n}$.

If a double sequence $\{S_{m,n}\}$ ($m, n = 0, 1, 2, \dots$) is convergent (*i.e.* in PRINGSHEIM's sense) or regularly convergent (*i.e.* $\{S_{m,n}\}$ is convergent and in addition $\lim_{m \rightarrow \infty} S_{m,n}$ and $\lim_{n \rightarrow \infty} S_{m,n}$ exist for each n and m respectively) to a value s , then the corresponding double series (1) is said to be convergent or regularly convergent respectively to the value s .

Replacing $\{S_{m,n}\}$ by $\{\sigma_{m,n}^{(\alpha,\beta)}\}$ in the above definitions we say that the series (1) is CESÀRO-summable or regularly CESÀRO-summable respectively, with order α, β to the value s , and we write

$$\sum_{i,j=0}^{\infty} a_{i,j} = s (C, \alpha, \beta) \quad \text{or} \quad \text{reg. } (C, \alpha, \beta)$$

respectively.

If a simple series

$$(2) \quad \sum_{i=0}^{\infty} c_i$$

is convergent to a value s , we say that the series (1) is Δ -convergent to the value s , and we write

$$\sum_{i,j=0}^{\infty} a_{i,j} = s(\Delta).$$

Many authors have discussed CESÀRO-summability for double series and obtained various results.¹⁾ For instance it is known that if $\alpha' > \alpha$ and $\beta' > \beta$, (C, α, β) -summability and boundedness of $\{\sigma_{m,n}^{(\alpha', \beta')}\}$ for a double series (1) imply (C, α', β') -summability of the series [3]. It is, however, easily verified that regular (C, α, β) -summability imply regular (C, α', β') -summability.

2. We consider relations between the summations above mentioned. When a double series (1) is convergent to a value s and also Δ -convergent to a value s' , it is not necessarily $s = s'$. This is shown by the example [5, p. 98]:

$$(a_{i,j}) = \begin{pmatrix} a_0 + b_0 & a_1 - b_0 & a_2 & a_3 & \cdots \\ -a_0 + b_1 & -a_1 - b_1 & -a_2 & -a_3 & \cdots \\ b_2 & -b_2 & 0 & 0 & \cdots \\ b_3 & -b_3 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where $a_n \rightarrow a$, $b_n \rightarrow b$ and $a + b \neq 0$. In fact this series is convergent to 0 and Δ -convergent to $a + b \neq 0$. Under these situations we have

Theorem 1. *Let the double series (1) be convergent to a value s and at the same time Δ -convergent to a value s' . If the condition*

$$(3) \quad \sum_{k=[n/2]}^n \sum_{i+j=k} |a_{i,j}| \rightarrow 0 \quad (n \rightarrow \infty)$$

is satisfied, then $s = s'$.

Proof. Let us denote by (i, j) a point in a plane and put $\Delta_n = \{(i, j) : i + j \leq n\}$ and $p_n = \{(i, j) : i \leq [n/2] \text{ and } j \leq [n/2]\}$. Then we have obviously by (3)

1) See [3] and [4].

$$\left| \sum_{(i,j) \in \mathcal{A}_n} a_{i,j} - \sum_{(i,j) \in \mathcal{P}_n} a_{i,j} \right| \leq \sum_{k=\lfloor n/2 \rfloor}^n \sum_{i+j=k} |a_{i,j}| \rightarrow 0 \quad (n \rightarrow \infty),$$

and thus we have $s = s'$.

3. For any two double sequences $\{S_{m,n}\}$ and $\{k_{m,n}\}$ ($m, n = 0, 1, 2, \dots$) we define a simple sequence $\{t_n\}$ ($n = 0, 1, 2, \dots$) as follows:

$$(4) \quad t_n = \sum_{i+j=n} k_{i,j} S_{i,j}.$$

We may consider the relation (4) as a transformation of a double sequence $\{S_{i,j}\}$ to a simple sequence $\{t_n\}$ by the matrix $(k_{i,j})$. Then we have

Lemma. *A necessary and sufficient condition in order that for any regularly convergent double sequence $\{S_{m,n}\}$ the sequence $\{t_n\}$ be convergent to the same value, is that the following hold:*

- 1°. $k_{i,j} \rightarrow 0$ ($i \rightarrow \infty, j = 0, 1, 2, \dots$) and $k_{i,j} \rightarrow 0$ ($j \rightarrow \infty, i = 0, 1, 2, \dots$),
- 2°. $\sum_{i+j=n} |k_{i,j}| < K$ (K is a positive constant independent of n),
- 3°. $\sum_{i+j=n} k_{i,j} \rightarrow 1$ ($n \rightarrow \infty$).

Proof. Sufficiency is first proved. Put $S_{i,j} = s + \varepsilon_{i,j}$, where $\{\varepsilon_{i,j}\}$ is regularly convergent to 0. Then we can find H such that

$$(5) \quad |\varepsilon_{i,j}| < H \quad (i, j = 0, 1, 2, \dots).$$

For an arbitrary $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$(6) \quad |\varepsilon_{i,j}| < \varepsilon \quad \text{for all } i, j \geq N,$$

and for any $n \geq N$ we have

$$\begin{aligned} t_n &= s \sum_{i+j=n} k_{i,j} + \sum_{i+j=n} k_{i,j} \varepsilon_{i,j} \\ &= s \sum_{i+j=n} k_{i,j} + \left(\sum_{i=0}^{N-1} + \sum_{i=N}^{n-N} + \sum_{i=n-N+1}^n \right) k_{i,n-i} \varepsilon_{i,n-i}. \end{aligned}$$

The first term on the right-hand side of the preceding tends to the value s by 3°. Now we can find $M = M(N)$ by 1° such that for all $n \geq M$

$$(7) \quad \sum_{i=0}^{N-1} |k_{i,n-i}| < \varepsilon, \quad \sum_{i=n-N+1}^n |k_{i,n-i}| = \sum_{i=0}^{N-1} |k_{n-i,i}| < \varepsilon.$$

Thus by 2° together with (5), (6) and (7) we have

$$\left| \left(\sum_{i=0}^{N-1} + \sum_{i=N}^{n-N} + \sum_{i=n-N+1}^n \right) k_{i, n-i} \varepsilon_{i, n-i} \right| < (2H + K) \varepsilon$$

for all $n \geq M(N) = M(\varepsilon)$.

In order to show that 1° and 3° are necessary we have only to consider particular sequences such as (i) $S_{p,j} = 1$ ($j = 0, 1, 2, \dots$), $S_{i,j} = 0$ ($i \neq p, j = 0, 1, 2, \dots$), (ii) $S_{i,p} = 1$ ($i = 0, 1, 2, \dots$), $S_{i,j} = 0$ ($j \neq p, i = 0, 1, 2, \dots$) for any fixed p , and (iii) $S_{i,j} = 1$ ($i, j = 0, 1, 2, \dots$). It is clear that these double sequences are regularly convergent.

To show the necessity of 2° we suppose that $t_n^* = \sum_{i+j=n} |k_{i,j}|$ is not bounded. Then there exists a sequence $\{n_k\}$ such that $n_k \uparrow \infty$ and

$$t_{n_k}^* > k^2 \quad k = 1, 2, \dots$$

Taking a monotone increasing function $\phi(n)$ of n , defined by

$$\phi(n) = k \quad \text{for} \quad n_k \leq n < n_{k+1} \quad k = 1, 2, \dots,$$

we consider a double sequence

$$S_{m,n} = \frac{1}{\phi(m+n)} \text{sign } k_{m,n}.$$

This is regularly convergent to 0 and hence $t_n \rightarrow 0$ by the assumption. On the other hand

$$t_{n_k} = \sum_{i+j=n_k} k_{i,j} S_{i,j} = t_{n_k}^* / \phi(n_k) > k \quad k = 1, 2, \dots,$$

i.e. $t_{n_k} \rightarrow \infty$, which is a contradiction.

4. We can deduce the following theorem from the above Lemma.

Theorem 2. *If the double series (1) is regularly CESÀRO-summable with order α, β to a value s , then the simple series (2) is CESÀRO-summable with order $\alpha + \beta + 1$ to the same value s .*

Proof. We may define $S_{m,n}^{(\alpha,\beta)}$ by the formula

$$\left(\sum_{i,j=0}^{\infty} a_{i,j} x^i y^j \right) (1-x)^{-(\alpha+1)} (1-y)^{-(\beta+1)} = \sum_{i,j=0}^{\infty} S_{i,j}^{(\alpha,\beta)} x^i y^j,$$

which is analogous to one in the case of simple series. Putting $x = y$ we have

$$\left(\sum_{n=0}^{\infty} c_n x^n\right) (1-x)^{-(\alpha+\beta+2)} = \sum_{n=0}^{\infty} \left(\sum_{i+j=n} S_{i,j}^{(\alpha,\beta)}\right) x^n$$

and

$$\sigma_n^{(\alpha+\beta+1)} = \sum_{i+j=n} A_i^{\alpha} A_j^{\beta} \sigma_{i,j}^{(\alpha,\beta)} / A_n^{\alpha+\beta+1}.$$

It is easily verified that the double sequence

$$k_{i,j} = A_i^{\alpha} A_j^{\beta} / A_{i+j}^{\alpha+\beta+1}$$

satisfies the conditions 1°, 2° and 3° in the Lemma. Thus we complete the proof.

Remark 1. From any two simple series

$$(8) \quad \sum_{i=0}^{\infty} a_i \quad \text{and} \quad \sum_{i=0}^{\infty} b_i,$$

we form by putting $a_{i,j} = a_i b_j$ a double series

$$(9) \quad \sum_{i,j=0}^{\infty} a_{i,j} = \sum_{i,j=0}^{\infty} a_i b_j.$$

Then $S_{m,n} = S'_m S''_n$, where $S'_m = \sum_{i=0}^m a_i$ and $S''_n = \sum_{i=0}^n b_i$. We call such a case “factorizable”. Then the following (i) and (ii) are equivalent:

$$(i) \quad \sum_{i=0}^{\infty} a_i = a(C, \alpha) \quad \text{and} \quad \sum_{i=0}^{\infty} b_i = b(C, \beta),$$

$$(ii) \quad \sum_{i,j=0}^{\infty} a_i b_j = ab \quad \text{reg.}(C, \alpha, \beta).$$

Hence in this case our Theorem asserts the well-known fact that the CAUCHY's product series of two simple series (8) is $(C, \alpha + \beta + 1)$ -summable when each series of (8) is (C, α) - and (C, β) -summable respectively [8, p. 228] [6, p. 19].

Remark 2. We can show here that the order $\alpha + \beta + 1$ in the conclusion of the Theorem cannot be replaced by $\alpha + \beta + 1 - \varepsilon$ ($\varepsilon > 0$). For simplicity we take $\alpha = \beta = 0$. We construct a double series (1) from a convergent series $\sum_{i=0}^{\infty} v_i = v$ by putting $a_{i,j} = v_{i+j} - v_{i+j+1}$ ($i, j = 0, 1, 2, \dots$) [2]. This double series is regularly convergent to v . Hence the corresponding series (2) is CESÀRO-summable with order 1 to v . But we can take particularly

$$\begin{aligned} v_i &= (-1)^i / \log(i+1) \quad i = 1, 2, \dots, \\ v_0 &= 1 \end{aligned}$$

and hence

$$c_n = \sum_{i+j=n} a_{i,j} = (n+1)(v_n - v_{n+1}) = (n+1)(-1)^n \frac{\log(n+1)(n+2)}{\log(n+1)\log(n+2)}.$$

Thus for $\delta < 1$ we have

$$c^n/n^\delta = O(n^{1-\delta}/\log n) \neq o(1),$$

and we see that the necessary condition for (C, δ) -summability of series $\sum_{i=0}^{\infty} c_i$ is not satisfied.

The following is an analogue of Theorem 2.

Theorem 3. *If the double series (1) is ABEL-summable²⁾ to a value s , then the simple series (2) is ABEL-summable to the same value s .*

Now we mention some results deduced from Theorem 2.

Corollary 1. *If the double series (1) is regularly convergent to a value s and at the same time Δ -convergent to a value s' , then $s = s'$ ³⁾.*

Corollary 2. *Let the double series (1) be regularly convergent. If the condition*

$$(10) \quad (i^2 + j^2) a_{i,j} < K \quad i, j = 0, 1, 2, \dots,$$

is satisfied, then the series is Δ -convergent.

Proof. By Theorem 2 the simple series (2) is $(C, 1)$ -summable and further we have from (10)

$$c_n = \sum_{i+j=n} a_{i,j} < K \sum_{i=0}^n \frac{1}{i^2 + (n-i)^2} < \frac{K'}{n},$$

where K' is a positive constant independent of n . Hence we complete the proof by use of the well-known Tauberian Theorem [8, p. 121].

5. We now deal with certain convergence factors. G. H. HARDY proved: In order that the double series

$$(11) \quad \sum_{i,j=0}^{\infty} k_{i,j} a_{i,j}$$

2) The definition of ABEL-summability for double series is given by K. KNOPP [7].

3) Corollary 1 is a well-known ABEL's theorem [8, p. 228] for a factorizable double series.

be regularly convergent for any regularly convergent double series (1) it is necessary and sufficient that the double sequence $\{k_{i,j}\}$ is bounded variation, *i.e.*

$$\begin{aligned} (a) \quad & \sum_{i,j=0}^{\infty} |\Delta_{i,j} k_{i,j}| < \infty, \\ (b) \quad & \sum_{i=0}^{\infty} |\Delta_i k_{i,j}| < \infty \quad j = 0, 1, 2, \dots, \\ (c) \quad & \sum_{j=0}^{\infty} |\Delta_j k_{i,j}| < \infty \quad i = 0, 1, 2, \dots, \end{aligned}$$

where $\Delta_i k_{i,j} = k_{i,j} - k_{i+1,j}$, $\Delta_j k_{i,j} = k_{i,j} - k_{i,j+1}$ and $\Delta_{i,j} k_{i,j} = \Delta_i \Delta_j k_{i,j} = \Delta_j \Delta_i k_{i,j}$.

We have the following

Theorem 4. *Let the double series (1) be regularly convergent. Then in order that the double series (11) be Δ -convergent it is sufficient that the double sequence $\{k_{i,j}\}$ is bounded variation and satisfies 1°, 2° and 3° in the lemma.*

Proof. Using ABEL's transformation we have

$$\begin{aligned} S_n^* &= \sum_{k=0}^n \sum_{i+j=k} k_{i,j} a_{i,j} = \sum_{i=0}^n \sum_{j=0}^{n-i} k_{i,j} a_{i,j} \\ &= \sum_{i=0}^{n-2} \sum_{j=0}^{n-2-i} (\Delta_{i,j} k_{i,j}) S_{i,j} + \sum_{j=0}^{n-1} (\Delta_j k_{n-1-j,j}) S_{n-1-j,j} + \sum_{i=0}^n r_{i,n-i} k_{i,n-i}, \end{aligned}$$

where $r_{i,n} = \sum_{j=0}^n a_{i,j}$ and $\Delta_j k_{n-1-j,j} = k_{n-1-j,j} - k_{n-1-j,j+1}$. We denote the three terms in the last expression by P_n , Q_n and R_n respectively.

By the regular convergence of $\{S_{m,n}\}$ there exists a constant H such that

$$(12) \quad |S_{i,j}| < H \quad i, j = 0, 1, 2, \dots.$$

From (a) we have

$$\sum_{k=0}^{\infty} \sum_{i+j=k} |\Delta_{i,j} k_{i,j}| < \infty$$

and see together with (12) that P_n is convergent as $n \rightarrow \infty$.

As for Q_n , since

$$\Delta_j k_{n-1-j,j} = \Delta_j k_{0,j} - \sum_{i=0}^{n-2-j} \Delta_{i,j} k_{i,j},$$

we have

$$\sum_{j=0}^{n-1} |\Delta_j k_{n-1-j,j}| \leq \sum_{j=0}^{n-1} |\Delta_j k_{0,j}| + \sum_{j=0}^{n-2} \sum_{i=0}^{n-2-j} |\Delta_{i,j} k_{i,j}|,$$

and again from (a) and (c) together with (12) we conclude that Q_n is convergent as $n \rightarrow \infty$.

Finally we see

$$R_n = \sum_{i=0}^n k_{i, n-i} S_{i, n-i} - \sum_{i=1}^n k_{i, n-i} S_{i-1, n-i}.$$

The first term on the right-hand side is convergent by our Lemma and for the second we have only to consider the double sequence

$$(S_{i,j}^*) = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ S_{0,0} & S_{0,1} & S_{0,2} & \cdots \\ S_{1,0} & S_{1,1} & S_{1,2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

instead of $\{S_{i,j}\}$.

Thus we complete the proof of the Theorem.

References

- [1] I. M. SHEFFER, Amer. Math. Monthly, **52** (1945), 365-376.
- [2] A. WILANSKY, *On the convergence of double series*, Bull. Amer. Math. Soc., **53** (1947), 793-799.
- [3] C. R. ADAMS, *On summability of double series*, Trans. Amer. Math. Soc., **34** (1932), 215-230.
- [4] C. R. ADAMS, *Transformations of double sequences with applications to CESÀRO summability of double series*, Bull. Amer. Math. Soc., **37** (1931), 741-748.
- [5] R. J. P. A. BROMWICH, *An introduction to the theory of infinite series*, London, 2nd ed. (1949).
- [6] E. KOGBETLIANTZ, *Sommation des séries et intégrales divergentes*, Mém. Sc. Math. **51** (1931).
- [7] K. KNOPP, *Limitierungs-Umkehrsätze für Doppelfolgen*, Mat. Z. **45** (1939), 573-589.
- [8] G. H. HARDY, *Divergent series*, Oxford (1949).