ON THE DENSITIES IN A TWO-DIMENSIONAL GENERALIZED SPACE.

By

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M. HAIMOVICI¹⁾ and L. A. SANTALÓ²⁾ made a study of integral geometry on a two-dimensional surface with Riemannian metric in the different ways. M. HAIMOVICI defined a density of a vector field and obtained a density of geodesics. By the geodesic polar coordinates L. A. SANTALÓ had densities of points, of geodesics and defined cinematic densities of geodesic segments, of congruent curves and attained to the POINCARÉ's formula on a surface. In his paper the congruence of two curves was extensively defined by means of their natural equations. We have taken an interest in these studies.

More previously than L. A. SANTALÓ, W. BLASCHKE³⁾ had most generally defined as density of geodesics in Finsler space and led the CROFTON's formula. With a view to have some contributions to the integral geometry in Finsler space, we shall adjust it by making use of the geometrical treatments established by É. CARTAN⁴⁾, analogously to the methods of HAIMOVICI and SANTALÓ in Riemannian surface.

At first we have defined a density of a vector field in Finsler space by the method of HAIMOVICI and get the CROFTON's formula as its application. In the treatment of a cinematic density we fall across a property of Finsler space. We define a cinematic density of geodesic segments with fixed length l by

$$dK = 2 [dG ds],$$

1) M. HAIMOVICI, *Geometrie integral sur les surfaces courbes*. Ann. Sc. Univ. Jassy, **23** (1936), 57-74.

2) L. A. SANTALÓ, *Integral geometry on surfaces*. Duke Math. Journ., **16** (1949), 361-375.

3) W. BLASCHKE, *Integralgeometrie XI: Zur Variationsrechnung*, Abhandl. Hans. Univ., **11** (1936), 359-366.

4) É. CARTAN, *Les espaces de Finsler*. Exposes de Geometrie, Paris, 1934.

where dG is the density of a vector field $a(x)$ and s is the distance from the center of a directive element $(x, a(x))$ of K (of G)—we shall call an element $(x, a(x))$ a directive element of a geodesic segment K (of a geodesic G) which $a(x)$ is a vector of the field and tangent to K (to G) at the point (x) —to the original point of K (of G). This density has the same invariant properties as the cinematic density in Riemannian surface and its representation by LANDSBERG's angle⁵⁾ is analogous to that in Riemannian surface. But the rotation of all the geodesic segment around its endpoint by a fixed LANDSBERG's angle does not preserve the invariance of the cinematic density. Therefore in order to define a cinematic density of polygonal lines, the congruency for these figures cannot be obtained by the LANDSBERG's angle. To preserve the invariance of the cinematic density under the rotation around the end point of segment, we define a new angle different from LANDSBERG's angle and obtain a cinematic density of oriented polygonal lines.

Furthermore making use of this angle we get a new curvature and a definition of the congruency of two curves as the limiting case of the polygonal lines. Then we have a cinematic density and attain the POINCARÉ's formula as in the case of L. A. SANTALÓ.

1. Density of a vector field. We consider a two-dimensional generalized space whose arc-length is given by

$$(1.1) \quad s = \int F(x, dx),$$

where the function $F(x, x')$ ($(x') \neq (0)$) is of class C^1 with respect to x, x' and positively homogeneous of degree 1 in x' 's. Following the method of É. CARTAN we introduce the fundamental tensor by

$$(1.2) \quad g_{ij}(x, x') = \frac{1}{2} \frac{\partial^2 (F(x, x')^2)}{\partial x'^i \partial x'^j}$$

and define a Euclidean connection, in which the covariant differential of a vector X^i is given by

5) L. BERWALD, *Über zwei dimensionale allgemeine metrische Räume*. Journ. f. reine u. angew. Math., **156** (1927), 211-222.

G. LANDSBERG, *Über die Krümmung in der Variationsrechnung*. Math. Ann., **65** (1908), 313-349.

$$\begin{aligned}
 (1.3) \quad \delta X^i &= dX^i + X^j (I_{jk}^i dx^k + C_{jk}^i dx'^k) \\
 &= dX^i + X^j (I_{jk}^{*i} dx^k + C_{jk}^i \delta x'^k)
 \end{aligned}$$

and that of the line-element by

$$(1.4) \quad \delta x'^i = dx'^i + G_k^i(x, x') dx^k,$$

where

$$\begin{aligned}
 (1.5) \quad C_{jlk} &= g_{il} C_{jk}^i = \frac{1}{2} \frac{\partial g_{jl}}{\partial x'^k}, \\
 I_{jlk} &= g_{il} I_{jk}^i = \frac{1}{2} \left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) + G_{lk}^m C_{jm} - G_j^m C_{lkm}, \\
 G_j^i &= x'^m I_{mj}^i.
 \end{aligned}$$

Hence we can denote (1.3) by

$$(1.3') \quad \delta X^i = X_{,k}^i dx^k + X_{;k}^i \delta x'^k,$$

where

$$X_{,k}^i = \frac{\partial X^i}{\partial x^k} + X^j I_{jk}^{*i}, \quad X_{;k}^i = \frac{\partial X^i}{\partial x'^k} + X^j C_{jk}^i.$$

Now in this space we consider a vector field $a(x) = (a^i(x), a^2(x))$ which depends only on the point (x) , where we assume the vector normalized with respect to itself:

$$(1.6) \quad g_{ij}(x, a) a^i a^j = 1.$$

And we construct a unit vector $b(x, x')$ orthogonal to the vector $a(x)$ with respect to the element (x, x') :

$$(1.7) \quad g_{ij}(x, x') a^i b^j = 0,$$

$$(1.8) \quad g_{ij}(x, x') b^i b^j = 1.$$

When the supporting element (x, x') is superposed on $(x, a(x))$ the vector b is transversal to the direction of the element (x, a) with respect to the same element.

Applying the covariant differentiation to (1.7) and using the relation $\delta g_{ij} = 0$, we get

$$g_{ij} \delta a^i b^j + g_{ij} a^i \delta b^j = 0.$$

We define two scalars at the element (x, a) as follows:

$$\begin{aligned}
 (1.9) \quad p &= g_{ij} \delta a^i b^j = -g_{ij} a^i \delta b^j & (x' \equiv a), \\
 \nu &= g_{ij} b^i dx^j & (x' \equiv a).
 \end{aligned}$$

Because of the superposition of the supporting element (x, x') on (x, a) we have $C_{jk}^i a^j = 0$, and since the vector a is independent of (x') , we obtain

$$a^i_{;k} = \frac{\partial a^i}{\partial x'^k} + C_{jk}^i a^j = 0 \quad (x' \equiv a).$$

Hence we see that the vector δa^i in (1.9) has the form

$$(1.10) \quad \delta a^i = a^i_{;k} dx^k.$$

Now, we define a density of the vector field by

$$(1.11) \quad dG = [p \nu].$$

From the above remark, it can be written in the form

$$(1.11') \quad dG = g_{ij} a^i_{;k} b^j g_{lm} b^l [dx^k dx^m] \quad (x' \equiv a).$$

Especially, if the vector field $a(x)$ is a unit tangent vector field of geodesics, we can follow the same arguments as in Riemannian surface. Namely at first

$$(1.12) \quad \begin{aligned} \delta a^i &= a^i_{;k} dx^k & (x' \equiv a) \\ &= 0 & (\text{When the direction of } dx = \text{direction of } a). \end{aligned}$$

Therefore, from the comparison of the coefficients of (1.7) and (1.12) we get

$$\frac{a^i_{;1}}{g_{1j} b^j} = \frac{a^i_{;2}}{g_{2j} b^j} = \frac{a^i_{;k} dx^k}{g_{kj} b^j dx^k} \quad (= \rho^i, \text{ say}),$$

accordingly

$$p = g_{ij} b^j \rho^i \nu.$$

Hence we obtain

$$dG = 0.$$

That is, we have the same conclusion as in Riemannian surface: *If the given vector field $a(x)$ is a unit tangent vector field of geodesics, the density of the vector field dG is zero.*

Since in our space we can draw through the element (x, a) one and only one geodesic G , we call the element (x, a) a *directive element* of G . A vector field on some domain $\mathfrak{D}$ is transported into a vector field on another domain $\mathfrak{D}'$, when we displace the

vector $a(x)$ in parallel to itself along the geodesic G of the directive element $(x, a(x))$ from the point x^i to some nearby point

$$\bar{x}^i = x^i + a^i(x) \varepsilon(x),$$

and the supporting line element x' at the point x^i in parallel to the point $\bar{x}^i$. Thus we get a mapping $\mathfrak{D} \rightarrow \mathfrak{D}'$, and we shall call it a *displacement*. Thus we have for the displacement

$$(1.13) \quad \begin{aligned} da^i + I^{*i}_{jk}(x, a) a^j a^k \varepsilon(x) &= 0, \\ dx'^i + G^i_k(x, x') a^k \varepsilon(x) &= 0. \end{aligned}$$

Denoting the transform of an object T by $\bar{T}$, we get by definition

$$(1.14) \quad \begin{aligned} (i) \quad \bar{a}^i(x) &= a^i(x) - I^{*i}_{jk}(x, a) a^j a^k \varepsilon(x), \\ (ii) \quad \bar{x}'^i &= x'^i - G^i_k(x, x') a^k \varepsilon(x), \\ (iii) \quad \bar{dx}^i &= dx^i + a^i \{ \varepsilon(x + dx) - \varepsilon(x) \} + da^i \varepsilon(x + dx), \end{aligned}$$

from which it follows easily that

$$\begin{aligned} \bar{dx}'^i &= dx'^i - \left(G^i_k \frac{\partial a^k}{\partial x^l} + \frac{\partial G^i_k}{\partial x^l} \cdot a^k \right) dx^l \varepsilon(x + dx) \\ &\quad - \frac{\partial G^i_k}{\partial x'^l} a^k dx'^l \varepsilon(x + dx) - G^i_k a^k \{ \varepsilon(x + dx) - \varepsilon(x) \}, \end{aligned}$$

where $\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - G^m_i(x, x') \frac{\partial}{\partial x'^m}$. Hence

$$(1.15) \quad \begin{aligned} \bar{\partial x}'^i &= \bar{dx}'^i + G^i_k(x, \bar{x}') \bar{dx}^k \\ &= \delta x'^i - \frac{\partial G^i_k}{\partial x'^l} a^k \delta x'^l \varepsilon(x + dx) + \left(\frac{\partial G^i_l}{\partial x^k} - \frac{\partial G^i_k}{\partial x'^l} \right) \times \\ &\quad \times a^k dx'^l \varepsilon(x + dx). \end{aligned}$$

In the same manner, we have the transform $\bar{\partial a}^i$ of ∂a^i as

$$\begin{aligned} \bar{\partial a}^i &= \bar{da}^i + \bar{a}^j \{ I^{*i}_{jk}(\bar{x}, \bar{x}') \bar{dx}^k + C^i_{jk}(\bar{x}, \bar{x}') \bar{dx}^{jk} \} \\ &= \partial a^i + \{ I^{*i}_{jk}(x, x') - I^{*i}_{jk}(x, a) \} a^j a^k \{ \varepsilon(x + dx) - \varepsilon(x) \} \\ &\quad + \left\{ I^{*i}_{jk}(x, x') - 2 I^{*i}_{jk}(x, a) - \left[\frac{\partial I^{*i}_{jh}}{\partial x^k} \right]_{(x, a)} a^h \right\} a^j da^k \varepsilon(x) \\ &\quad + \left\{ \left[\frac{\partial I^{*i}_{jk}}{\partial x^l} \right]_{(x, x')} + C^i_{jm}(x, x') \left[\frac{\partial G^m_k}{\partial x^l} - \frac{\partial G^m_l}{\partial x^k} \right]_{(x, x')} - \left[\frac{\partial I^{*i}_{jl}}{\partial x^k} \right]_{(x, a)} \right. \\ &\quad \left. - I^{*i}_{mk}(x, x') I^{*m}_{jl}(x, a) \right\} a^j a^l dx^k \varepsilon(x) \end{aligned}$$

$$+ \left\{ \left[\frac{\partial C_{jk}^i}{\partial x^l} \right]_{(x, x')} - C_{mj}^i(x, x') \left[\frac{\partial G_l^m}{\partial x'^k} \right]_{(x, x')} - C_{mk}^i(x, x') I^{*m}_{jl}(x, a) \right\} a^j a^l \delta x'^k \varepsilon(x).$$

When we superpose the element (x, x') upon the element (x, a) , we have

$$\begin{aligned} \overline{\partial a^i} &= \partial a^i - I^{*i}_{jk}(x, a) a^j \partial a^k \varepsilon(x) \\ &\quad + [R^i_{jkl}]_{(x, a)} a^j a^l dx^k \varepsilon(x) + [L^i_{jkl}]_{(x, a)} a^j a^l \delta a^k \varepsilon(x), \end{aligned}$$

where

$$\begin{aligned} R^i_{jkl} &= \frac{\partial I^{*i}_{jk}}{\partial x^l} - \frac{\partial I^{*i}_{jl}}{\partial x^k} + C_{jm}^i \left(\frac{\partial G_k^m}{\partial x^l} - \frac{\partial G_l^m}{\partial x^k} \right) + I^{*i}_{ml} I^{*m}_{jk} - I^{*i}_{mk} I^{*m}_{jl}, \\ L^i_{jkl} &= \frac{\partial C_{jk}^i}{\partial x^l} - C_{jm}^i \frac{\partial G_l^m}{\partial x'^k} - C_{mk}^i I^{*m}_{jl} - \frac{\partial I^{*i}_{jl}}{\partial x'^k}. \end{aligned}$$

Since the covariant differential DX^i of the vector field $X^i(x, x')$ for the displacement may be written in the form

$$DX^i = \overline{X^i} - X^i + I^{*i}_{jk}(x, x') X^j a^k \varepsilon(x),$$

we see that

$$(1.16) \quad D(\partial a^i) = R^i_{jkl}(x, a) a^j a^l dx^k \varepsilon(x) \quad (x' \equiv a),$$

because we can see easily that

$$L^i_{jkl} x'^j x'^l = 0.$$

Using the relations $Dg_{ij}=0$, $Db^i=0$, we have

$$Dp = g_{ij} b^j D(\partial a^i), \quad D\nu = g_{ij} b^i D(dx^j).$$

Therefore the covariant differential of dG is

$$\begin{aligned} D(dG) &= [Dp]_\nu + [p] D\nu \\ &= 0 \quad (x' \equiv a). \end{aligned}$$

Hence we can conclude that *a density of a vector field is invariant under the displacement of each vector $a(x)$ of the field along each geodesic of directive element $(x, a(x))$.*

And consequently *the density of a vector field $a(x)$ can be considered as a density of the geodesic curve G with the directive element $(x, a(x))$.*

2. Crofton's formula and Poincaré's integral. In the first place we take a scalar

$$L(x, x') = g_{ij}(x, x') a^i a^j$$

and evaluate its partial derivatives:

$$\left[\frac{\partial L}{\partial x'^k} \right]_{(x, a)} = \left[\frac{\partial g_{ij}}{\partial x'^k} \right]_{(x, a)} a^i a^j = 0.$$

Hence we have

$$\begin{aligned} \left[\frac{\partial L}{\partial x^k} \right]_{(x, a)} &= \left[\frac{\partial g_{ij}}{\partial x^k} a^i a^j + 2 g_{ij} a^i \frac{\partial a^j}{\partial x^k} \right]_{(x, a)} \\ &= \left[\frac{\partial g_{ij}}{\partial x^k} a^i a^j + \frac{\partial g_{ij}}{\partial x'^m} \frac{\partial a^m}{\partial x^k} a^i a^j + 2 g_{ij} a^i \frac{\partial a^j}{\partial x^k} \right]_{(x, a)} \\ &= \frac{\partial L(x, a)}{\partial x^k} \\ &= 0. \end{aligned}$$

Accordingly

$$(2.1) \quad \delta L(x, x') = 2 g_{ij} \delta a^i a^j = 0 \quad (x' \equiv a).$$

On the other hand we can write (1.7) in the form

$$-\frac{g_{1j} b^j}{a^2} = \frac{g_{2j} b^j}{a^1} = \rho$$

or

$$(2.2) \quad g_{1j} b^j = -\rho a^2, \quad g_{2j} b^j = \rho a^1.$$

And by substitution into (1.8) we get

$$(2.3) \quad \rho(a^1 b^2 - a^2 b^1) = 1.$$

On the other hand we know that

$$(2.4) \quad \sqrt{g}(a^1 b^2 - a^2 b^1) = 1, \quad g = |g_{ij}(x, a)|;$$

therefore we have

$$(2.5) \quad \rho^2 = g.$$

From these relations we can derive some other forms of the density dG . Substituting (2.2) into (1.11'), we get

$$(2.6) \quad dG = \rho g_{ij} a^i a^k b^j [dx^1 dx^2],$$

and making use of (2.2) and (2.4)

$$(2.7) \quad dG = g(a^1 a_{,k}^2 - a^2 a_{,k}^1) a^k [dx^1 dx^2].$$

Moreover, from (2.1) and (2.6) we have

$$dG = \rho(a^1 b^2 - a^2 b^1) (g_{i2} a_{,1}^i - g_{i1} a_{,2}^i) [dx^1 dx^2],$$

and accordingly

$$(2.8) \quad \begin{aligned} dG &= (g_{i2} a_{,1}^i - g_{i1} a_{,2}^i) [dx^1 dx^2] \\ &= \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \right) [dx^1 dx^2]. \end{aligned}$$

If the vectors $a(x)$ are the unit tangent vectors of the parameter curves, one of the covariant components of $a(x)$, for instance a_2 , is zero and then the density dG has the form

$$(2.9) \quad dG = - \frac{\partial(g_{1i} a^i dx^1)}{\partial x^2} dx^2.$$

Therefore following HAIMOVICI's arguments *we have* CROFTON's formula concerning a piecewise convex curve C of length L :

$$(2.10) \quad \int_{G \cdot C \neq 0} n(G, C) = 2L,$$

where $N(G, C)$ is the number of intersections of G and C , and dG is the density of the geodesic curve G . Furthermore we can extend this formula to a rectifiable continuous curve in conformity to the method of W. MAAK.

In the next place we have the analogous theorem on the POINCARÉ's integral as in a Riemannian surface: *Let x^i, p_i be a coordinates in the phase space; then the POINCARÉ's integral*

$$(2.11) \quad I = \int [dp_i dx^i]$$

is invariant under the displacement of each point (in the domain of the integration) along a trajectory whose equations are

$$(2.12) \quad \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = - \frac{\partial H}{\partial x^i}.$$

Now we consider that (2.12) is the equations of the motion of a point on a generalized surface, and put

$$H = \frac{1}{2} g^{ij}(x, x') p_i p_j,$$

where g^{ij} is a reciprocal element of g_{ij} . Hence if we superpose the element (x, x') on (x, a) we can verify that the trajectory is geodesic:

$$\frac{d^2 x^i}{dt^2} + G^i\left(x, \frac{dx}{dt}\right) = 0,$$

and that the velocity is uniform.

Then we take the domain of the integration (2.11) by

$$(2.13) \quad p_i = g_{ij}(x, x') a^j,$$

where a^i is the velocity of a point. When a point is displaced along the geodesic of directive element (x, a) , the covariant differential of p_i is

$$\begin{aligned} \delta p_i &= g_{ij} \delta a^j \\ &= dp_i - p_j (I_{ik}^{*j} dx^k + C_{ik}^j \delta x'^k). \end{aligned}$$

Therefore we get

$$dp_i = g_{ij} \delta a^j + a^l I_{ilj}^{*k} dx^j + a^l C_{ilj} \delta x'^j.$$

By virtue of the symmetry of I_{ij}^{*k} with respect to the indices i, k and the relation (1.10) we get, when we superpose x' on $a(x)$,

$$\begin{aligned} [dp_i dx^i] &= g_{ij} [\delta a^i dx^j] \\ &= (g_{2j} a^j_{,1} - g_{1j} a^j_{,2}) [dx^1 dx^2] \quad (x' \equiv a). \end{aligned}$$

Hence we can reduce the POINCARÉ's integral into the integral of a density of a vector field dG , hence according to the invariance of the density dG we conclude the theorem.

3. Cinematic densities. On our vector field which depends only on the point (x) , we assume that each element of a manifold of geodesics of directive element $(x, a(x))$ corresponds one and only one vector $a(x)$ and that the manifold is differentiable with respect to (x) . Now we consider a set of oriented geodesic segments K of fixed length l on each geodesic. Denoting by s the distance from the center (x) of directive element (x, a) of a geodesic G (of a segment K) to the original point P of the segment K , each segment K determined by s and the directive element $(x, a(x))$, so that the segments K form a three dimensional (x^1, x^2, s) -manifold.

Previously we defined a density of a vector field by a differential invariant on x^1, x^2 , hence we now define a cinematic density by

$$(3.1) \quad dK = 2 [dG ds].$$

Then we see that *this cinematic density does not depend on a selection of the vector field used in our definition of the density dG .*

Displacing each vector $a(x)$ along the geodesic of directive element (x, a) , we have a new vector field and denote it by $\bar{a}$. Then according to the invariance of the density dG we have

$$\bar{dG} = dG.$$

We denote the distances by $s, \bar{s}$ from the two centers of the directive elements of K to the original point P and put $\phi(x)$ as the distance of this displacement, that is,

$$\bar{s} = s + \phi(x^1, x^2).$$

Then we get

$$d\bar{s} = ds \quad (\text{mod } dx^1, dx^2).$$

Hence we have

$$\begin{aligned} \bar{dK} &= 2 [\bar{dG} ds] = 2 [dG ds] \\ &= dK. \end{aligned} \quad q.e.d.$$

Now we shall give a geometrical expression to a cinematic density dK of oriented geodesic segments. Because of the above mentioned independence of the selection of a vector field $a(x)$, we can take a geodesic segment K of directive element (x, a) of original point (x) , and a neighbouring geodesic segment K' of directive element $(x + dx, a(x + dx))$. Using the distance ds from the center $(x + dx)$ to the original point of K' we can write an area element dP at $P(x)$ as

$$dP = \sqrt{g(x, a)} ds (a^1 dx^2 - a^2 dx^1) \quad (x' \equiv a)$$

up to second order. On the other hand the LANDSBERG's angle $d\theta$ determined by the directions of the segments K, K' is

$$\begin{aligned} d\theta &= \sqrt{g(x, a)} (a^1 \partial a^2 - a^2 \partial a^1) \\ &= \sqrt{g(x, a)} (a^1 a^2_{,k} - a^2 a^1_{,k}) dx^k \quad (x' \equiv a). \end{aligned}$$

Hence we have

$$(3.2) \quad \begin{aligned} [dP d\theta] &= g(a^1 a^2_{,k} - a^2 a^1_{,k}) a^k [dx^1 dx^2 ds] \quad (x' \equiv a) \\ &= [dG ds]. \end{aligned}$$

Thus making use of LANDSBERG's angle we have an analogous expression of a cinematic density as in Riemannian surface: *the half of the cinematic density dK of oriented geodesic segments is a exterior product of an area element dP at the original point with the LANDSBERG's angle $d\theta$ determined by their directions at their original points.*

In the next place we shall consider a cinematic density of the polygonal lines. For this purpose we must suitably define a congruency of polygonal lines. The set of oriented geodesic segments is a three dimensional (x^1, x^2, s) - or (x^1, x^2, θ) -manifold and its cinematic density dK is defined by an invariant differential form:

$$dK = 2 [dG ds].$$

Now we consider some manifolds $\mathfrak{M}_i$ ($i=0, 1, \dots, m$) whose element is a oriented geodesic segment of fixed length and assume that there exist the density-preserving transformations T_k ($k=1, \dots, m$) from $\mathfrak{M}_0$ to $\mathfrak{M}_k$. Then we call the pair of geodesic segments $(K_0, K_1, \dots, K_m) = (K_0, T_1 K_0, \dots, T_m K_0)$ a *polygonal line*, where K_i are the element of $\mathfrak{M}_i$. The set of these polygonal lines is a three dimensional (x^1, x^2, θ) -manifold. Hence we can define a cinematic density of polygonal lines by

$$(3.3) \quad dK = 2 [dP d\theta],$$

where dP means an area element at the original point P of a polygonal line and $d\theta$ is the LANDSBERG's angle determined by their directions at the original points.

With a view to affirm the above assumption we shall consider the density-preserving transformation T . Since the cinematic density dK does not depend on the selection of a vector field used in the definition, we can take the transformation T as infinitesimal

$$(3.4) \quad \begin{aligned} \bar{x}^i &= x^i, \\ \bar{x}^i &= x'^i + \varepsilon f^i(x, x'). \end{aligned}$$

But the area element dP is given by

$$dP = \sqrt{g(x, x')} dx^1 dx^2:$$

therefore the density-preserving transformation T is characterized by the invariance of

$$\sqrt{g(x, x')} d\theta = G(x, x') (x'^1 dx'^2 - x'^2 dx'^1) \pmod{dx^1, dx^2},$$

where $G(x, x') = F(x, x') F_1(x, x')$.

Denoting the difference under this infinitesimal transformation by Δ , we have

$$(3.5) \quad \begin{aligned} \Delta x'^i &= \varepsilon f^i(x, x'), \\ \Delta dx'^i &= \varepsilon df^i(x, x'). \end{aligned}$$

Then we obtain the following relation:

$$\begin{aligned} \Delta[G(x, x')(x'^1 dx'^2 - x'^2 dx'^1)] \\ = \varepsilon \frac{\partial G}{\partial x'^i} f^i (x'^1 dx'^2 - x'^2 dx'^1) + \varepsilon G(f^1 dx'^2 - f^2 dx'^1 + x'^1 df^2 - x'^2 df^1). \end{aligned}$$

Therefore the invariance of the density dK can be characterized by

$$\begin{aligned} -x'^2 \frac{\partial G}{\partial x'^i} f^i + G \left(-f^2 + x'^1 \frac{\partial f^2}{\partial x'^1} - x'^2 \frac{\partial f^1}{\partial x'^1} \right) &= 0, \\ x'^1 \frac{\partial G}{\partial x'^i} f^i + G \left(f^1 + x'^1 \frac{\partial f^2}{\partial x'^2} - x'^2 \frac{\partial f^1}{\partial x'^2} \right) &= 0. \end{aligned}$$

G being a positively homogeneous of degree -2 , we have

$$\begin{aligned} \frac{\partial G}{\partial x'^1} (x'^1 \Delta x'^2 - x'^2 \Delta x'^1) + G \frac{\partial}{\partial x'^1} (x'^1 \Delta x'^2 - x'^2 \Delta x'^1) &= 0, \\ \frac{\partial G}{\partial x'^2} (x'^1 \Delta x'^2 - x'^2 \Delta x'^1) + G \frac{\partial}{\partial x'^2} (x'^1 \Delta x'^2 - x'^2 \Delta x'^1) &= 0, \end{aligned}$$

that is,

$$(3.6) \quad \frac{\partial}{\partial x'^1} [G(x'^1 \Delta x'^2 - x'^2 \Delta x'^1)] = \frac{\partial}{\partial x'^2} [G(x'^1 \Delta x'^2 - x'^2 \Delta x'^1)] = 0.$$

Therefore we affirm that *a transformation T preserve a density dK if and only if $G(x'^1 \Delta x'^2 - x'^2 \Delta x'^1)$ depends only on the point.*

Especially in a Riemannian surface the determinat g depends only on (x) and $G(x'^1 \Delta x'^2 - x'^2 \Delta x'^1) = \sqrt{g} d\theta$; the desired transformation is a rotation in the sense of ordinary angle.

But then at a fixed point on our generalized space, if we consider the integral

$$(3.7) \quad \phi(x) = \int g(x, x') d\theta$$

extended over all directions, we have a scalar density $\phi(x)$. In a Riemannian surface it is $2\pi\sqrt{g}$. Now in our space we define a new angle by

$$(3.8) \quad \Delta\omega = \frac{2\pi G(x, x') (x'^1 \Delta x'^2 - x'^2 \Delta x'^1)}{\phi(x)}.$$

And we take the transformations of the manifolds of oriented geodesic segments as the rotations in the sense of the newly defined angle. Then we see that this transformation preserves our cinematic density dK .

In the generalized space the *congruent polygonal lines* are defined by the properties that they have the same side-lengths and the same vertical angle measured by the newly defined angle. Then we can define a cinematic density for the congruent polygonal lines by (3.3):

$$dK = 2 [dP d\theta].$$

Furthermore, if we define a curvature of a curve by

$$(3.9) \quad \kappa(s) = \lim_{\Delta s \rightarrow 0} \frac{\Delta\omega}{\Delta s},$$

where $\Delta\omega$ is a newly defined angle between the two adjacent tangents, then we can consider the *congruency* of general curves by means of natural equations. Thus we define a cinematic density of congruent curves by the same expression

$$(3.10) \quad dK = 2[dP d\theta].$$

Now we can extend the POINCARÉ'S formula to the figures in the generalized space: *denoting by $n(K, C)$ the number of intersections of a fixed rectifiable curve C of length L_0 and a congruent curve K of length L we have:*

$$(3.11) \quad \int_{K \cdot C \neq 0} n(K, C) dK = 4 L_0 L,$$

where dK is the cinematic density of curves K .