REDUCTION FORMULAS FOR STEENROD'S D_i IN THE CUBIC SINGULAR COHOMOLOGY THEORY

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§1. STEENROD'S reduced power operations $\mathcal{P}_p^k$ for a cell complex K are defined by a set of operations $D_i: K \rightarrow K \times K \times \dots \times K$ (n -fold product) with the usual boundary formula:

$$(1.1) \quad \partial D_i + (-1)^{i+1} D_i \partial = a_i D_{i-1}.$$

As is well known¹⁾ these operations may be introduced also in the singular theory, for example, by a universal simplicial formula. In this paper we shall give a universal cubical formula for D_i . For $u \in Q_q(X)$, $D_i u$ will be of the form:

$$(1.2) \quad D_i u = \sum a(\lambda_1, \lambda_2, \dots, \lambda_n) \lambda_1 u \otimes \lambda_2 u \otimes \dots \otimes \lambda_n u,$$

where λu is a $|\lambda|$ -dimensional face of u and $a(\lambda_1, \lambda_2, \dots, \lambda_n)$ are integral coefficients. The form (1.2) is universal in the sense that it is independent of the special u and may be written symbolically as:

$$(1.3) \quad D_i(\bar{q}) = \sum a(\lambda_1, \lambda_2, \dots, \lambda_n) \lambda_1 \times \lambda_2 \times \dots \times \lambda_n.$$

It is convenient to identify a boundary operator λ with a vector $A = (a_1, a_2, \dots, a_q)$ such that $a_j = 0, 1$ or t and such that

$$(1.4) \quad (\lambda u) \underset{|\lambda| \text{ times}}{(t \dots t)} = u(a_1, a_2, \dots, a_q).$$

Then (1.3) becomes

$$(1.5) \quad D_i(\bar{q}) = \sum a(A_1, A_2, \dots, A_n) A_1 \times A_2 \times \dots \times A_n,$$

or rather in matrix form

$$(1.6) \quad D_i(\bar{q}) = \sum a(A_1, A_2, \dots, A_n) \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \sum a_j A^j,$$

1) See [4], §7, Note.

where A^j are (n, q) -matrices whose elements are 0, 1 or t .

Since $D_i(\bar{1})$ was essentially given by N. E. STEENROD in [3], our task is to deduce the following reduction formulas:

$$D_{2i+1}(\overline{q+1}) = \sum_{j=0}^i \{D_{2j+1}(\bar{1}) \cdot \rho D_{2i-2j}(\bar{q}) - D_{2i-2j}(\bar{1}) \cdot D_{2j+1}(\bar{q})\},$$

and

$$D_{2i}(\overline{q+1}) = \sum_{j=0}^i D_{2j}(\bar{1}) \cdot D_{2i-2j}(q) + \sum_{j=0}^{i-1} D_{2j+1}(\bar{1}) * D_{2i-2j-1}(q),$$

where operations $\cdot$, ρ and $*$ are defined in the next §§.

As an immediate consequence of these formulas we have the following important property:

PROPERTY 1. *In (1.4) $a_j \neq 0$ only if A^j contains no column consisting of 0 and 1 only.*

This property serves to prove that D_i are operations not only in $Q(X)$ but also its normalized complex $Q^N(X)$. Thus in the usual way we may define operations $\mathcal{O}_p^k: H^q(X, Y; Z_p) \rightarrow H^{q+2k(p-1)}(X, Y; Z_p)$ for any pair of spaces (X, Y) .

That they are exactly the STEENROD operations for CW -complexes (X, Y) is seen since they satisfy the three axioms (2.1), (2.2) and (2.4) and consequently all other axioms in [4].

Our observation is especially useful in the homology theory of fibre bundles. For example, with respect to SERRE's filtration we have the following

THEOREM. *If $f_j \in A^{*a_j}$ ($j = 1, 2, \dots, n$) we have*

$$D_i(f_1 \times f_2 \times \dots \times f_n) \in A^{*\text{Max}[(a_1+a_2+\dots+a_n)-i, 0]}$$

which generalizes the relation $A^{*a_1} \cdot A^{*a_2} \subset A^{*a_1+a_2}$ and was used repeatedly in [5].

§2. Let $A = (a_{ij})$ be an (n, q) -matrix whose elements are 0, 1 or t . Its rows and columns will be denoted by A_i ($i = 1, \dots, n$) and $\bar{A}_j$ ($j = 1, \dots, q$), respectively. We call the number of t 's contained in A the dimension of A , and denote it by $|A|$.

The free abelian group generated by such (n, q) -matrices will be denoted by $\mathfrak{A}_q^n$ ($n, q \geq 1$). Put $\mathfrak{A}^n = \sum_q \mathfrak{A}_q^n$, $\mathfrak{A}_q = \sum_n \mathfrak{A}_q^n$ and $\mathfrak{A} = \sum_n \mathfrak{A}^n$ (direct sum).

Now, we shall define several operations in $\mathfrak{A}$; first of all, define these operations on generators, and then extend them by linearity all over $\mathfrak{A}$. (In the following definitions (i)–(iv), $A, B, \dots$ mean generators.)

(i) If $A = (a_{ij}) \in \mathfrak{A}_q^m$ and $B = (b_{ij}) \in \mathfrak{A}_q^n$,
then $A \times B = C = (c_{ij}) \in \mathfrak{A}_q^{m+n}$, where

$$c_{ij} = \begin{cases} a_{ij} & 1 \leq i \leq m, \\ b_{i-m, j} & m+1 \leq i \leq m+n. \end{cases}$$

(ii) If $A \in \mathfrak{A}_q^n$, $B \in \mathfrak{A}_r^n$, then $A \cdot B = (-1)^{\varepsilon(A, B)} C \in \mathfrak{A}_{q+r}^n$, where

$$c_{ij} = \begin{cases} a_{ij} & 1 \leq j \leq q, \\ b_{i, j-q} & q+1 \leq j \leq q+r, \end{cases}$$

and

$$\varepsilon(A, B) = \sum_{n \geq i > j \geq 1} |A_i| |B_j|.$$

(iii) If $A \in \mathfrak{A}_q^n$. Define $\rho A = (-1)^{\varepsilon(A)} B \in \mathfrak{A}_q^n$, where

$$b_{ij} = \begin{cases} a_{nj} & i = 1, \\ a_{i-1, j} & i > 1 \end{cases}$$

and

$$\varepsilon(A) = \sum_{j=1}^{n-1} |A_n| |A_j|.$$

Then the cyclic group π_n of order n generated by ρ may be considered as a group of operators on $\mathfrak{A}_q^n$.

(iv) If $A \in \mathfrak{A}_q^n$, then $\partial A = \sum_{i, j} (-1)^{\tau_{ij}} (\eta_{ij}^0 A - \eta_{ij}^1 A)$, where

$$\eta_{ij}^\varepsilon A = \begin{cases} 0, & \text{if } a_{ij} \neq t, \\ (b_{kl}), & \text{if } a_{ij} = t, \end{cases}$$

$$b_{kl} = \begin{cases} a_{kl} & \text{if } (k, l) \neq (i, j), \\ \varepsilon & \text{if } (k, l) = (i, j), \end{cases} \quad \varepsilon = 0 \text{ or } 1$$

$$\tau_{ij} = \sum_{k < i} |A_k| + |a_{ij}|;$$

and $|a_{ij}|$ = number of l 's such that $l < j$ and $a_{il} = t$.

Then concerning with these operations, we have the following properties:

$$(2.1.1) \quad (A \times B) \times C = A \times (B \times C).$$

$$(2.1.2) \quad (A \cdot B) \cdot C = A \cdot (B \cdot C).$$

$$(2.2.1) \quad (A + B) \times C = A \times C + B \times C; \quad A \times (B + C) = A \times B + A \times C.$$

$$(2.2.2) \quad (A + B) \cdot C = A \cdot C + B \cdot C; \quad A \cdot (B + C) = A \cdot B + A \cdot C.$$

$$(2.3) \quad A = A_1 \times A_2 \times \dots \times A_n \quad \text{for } A \in \mathfrak{A}^n.$$

$$(2.4) \quad (A_1 \times A_2 \times \dots \times A_n) \cdot (B_1 \times B_2 \times \dots \times B_n) \\ = (-1)^{\varepsilon(A, B)} (A_1 \cdot B_1) \times (A_2 \cdot B_2) \times \dots \times (A_n \cdot B_n).$$

$$(2.5.1) \quad \partial(A \times B) = \partial A \times B + (-1)^{|A|} A \times \partial B.$$

$$(2.5.2) \quad \partial(A \cdot B) = \partial A \cdot B + (-1)^{|A|} A \cdot \partial B.$$

$$(2.6) \quad \partial \partial = 0.$$

$$(2.7) \quad \partial \rho = \rho \partial.$$

$$(2.8) \quad \rho(A \cdot B) = \rho A \cdot \rho B.$$

§3. PROOFS. (2.1.1)–(2.5.1) are immediate consequences of the definitions. (2.6) may be proved as usual. Here, we shall prove (2.5.2), (2.7) and (2.8).

PROOF of (2.5.2).

$$\begin{aligned} \partial(A \cdot B) &= (-1)^{\varepsilon(A, B)} \partial \{ (A_1 \cdot B_1) \times \dots \times (A_n \cdot B_n) \} \\ &= (-1)^{\varepsilon(A, B)} \left\{ \sum_{i=1}^n (-1)^{\sum_{j<i} |A_j \cdot B_j|} (A_1 \cdot B_1) \right. \\ &\quad \times \dots \times \partial(A_i \cdot B_i) \times \dots \times (A_n \cdot B_n) \} \\ &= (-1)^{\varepsilon(A, B)} \sum_{i=1}^n (-1)^{\sum_{j<i} (|A_j| + |B_j|)} (A_1 \cdot B_1) \times \dots \times (\partial A_i \cdot B_i) \\ &\quad \times \dots \times (A_n \cdot B_n) + (-1)^{\varepsilon(A, B)} \sum_{i=1}^n (-1)^{\sum_{j<i} (|A_j| + |B_j|) + |A_i|} (A_1 \cdot B_1) \\ &\quad \times \dots \times (A_i \cdot \partial B_i) \times \dots \times (A_n \cdot B_n). \\ \partial A \cdot B &= \sum_{i=1}^n (-1)^{\sum_{j<i} |A_j|} (A_1 \times \dots \times \partial A_i \times \dots \times A_n) \cdot (B_1 \times \dots \times B_n) \\ &= (-1)^{\varepsilon(A, B)} \sum_{i=1}^n (-1)^{\sum_{j<i} |A_j| - \sum_{j<i} |B_j|} (A_1 \cdot B_1) \\ &\quad \times \dots \times (\partial A_i \cdot B_i) \times \dots \times (A_n \cdot B_n). \\ (-1)^{|A|} A \cdot \partial B &= (-1)^{|A|} \sum_{i=1}^n (-1)^{\sum_{j<i} |B_j|} (A_1 \times \dots \times A_n) \cdot (B_1 \times \dots \times \partial B_i \\ &\quad \times \dots \times B_n) = (-1)^{|A|} \sum_{i=1}^n (-1)^{\sum_{j<i} |B_j| + \varepsilon(A, B) - \sum_{j>i} |A_j|} (A_1 \cdot B_1) \\ &\quad \times \dots \times (A_i \cdot \partial B_i) \times \dots \times (A_n \cdot B_n) \\ &= (-1)^{\varepsilon(A, B)} \sum_{i=1}^n (-1)^{\sum_{j<i} |B_j| + \sum_{j \leq i} |A_j|} (A_1 \cdot B_1) \\ &\quad \times \dots \times (A_i \cdot \partial B_i) \times \dots \times (A_n \cdot B_n). \end{aligned}$$

Therefore we have $\partial(A \cdot B) = \partial A \cdot B + (-1)^{|A|} A \cdot \partial B.$

PROOF of (2.7).

Let $A = (a_{ij}) \in \mathfrak{A}^n$, then

$$\partial \rho A = (-1)^{\varepsilon(A)} \{ \partial A_n \times A_1 \times \dots \times A_{n-1} + (-1)^{|A_n|} A_n \times \partial(A_1 \times \dots \times A_{n-1}) \},$$

and

$$\begin{aligned} \rho \partial A &= \rho \{ \partial(A_1 \times \dots \times A_{n-1}) \times A_n + (-1)^{\sum_{j=1}^{n-1} |A_j|} A_1 \times \dots \times A_{n-1} \times \partial A_n \} \\ &= (-1)^{\varepsilon(A) - |A_n|} A_n \times \partial(A_1 \times \dots \times A_{n-1}) + (-1)^{\varepsilon(A)} \partial A_n \\ &\quad \times (A_1 \times \dots \times A_{n-1}). \end{aligned}$$

Hence we have $\partial \rho A = \rho \partial A$, and the result follows.

PROOF of (2.8).

$$\rho(A \cdot B) = (-1)^{\varepsilon(A, B) + \varepsilon(A \cdot B)} (A_n \cdot B_n) \times (A_1 \cdot B_1) \times \dots \times (A_{n-1} \cdot B_{n-1})$$

and

$$\rho A \cdot \rho B = (-1)^{\varepsilon(A) + \varepsilon(B) + \varepsilon(\rho A, \rho B)} (A_n \cdot B_n) \times (A_1 \cdot B_1) \times \dots \times (A_{n-1} \cdot B_{n-1}).$$

Therefore, it will be sufficient for our purpose, to show

$$\varepsilon(A, B) + \varepsilon(A \cdot B) = \varepsilon(A) + \varepsilon(\rho A, \rho B) \pmod{2},$$

and this may be proved by direct calculation.

§4. First of all, we define homomorphisms $\eta_\alpha^\varepsilon : \mathfrak{A}_q^n \rightarrow \mathfrak{A}_{q+1}^n$, $\varepsilon = 0$ or 1, and $\alpha = 1, 2, \dots, q+1$ as follows:

$$\eta_\alpha^\varepsilon(a_{ij}) = (b_{kl}),$$

where

$$b_{kl} = \begin{cases} a_{kl} & \text{for } l < \alpha, \\ \varepsilon & \text{for } l = \alpha, \\ a_{k, l+1} & \text{for } l > \alpha. \end{cases}$$

Put

$$\eta_{\alpha_r}^{\varepsilon_r} \dots \eta_{\alpha_1}^{\varepsilon_1} = \eta_{\alpha_r}^{\varepsilon_r} \circ \dots \circ \eta_{\alpha_1}^{\varepsilon_1}.$$

Then we define a sequence of homomorphisms $D_i : \mathfrak{A}_q^1 \rightarrow \mathfrak{A}_q^n$ satisfying usual boundary formula, i. e.,

$$(i) \quad \partial D_i + (-1)^{i+1} D_i \partial = \begin{cases} \bar{\Delta} D_{i-1}, & \text{for even } i, \\ \Delta D_{i-1}, & \text{for odd } i; \end{cases}$$

where

$$\bar{\Delta} = \sum_{\nu=0}^{n-1} \rho^\nu, \quad \text{and} \quad \Delta = \rho - 1.$$

(ii)

$$\eta \circ D_i = D_i \circ \eta.$$

(iii)

$$D_i = 0 \quad \text{for } i < 0.$$

In fact these D_i are given explicitly for $q = 1$ by STEENROD [3]. Precisely speaking, he gives an equivariant chain map²⁾ $\phi_*: W \otimes I \rightarrow I^n$ as follows:

$$\begin{aligned} \phi_*(e_0 \otimes U) &= U^n, \quad \phi_*(e_0 \otimes V) = V^n; \\ \phi_*(e_j \otimes U) &= \phi_*(e_j \otimes V) = 0 \quad \text{for } j > 0; \\ (*) \quad \phi_*(e_{2i} \otimes I) &= i! \sum_{a,b} U^{a_0} I V^{b_0} I U^{a_1} I V^{b_1} \dots I U^{a_i} I V^{b_i}, \\ (**) \quad \phi_*(e_{2i+1} \otimes I) &= i! \sum_{a,b} I U^{a_0} I V^{b_0} I U^{a_1} I V^{b_1} \dots I U^{a_i} I V^{b_i}, \end{aligned}$$

where the exponents a, b are ≥ 0 , and the sum of a 's and b 's = $n - 2i - 1$ for $2i$ and $n - 2i - 2$ for $2i + 1$, and in both cases the sum is taken over all possible distinct choices of the a 's and b 's.

In our terms, they are

$$D_0(\varepsilon) = \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix} n\text{-factors}, \quad D_i(\varepsilon) = 0 \quad \text{for } i > 0, \\ \varepsilon = 0 \text{ or } 1;$$

and $D_i(1) = i! \sum A^{a,b} \in \mathfrak{A}_1^n$, $|A^{a,b}| = i + 1$, $A^{a,b}$ is obtained from (*) or (**) by transposing rows into columns and replacing U, V and I by 0, 1 and t respectively.

Easy consequences:

$$(4.1) \quad D_0(\bar{1}) = \sum_{j=1}^n \xi_j, \quad \xi_j = \begin{pmatrix} 0 \\ \vdots \\ t \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad j\text{-th.}$$

$$(4.2) \quad D_i(\bar{1}) = 0 \quad \text{for } i > n - 1.$$

$$(4.3) \quad D_{n-1}(\bar{1}) = \left[\frac{n-1}{2} \right]! \begin{pmatrix} t \\ \vdots \\ \vdots \\ t \end{pmatrix}; \quad [\quad] \text{ means Gauss' symbol.}$$

For $q \geq 1$ we may define them by the next reduction formulas:

$$(4.4) \quad D_i(\gamma_j^\varepsilon \bar{q}) = \gamma_j^\varepsilon D_i(\bar{q}),$$

$$(4.5.1) \quad D_{2i}(\overline{q+1}) = \sum_{j=0}^i D_{2j}(\bar{1}) \cdot D_{2i-2j}(\bar{q}) + \sum_{j=0}^{i-1} D_{2j+1}(\bar{1}) * D_{2i-2j-1}(\bar{q}),$$

$$(4.5.2) \quad D_{2i+1}(\overline{q+1}) = \sum_{j=0}^i \{ D_{2j+1}(\bar{1}) \cdot \rho D_{2i-2j}(\bar{q}) - D_{2i-2j}(\bar{1}) \cdot D_{2j+1}(\bar{q}) \},$$

2) Notations as in [2].

where $\bar{q} = (t, \dots, t) \in \mathfrak{A}_q^1$, and $D_j(\bar{a}) * D_k(\bar{b}) = \sum_{n-1 \geq \lambda \geq \mu \geq 0} \rho^\lambda D_j(\bar{a}) \cdot \rho^\mu D_k(\bar{b})$.

REMARKS.

(1) For $i = 0$, from (4.5.1) we have the following formula :

$$D_0(\bar{q}) = \{D_0(\bar{1})\}^q \text{ for } q \geq 1 \text{ (power in } \cdot \text{-product).}$$

(2) When $n = 2$, since $D_1(\bar{1}) = \begin{pmatrix} t \\ t \end{pmatrix}$ by (4.3), it holds

$$D_1(\bar{1}) = -\rho D_1(\bar{1}).$$

Therefore (4.5.1) and (4.5.2) may be written in the form

$$D_i(\bar{q} + \bar{1}) = (-1)^{i+1} \{D_1(\bar{1}) \cdot \rho D_{i-1}(\bar{q}) - D_0(\bar{1}) \cdot D_i(\bar{q})\}.$$

Thus we obtain the formulas for $n = 2$.³⁾

$$\begin{aligned} D_{2i}(\bar{q}) &= \sum_k (-1)^\varepsilon D_0(\bar{1})^{k_1} \cdot D_1(\bar{1}) \cdot (\rho D_0(\bar{1}))^{k_2} \cdot D_1(\bar{1}) \cdot D_0(\bar{1})^{k_3} \cdot D_1(\bar{1}) \cdot \\ &\quad \cdot (\rho D_0(\bar{1}))^{k_4} \cdot D_1(\bar{1}) \cdot \dots \cdot (\rho D_0(\bar{1}))^{k_{2i}} \cdot D_1(\bar{1}) \cdot D_0(\bar{1})^{k_{2i+1}} (i > 0) \\ D_{2i+1}(\bar{q}) &= \sum_k (-1)^{\varepsilon'} D_0(\bar{1})^{k_1} \cdot D_1(\bar{1}) \cdot (\rho D_0(\bar{1}))^{k_2} \cdot D_1(\bar{1}) \cdot D_0(\bar{1})^{k_3} \cdot D_1(\bar{1}) \cdot \\ &\quad \cdot (\rho D_0(\bar{1}))^{k_4} \cdot D_1(\bar{1}) \cdot \dots \cdot D_0(\bar{1})^{k_{2i+1}} \cdot D_1(\bar{1}) \cdot (\rho D_0(\bar{1}))^{k_{2i+2}} (i \geq 0). \end{aligned}$$

where $\varepsilon = \sum_{j=1}^i k_{2j}$ and $\varepsilon' = \sum_{j=0}^i k_{2j+1}$.

The exponents k_j are ≥ 0 , and the sum of the k_j 's is $q - 2i$ in the first case, and $q - 2i - 1$ in the second. In both cases the sum is taken over all possible distinct choices of the k_j 's.

§5. Before proceeding to prove the validity of these formulas, we give several lemmas which are convenient for our purpose.

LEMMA 1. $\rho\eta = \eta\rho$.

LEMMA 2. *It holds*

$$(4.6.1) \quad \begin{aligned} D_{2i+1}(\partial\bar{q} + \bar{1}) &= \partial D_0(\bar{1}) \cdot D_{2i+1}(\bar{q}) \\ &\quad - \sum_{j=0}^i \{D_{2j+1}(\bar{1}) \cdot \rho D_{2i-2j}(\partial\bar{q}) - D_{2i-2j}(\bar{1}) \cdot D_{2j+1}(\partial\bar{q})\}, \end{aligned}$$

and

$$(4.6.2) \quad \begin{aligned} D_{2i}(\partial\bar{q} + \bar{1}) &= \partial D_0(\bar{1}) \cdot D_{2i}(\bar{q}) \\ &\quad - \left\{ \sum_{j=0}^i D_{2j}(\bar{1}) \cdot D_{2i-2j}(\partial\bar{q}) + \sum_{j=0}^i D_{2j+1}(\bar{1}) \cdot D_{2i-2j-1}(\partial\bar{q}) \right\}. \end{aligned}$$

LEMMA 3. $\partial(A * B) = \partial A * B + (-1)^{|A|} A * \partial B$.

LEMMA 4. $\bar{\partial}(A * B) = A \cdot \bar{\partial}B - \bar{\partial}A \cdot B$.

3) Mr. Araki gets essentially the same result, C^f . [6].

LEMMA 5. $\mathcal{A}A * B = A \cdot \bar{\mathcal{A}}B - \bar{\mathcal{A}}(A \cdot B).$

LEMMA 6. $A * \mathcal{A}B = -\bar{\mathcal{A}}A \cdot B + \bar{\mathcal{A}}(A \cdot \rho B).$

LEMMA 1 is easily seen by definition.

Concerning with LEMMA 2, both formulas may be shown by the same way, therefore we give here only the proof of (3.2.1).

Put $\gamma_k = \gamma_k^0 - \gamma_k^1.$

At first, for $q = 1$

$$\begin{aligned} D_{2i+1}(\partial\bar{2}) &= D_{2i+1}(-\gamma_1\bar{1} + \gamma_2\bar{1}) = -\gamma_1 D_{2i+1}(\bar{1}) + \gamma_2 D_{2i+1}(\bar{1}) \\ &= \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} D_{2i+1}(\bar{1}) - \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} D_{2i+1}(\bar{1}) + D_{2i+1}(\bar{1}) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - D_{2i+1}(\bar{1}) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \partial D_0(\bar{1}) \cdot D_{2i+1}(\bar{1}) - D_{2i+1}(\rho D_0(\partial\bar{1})). \end{aligned}$$

Assume that (4.6.1) holds for all integers less than q . Then we have

$$\begin{aligned} D_{2i+1}(\partial\bar{q}+1) &= D_{2i+1}\left(\sum_{k=1}^{q+1} (-1)^k \gamma_k \bar{q}\right) = -\gamma_1 D_{2i+1}(\bar{q}) + \sum_{k=2}^{q+1} (-1)^k \gamma_k D_{2i+1}(\bar{q}) \\ &= -\gamma_1 D_{2i+1}(\bar{q}) + \sum_{k=2}^{q+1} (-1)^k \gamma_k \sum_{j=0}^i \{D_{2i+1}(\bar{1}) \cdot \rho D_{2i-2j}(\bar{q}-1) \\ &\quad - D_{2i-2j}(\bar{1}) \cdot D_{2j+1}(\bar{q}-1)\} \\ &= -\gamma_1 D_{2i+1}(\bar{q}) - \sum_{j=0}^i \{D_{2j+1}(\bar{1}) \cdot \rho D_{2i-2j}(\sum_{k=1}^q (-1)^k \gamma_k (\bar{q}-1)) \\ &\quad - D_{2i-2j}(\bar{1}) \cdot D_{2j+1}(\sum_{k=1}^q (-1)^k \gamma_k (\bar{q}-1))\} \\ &= \partial D_0(\bar{1}) \cdot D_{2i+1}(\bar{q}) - \sum_{j=0}^i \{D_{2j+1}(\bar{1}) \cdot \rho D_{2i-2j}(\partial\bar{q}) \\ &\quad - D_{2i-2j}(\bar{1}) \cdot D_{2j+1}(\partial\bar{q})\}. \end{aligned}$$

LEMMA 3 is also an easy consequence of definitions and (2.7).

PROOF of LEMMA 4.

$$\begin{aligned} \mathcal{A}(A * B) &= \rho(A * B) - A * B \\ &= \sum_{n-1 \geq \lambda \geq \mu \geq 0} \rho^{\lambda+1} A \cdot \rho^{\mu+1} B - \sum_{n-1 \geq \lambda \geq \mu \geq 0} \rho^{\lambda} A \cdot \rho^{\mu} B \\ &= \sum_{\mu=1}^n A \cdot \rho^{\mu} B - \sum_{\lambda=0}^{n-1} \rho^{\lambda} A \cdot B = A \cdot \bar{\mathcal{A}}B - \bar{\mathcal{A}}A \cdot B. \quad \text{Q.E.D.} \end{aligned}$$

Since proofs of LEMMAS 5 and 6 are very similar with that of the LEMMA 4, we will omit the proof.

Proof of (4.5.1).

$$\begin{aligned}
\partial \left(\sum_{j=0}^i D_{2j}(\bar{1}) \cdot D_{2i-2j}(\bar{q}) \right) &= \partial (D_0(\bar{1}) \cdot D_{2i}(\bar{q}) + D_{2i}(\bar{1}) \cdot D_0(\bar{q})) \\
&\quad + \sum_{j=1}^{i-1} \partial (D_{2j}(\bar{1}) \cdot D_{2i-2j}(\bar{q})) \\
&= \partial D_0(\bar{1}) \cdot D_{2i}(\bar{q}) - D_0(\bar{1}) \cdot \bar{\Delta} D_{2i-1}(\bar{q}) - D_0(\bar{1}) \cdot D_{2i}(\partial \bar{q}) \\
&\quad + \bar{\Delta} D_{2i-1}(\bar{1}) \cdot D_0(\bar{q}) - D_{2i}(\bar{1}) \cdot D_0(\partial \bar{q}) \\
&\quad + \sum_{j=1}^{i-1} \{ \bar{\Delta} D_{2j-1}(\bar{1}) \cdot D_{2i-2j}(\bar{q}) - D_{2j}(\bar{1}) \cdot \bar{\Delta} D_{2i-2j-1}(\bar{q}) \\
&\quad \quad - D_{2j}(\bar{1}) \cdot D_{2i-2j}(\partial \bar{q}) \} \\
&= \partial D_0(\bar{1}) \cdot D_{2i}(\bar{q}) - \sum_{j=0}^i D_{2j}(\bar{1}) \cdot D_{2i-2j}(\partial \bar{q}) \\
&\quad + \sum_{j=0}^{i-1} \{ \bar{\Delta} D_{2j+1}(\bar{1}) \cdot D_{2i-2j-2}(\bar{q}) - D_{2j}(\bar{1}) \cdot \bar{\Delta} D_{2i-2j-1}(\bar{q}) \}. \\
\partial \sum_{j=0}^{i-1} \{ D_{2j+1}(\bar{1}) * D_{2i-2j-1}(\bar{q}) \} &= \sum_{j=0}^{i-1} \{ \partial D_{2j+1}(\bar{1}) * D_{2i-2j-1}(\bar{q}) \\
&\quad + D_{2j+1}(\bar{1}) * \partial D_{2i-2j-1}(\bar{q}) \} \\
&= \sum_{j=0}^{i-1} \{ D_{2j}(\bar{1}) * D_{2i-2j-1}(\bar{q}) + D_{2j+1}(\bar{1}) * \bar{\Delta} D_{2i-2j-1}(\bar{q}) \\
&\quad \quad - D_{2j+1}(\bar{1}) * D_{2i-2j-1}(\partial \bar{q}) \} \\
&= \sum_{j=0}^{i-1} \{ D_{2j}(\bar{1}) \cdot \bar{\Delta} D_{2i-2j-1}(\bar{q}) - \bar{\Delta} (D_{2j}(\bar{1}) \cdot D_{2i-2j-1}(\bar{q})) \\
&\quad \quad - \bar{\Delta} D_{2j+1}(\bar{1}) \cdot D_{2i-2j-2}(\bar{q}) \\
&\quad \quad + \bar{\Delta} (D_{2j+1}(\bar{1}) \cdot \rho D_{2i-2j-2}(\bar{q})) - D_{2j+1}(\bar{1}) * D_{2i-2j-1}(\partial \bar{q}) \} \\
&= - \sum_{j=0}^{i-1} \{ D_{2j+1}(\bar{1}) * D_{2i-2j-1}(\partial \bar{q}) \} + \sum_{j=0}^{i-1} \bar{\Delta} \{ D_{2j+1}(\bar{1}) \cdot \rho D_{2i-2j-2}(\bar{q}) \\
&\quad \quad - D_{2j}(\bar{1}) \cdot D_{2i-2j-1}(\bar{q}) \} - \sum_{j=0}^{i-1} \{ \bar{\Delta} D_{2j+1}(\bar{1}) \cdot D_{2i-2j-2}(\bar{q}) \\
&\quad \quad - D_{2i}(\bar{1}) \cdot \bar{\Delta} D_{2i-2j-1}(\bar{q}) \}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\partial D_{2i}(\bar{q} + \bar{1}) &= \partial D_0(\bar{1}) \cdot D_{2i}(\bar{q}) - \sum_{j=0}^i D_{2j}(\bar{1}) \cdot D_{2i-2j}(\bar{q}) \\
&\quad + \sum_{j=0}^{i-1} D_{2j+1}(\bar{1}) * D_{2i-2j-1}(\partial \bar{q}) \\
&\quad + \sum_{j=0}^{i-1} \{ D_{2j+1}(\bar{1}) \cdot \rho D_{2i-2j-2}(\bar{q}) - D_{2j}(\bar{1}) \cdot D_{2i-2j-1}(\bar{q}) \}
\end{aligned}$$

$= D_{2i}(\partial \bar{q} + \bar{1}) + \bar{J} D_{2i-1}(\bar{q} + \bar{1})$, by (4.6.2) and the fact that

$$\sum_{j=0}^{i-1} D_{2j}(\bar{1}) \cdot D_{2i-2j-1}(\bar{q}) = \sum_{j=0}^{i-1} D_{2i-2j-2}(\bar{1}) \cdot D_{2j+1}(\bar{q}). \quad \text{Q.E.D.}$$

(4.5.2) may be proved similarly, using (4.6.1), LEMMA 4 and elementary properties.

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