NORMAL COORDINATES IN KÄHLERIAN SPACES

By
Tsuneo Suguri and Seitaro Ueno
(Received June 28, 1955)

§0. Introduction. In an analytic Riemannian space $V^n$ in the sense of E. Cartan [1], by making use of a normal coordinates defined in some neighborhood of a point $0 \in V^n$, he determined the fundamental quadratic form from the solution of the fundamental differential equations which are derived from the equations of the structure. In other words, he showed that the fundamental quadratic form at any point $P$ of the neighborhood is uniquely determined if the numerical values at 0 of curvature tensor and all its successive covariant derivatives are known.

We shall show that the same arguments for Kählerian space can be extended. The equations of structure for Hermitean space were given by S. S. Chern [2], but it seems to us that they do not answer our purposes. In this paper we introduce another type of equations of structure which are equivalent to those of S. S. Chern, but are more convenient to our purposes.

We discussed also the symmetric Kählerian spaces as the applications of our equations of structure and the fundamental differential equations. These theories are almost analogous to those of E. Cartan [1].

§1. Equations of structure of S. S. Chern for Hermitean spaces.
Let $M^{(n)}$ be a compact complex analytic Hermitean space of complex dimension $n$, and its intrinsic Hermitean differential form with respect to an admissible complex local coordinate system $z^{(1)}$ in each neighborhood be defined by

1) Latin indices $i, j, k, \ldots$ take the values $1, 2, \ldots, n$; while the Greek indices $\alpha, \beta, \gamma, \ldots$ take the values $1, 2, \ldots, n, n+1, \ldots, 2n$. The symbol such as $n+i$ means $n+i$ for $\alpha = i$, and $i$ for $\alpha = n+i$. 
(1.1) \[ ds^2 = a_{ij} dz^i d\bar{z}^j, \quad (a_{ij} = \overline{a_{ji}}). \]

Then we can take \( n \) linearly independent differential forms \( \varphi_i \) in the coordinate neighborhood so as to satisfy

(1.2) \[ ds^2 = \sum_{i=1}^{n} \varphi_i \varphi_i. \]

They are determined up to unitary transformations. After the suitable considerations S. S. Chern [2] determines the following equations of structure:

(1.3) \[
\begin{align}
\{ d\varphi_j &= \varphi_k \wedge \varphi_{kj} + \Omega_{j}^k, \\
\{ d\varphi_{jk} &= \varphi_{jh} \wedge \varphi_{hj} + \Omega_{jk}, \\
\{ \Omega_{j}^k &= A_{jkh} \varphi_h \wedge \varphi_k, \\
\{ \Omega_{jk} &= B_{jkh} \varphi_h \wedge \varphi_k,
\end{align}
\]

where

(1.5) \[ \varphi_{jk} + \varphi_{kj} = 0, \quad \Omega_{jk} + \Omega_{kj} = 0, \]

(1.6) \[ A_{jkh} = -A_{kjh}, \quad B_{jkh} = B_{kjh}. \]

Furthermore Bianchi identities are given by

(1.7) \[
\begin{align}
\{ d\varphi_j - \Omega_{hj} \wedge \varphi_h + \varphi_{hj} = 0, \\
\{ d\Omega_{jk} + \Omega_{jh} \wedge \varphi_h - \varphi_{jh} \wedge \Omega_{hk} = 0.
\end{align}
\]

The differential forms \( \Omega_j, \Omega_{jk} \) are called respectively the \textit{torsion form} and \textit{curvature form} and the tensors \( A_{jkh}, B_{jkh} \) are called the \textit{torsion tensor} and \textit{curvature tensor}.

We consider the compact Riemannian space \( V^{2n} \) of real dimension \( 2n \) which is a real representation of our Hermitean space \( M^{(n)} \), then its intrinsic quadratic differential form is of the following form:

(1.8) \[ ds^2 = h_{ab} dx^a dx^b = \sum_{\alpha=1}^{2n} \omega^\alpha_n. \]

We have the well-known equations of structure for \( V^{2n} \)

(1.9) \[
\begin{align}
\{ d\omega_a &= \omega_{\beta} \wedge \omega_{\beta a}, \\
\{ d\omega_{ab} &= \omega_{a\beta} \wedge \omega_{\beta b} + \Omega_{ab},
\end{align}
\]

(1.10) \[ \omega_{a\beta} + \omega_{\beta a} = 0, \quad \omega_{\alpha\beta} + \omega_{\beta \alpha} = 0, \]

(1.11) \[ \Omega_{\alpha\beta} = R_{\alpha\beta\gamma} \omega_{\gamma} \wedge \omega_{\beta}. \]

The relations between the differential forms \( \varphi_j, \varphi_{jk} \) and \( \omega_a, \omega_{a\beta} \) are given by S. S. Chern [3] as follows:

(1.12) \[ \varphi_j = \omega_j + \sqrt{-1} \omega_{n+j}, \]
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\begin{equation}
\varphi_{jk} = \omega_{jk} + \sqrt{-1} \omega_{j n+k}^\ast + \frac{\sqrt{-1}}{2} (T_{k hj} + T_{j nk}) \varphi^*_h - \frac{\sqrt{-1}}{2} T_{h kj} \varphi^*_h,
\end{equation}

or

\begin{equation}
\varphi^*_{jk} = \omega^*_{jk} + \sqrt{-1} \omega^*_{j n+k} + \frac{\sqrt{-1}}{2} (T^*_{k hj} + T^*_{j nk}) \omega^*_h - \frac{\sqrt{-1}}{2} (T^*_{h kj} + T^*_{k jh} + T^*_{j nk}) \omega^*_{h+k},
\end{equation}

where the tensor \( A_{jnk} \) is defined by

\begin{equation}
A_{jnk} = \frac{\sqrt{-1}}{2} T_{sk j}.
\end{equation}

The following relations are important

\begin{equation}
\begin{cases}
\omega_{j n+k} - \omega^*_{k n+j} = \frac{1}{2} (T_{j nk} \varphi^*_h + T_{j kh} \varphi^*_h), \\
- \omega^*_{j k} + \omega^*_{n+j n+k} = - \frac{\sqrt{-1}}{2} (T^*_{j nk} \varphi^*_h - T^*_{j kh} \varphi^*_h).
\end{cases}
\end{equation}

Therefore, if we consider a Kählerian space, that is if the torsion form vanishes identically we have

\[ A_{jnk} = 0 \]

and

\[ \varphi_{jk} = \omega_{jk} + \sqrt{-1} \omega_{j n+k}^\ast. \]

Hence for the Riemannian space \( V^{2n} \) which is a real representation of our Kählerian space \( M^{(\omega)} \), we have the following relations

\[ \omega_{j n+k} = \omega^*_{k n+j}, \quad \omega^*_{j k} = \omega^*_{n+j n+k}. \]

§2. Equations of structure of another type for Hermitean spaces. The equations of structure in the last paragraph are somewhat like to those of Riemannian space, but they are not convenient to our purposes because of the identities (1.5), (1.6). That is, the identities (1.5), (1.6) of the differential forms \( \varphi_{jk} \), \( \omega_{jk} \) and the curvature tensor \( B_{j k l} \) are not similar to the corresponding identities for Riemannian space. Now we introduce the equations of structure of another type which are equivalent to the equations in §1.

(i) Instead of the differential forms \( \psi_i, \overline{\psi}_i ; \omega_i, \overline{\omega}_i \), we define the differential forms \( \theta_i, \overline{\theta}_i \) by
\(\theta_a: \begin{cases} \theta_i = \psi_i, \\ \theta_{n+i} = \overline{\psi}_i. \end{cases}\)

\(\theta_{\alpha}: \begin{cases} \theta_{ij} = \Omega_{ij}, \\ \theta_{n+i+i} = \overline{\Omega}_{ij}. \end{cases}\)

(ii) We replace the differential forms \(\psi_{ij}, \overline{\psi}_i, \Omega_{ij}, \overline{\Omega}_{ij}\) by \(\theta_{\alpha\beta}, \theta_{\alpha\beta}\)

\[\theta_{\alpha\beta}: \begin{cases} \theta_{ij} = \theta_{n+i+i+j} = 0, \\ \theta_{n+i+j} = \psi_{ij}, \theta_{n+i+j} = \overline{\psi}_{ij}. \end{cases}\]

\[\theta_{\alpha\beta}: \begin{cases} \theta_{ij} = \theta_{n+i+i+j} = 0, \\ \theta_{n+i+j} = \Omega_{ij}, \theta_{n+i+j} = \overline{\Omega}_{ij}. \end{cases}\]

Remembering the relations (1.5) we can easily see the following equations

\(\theta_{\alpha\beta} + \theta_{\beta\alpha} = 0, \quad \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0.\)

(iii) Now we consider the following objects \(\tilde{t}^i, \tilde{t}_a, \ldots\) with \(2n\) indices and call it a vector, a tensor, ...

\[t^i \rightarrow \tilde{t}^i: \begin{cases} \tilde{t}^i = t^i, \\ \tilde{t}_{n+i} = \overline{t}_i. \end{cases}\]

\[t_i \rightarrow \tilde{t}_{n+i}: \begin{cases} \tilde{t}_{n+i} = t_i, \\ \tilde{t}_i = \overline{t}_i. \end{cases}\]

\[t^{ij} \rightarrow \tilde{t}^{\alpha\beta}: \begin{cases} \tilde{t}^{ij} = t^{ij}, \quad \text{otherwise } = 0, \\ \tilde{t}^{i+i+n+i} = \overline{t}^{ij}. \end{cases}\]

\[t^i_j \rightarrow \tilde{t}^{i}_{n+i}: \begin{cases} \tilde{t}^i_{n+i} = t^i_j, \quad \text{otherwise } = 0. \end{cases}\]

(iv) In this manner we define the new tensors \(\tilde{A}_{\alpha\beta}, \tilde{B}_{\alpha\beta}\) from the tensors \(A_{jk}, B_{jk\ell}\): \(\tilde{A}_{\alpha\beta}: \begin{cases} \tilde{A}_{jk} = A_{jk}, \\ \tilde{A}_{n+i+n+j+n+k} = \overline{A}_{jk}, \quad \text{otherwise } = 0, \end{cases}\)

\(\tilde{B}_{\alpha\beta}: \begin{cases} \tilde{B}_{jk} = B_{jk}, \\ \tilde{B}_{n+i+n+j+n+k} = \overline{B}_{jk}, \quad \text{otherwise } = 0. \end{cases}\)

Then we can get the following relations in place of (1.6)

\[\tilde{A}_{\alpha\beta} = -\tilde{A}_{\alpha\beta}, \quad \tilde{B}_{\alpha\beta} = \tilde{B}_{\alpha\beta}.\]
Hence setting

\[ (2.7) \quad \theta_{\bar{a}B\bar{t}\bar{o}} = \bar{B}_{\bar{a}n+\bar{B}\bar{t}\bar{o}} - \bar{B}_{Bn+aT\bar{o}}, \]

we get

\[ (2.8) \quad \theta_{\bar{a}B\bar{t}\bar{o}} = -\theta_{\bar{a}nT\bar{o}} = -\theta_{\bar{a}B\bar{t}} \]

and the following representations of the torsion, and the curvature forms

\[ (2.9) \quad \theta_{\bar{a}} = \bar{A}_{\bar{a}B\bar{t}} \theta_{\bar{B}} \wedge \theta_{\bar{T}} , \]

\[ \theta_{\bar{n}\bar{B}} = \frac{1}{2} \theta_{\bar{a}B\bar{t}} \theta_{\bar{T}} \wedge \theta_{\bar{n}} . \]

We call $\bar{A}_{\bar{a}B\bar{t}}$, $\theta_{\bar{a}B\bar{t}}$ the **torsion tensor** and **curvature tensor** in our sense.

From the above preparations we can replace the equations of structure (1.3) with the following **equations of structure in our sense**

\[ (2.10) \quad \begin{cases} d\theta_{\bar{a}} = \theta_{\bar{a}} \wedge \theta_{\bar{a}n+a} + \theta_{\bar{a}} , \\ d\theta_{\bar{n}\bar{B}} = \theta_{\bar{n}\bar{a}} \wedge \theta_{\bar{n}a+\bar{B}} + \theta_{\bar{n}\bar{B}} . \end{cases} \]

Making use of the exterior derivative we get the Bianchi identities

\[ (2.11) \quad \begin{cases} d\theta_{\bar{a}} + \theta_{\bar{a}} \wedge \theta_{\bar{a}n+a} - \theta_{\bar{a}} \wedge \theta_{\bar{a}n+a} = 0 , \\ d\theta_{\bar{n}\bar{B}} + \theta_{\bar{n}\bar{B}} \wedge \theta_{\bar{n}a+\bar{B}} - \theta_{\bar{n}\bar{B}} \wedge \theta_{\bar{n}a+\bar{B}} = 0 . \end{cases} \]

When we consider a Kählerian space, we see that the Bianchi identities take the following forms because of $\theta_{\bar{a}} = 0$:

\[ (2.12) \quad \begin{cases} (i) \quad \theta_{\bar{a}B\bar{t}\bar{o}} + \theta_{\bar{a}T\bar{o}} - \theta_{\bar{a}B\bar{t}} = 0 , \\ (ii) \quad \theta_{\bar{a}B\bar{t}\bar{o}} + \theta_{\bar{a}T\bar{o}} + \theta_{\bar{a}B\bar{t}} = 0 . \end{cases} \]

Where the semi-colon means a covariant derivative, that is, the covariant differentials $D\tilde{t}^o$ and $D\tilde{t}_n$ of a contravariant vector $\tilde{t}^o$ and of a covariant vector $\tilde{t}_n$ are respectively defined by

\[ (2.13) \quad \begin{cases} D\tilde{t}^o = d\tilde{t}^o + \tilde{t}^o \theta_{\bar{n}a+\bar{o}} , \\ D\tilde{t}_n = d\tilde{t}_n - \tilde{t}_n \theta_{\bar{n}a+\bar{o}} . \end{cases} \]

As in the Riemannian spaces, it follows easily from (2.8) and (2.12) i) that

\[ (2.14) \quad \theta_{\bar{a}B\bar{t}\bar{o}} = \theta_{\bar{T}\bar{a}B\bar{t}} . \]

§3. **Normal coordinates in Kählerian spaces.** From now we shall consider an Hermitean space $M^{(\alpha)}$ without torsion, i.e. a Kählerian space. Making use of a suitably defined real valued
function $U(z, \bar{z})$, the fundamental tensor $a_{ij}(z, \bar{z})$ and the coefficient of connection $l^i_{jk}(z, \bar{z})$ can be written respectively in the form:

$$a_{ij}(z, \bar{z}) = \frac{\partial^i U(z, \bar{z})}{\partial z^j \partial \bar{z}^k},$$

$$l^i_{jk}(z, \bar{z}) = a^{ki} \frac{\partial a_{jk}}{\partial z^l} = a^{ki} \frac{\partial^i U(z, \bar{z})}{\partial z^j \partial \bar{z}^k}.\)$$

A curve

$$z^i = z^i(s), \quad s: \text{ arc length},$$

defined in some coordinate neighborhood of a point $0 \in M^{(0)}$ is called a geodesic in a space, if it satisfies the system of differential equations

$$\frac{d^2 z^i}{ds^2} + l^i_{jk} \frac{dz^j}{ds} \frac{dz^k}{ds} = 0.$$

Take the point $0$ as origin, and let

$$z^i = z^i(s, a)$$

be the solution of (3.3) satisfying the initial condition

$$z^i = 0, \quad \frac{dz^i}{ds} = a^i, \text{ for } s = 0.$$

Then it is well known that there exist coordinate systems $z^{*i}$ defined in some neighborhood $U$ of the point $0$ such that the equations (3.4) reduce to

$$z^{*i} = a^i s.$$

We call $z^{*i}$ the normal coordinates at the point $0 \in M^{(0)}$. The following facts may be proved as in the Riemannian spaces:

(i) There exists one and only one geodesic through $0$ and tangent to a preassigned direction $a^i$ at the point $0$.

(ii) There determines a unique geodesic joining two points $0$ and $P \in U$.

(iii) Every point $P \in U$ has a neighborhood covered by a normal coordinate system at $P$ and containing $U$.

(iv) The geodesic joining $0$ and $P$ is the only curve of shortest length in $M^{(0)}$ joining these two points.
We take a unitary orthogonal frame $R_0$ at the point 0 and displace it parallelly along the geodesic $\tilde{OP}$ to any point $P \in U$, thus we get a family of unitary orthogonal frames $R_p$ in the neighborhood $U$. The family $(R_p)$ determined by the original frame $R_0$ is called adapted to the normal coordinate. Therefore every tangent vector of a geodesic $\gamma$ through the point 0 has the same components with respect to the frame of the adapted family.

In the following, unless otherwise stated, we confine ourselves to normal coordinate system, and denote it simply by $z^i$, etc. instead of $z^{*i}$, etc.

For the normal coordinate $z^i$ we see from (3.3) and (3.6) that

\[(3.7) \quad I^{i}_{jk} a^j a^k = 0,\]

and therefore

\[(3.8) \quad I^{i}_{jk} z^j z^k = 0.\]

Using the analogous theorem to S. S. Chern [4, Chap. V, Theorem 1.2], we can easily find because of (8.8) that

\[(3.9) \quad B_{ijk} z^j z^k = 0.\]

By making use of the manner explained in §2, we define the following objects:

\[z^i \rightarrow \xi^a: \quad \begin{cases} \xi^i = z^i, \\ \xi^{n+i} = \tilde{z}^i, \end{cases}\]

\[a^i \rightarrow A^a: \quad \begin{cases} A^i = a^i, \\ A^{n+i} = \tilde{a}^i, \end{cases}\]

\[I^{i}_{jk} \rightarrow A_{ij}^a: \quad \begin{cases} A_{jk}^i = I^{i}_{jk}, \\ A_{n+i,j+n+k}^i = \tilde{T}_{jk}, \end{cases}\]

otherwise $= 0$.

Then the equations of a geodesic through a point 0 are

\[(3.10) \quad \xi^a = A^a s\]

in terms of a normal coordinate system at the point 0. The relation

\[(3.11) \quad A_{ij}^a \xi^a \xi^i = 0\]

equivalent to (3.8) also characterizes the coordinate system $\xi^a$ as normal.

§4. **Fundamental differential equations.** Using a normal coordinate system defined in a neighborhood of a point $0 \in M^{(0)}$ (being a Kählerian space), a geodesic passing through the point 0 can be shown by the equations
\( (4.1) \)
\[ \zeta^a = A^a t, \]
where \( A^a \) are the components of the tangent vector of a geodesic at 0 with respect to a unitary orthogonal frame \( R_n \), that is they are the constant components of the tangent vector relative to an adapted family of frames, and the parameter \( t \) means a distance along the geodesic from the origin 0.

Now we shall consider the fundamental differential forms \( \theta_{a}, \zeta, d\zeta \) and \( \theta_{ab}, \zeta, d\zeta \) with respect to our normal coordinate system. Let \( \theta_{a}^*, \theta_{ab}^* \) be the linear forms of \( dA^a \) obtained from \( \theta_{a}, \theta_{ab} \) by putting \( dt = 0 \), then we have the following equations in conformity to Riemannian spaces

\[ (4.2) \]
\[ \begin{align*}
\theta_{a}(t, A; dt, dA) &= A^a dt + \theta_{a}^*(t, A; dA), \\
\theta_{ab}(t, A; dt, dA) &= \theta_{ab}^*(t, A; dA),
\end{align*} \]
where we have at the origin \( t = 0 \)

\[ (4.3) \]
\[ \theta_{a}^*(0, A; dA) = 0, \quad \theta_{ab}^*(0, A; dA) = 0. \]

Substituting the equations (4.2) into the equations of structure (2.10), we get

\[ (4.4) \]
\[ \begin{align*}
dA^b \wedge dt + dt \wedge \frac{\partial \theta_{a}^*}{\partial t} + d\theta_{ab}^* &= (A^a dt + \theta_{a}^*) \wedge \theta_{tn+b}^*, \\
dt \wedge \frac{\partial \theta_{ab}^*}{\partial t} + d\theta_{ab}^* &= \theta_{ba}^* \wedge \theta_{n+ab}^* + \frac{1}{2} \theta_{tna}^* (A^a dt + \theta_{a}^*) \wedge (A^b dt + \theta_{b}^*),
\end{align*} \]
where \( d\theta_{a}^*, d\theta_{ab}^* \) are the exterior derivatives of \( \theta_{a}^*, \theta_{ab}^* \) regarding \( t \) as a parameter. Comparing the terms of \( dt \) we can easily see the following differential equations:

\[ (4.5) \]
\[ \begin{align*}
\frac{\partial \theta_{a}^*}{\partial t} &= dA^a + A^a \theta_{tn+b}^*, \\
\frac{\partial \theta_{ab}^*}{\partial t} &= \theta_{tna}^* A^b \theta_{a}^*. 
\end{align*} \]

Following E. Cartan we call them the fundamental differential equations. When we solve the fundamental differential equations under the initial condition (4.3), we get Pfaffian forms \( \theta_{a}^* \) and \( \theta_{ab}^* \). Hence replacing the result into (4.2), we have the fundamental differential forms \( \theta_{a}(\zeta, d\zeta) \) and \( \theta_{ab}(\zeta, d\zeta) \) with respect to our normal coordinate system.

It is obviously shown from (4.5) that \( \theta_{a}^* \) satisfies the differential equation of the second order:
(4.6) \[ \frac{\partial^2 \theta^*_\sigma}{\partial t^2} = \partial_{\gamma + 2\alpha} \delta^\gamma \theta^*_\sigma \]

and the relations

(4.7) \[ \theta^*_\beta = 0, \quad \frac{\partial \theta^*_\beta}{\partial t} = dA^\beta, \text{ for } t = 0. \]

Therefore we can completely determine \( \theta^*_\beta \) as the solution of (4.6) with the initial condition (4.7). Consequently we can state the following important

**Theorem 1.** If we know the curvature tensor \( \theta_{\alpha \beta \gamma} \) with respect to the normal coordinates \( \zeta^\alpha \) relative to the unitary orthogonal frame \( R_0 \) of an adapted family defined in a neighborhood of a point 0 in Kählerian space \( M^{(0)} \), we can uniquely determine the fundamental differential forms \( \theta_\beta (\zeta, d\zeta) \) with respect to the normal coordinate system at the origin 0.

In Kählerian space the fundamental tensor \( a_{ij} (z, \bar{z}) \) are defined by the power series of \( z^i, \bar{z}^\bar{i} \). When we extend the quantities \( a_{ij} (z, \bar{z}) \) and \( a^*\bar{a} \) to \( g_{\alpha \beta} (\zeta) \) and \( g^{*\beta} (\zeta) \) by the manner explained in §2:

(4.8) \[ g_{\alpha \beta} (\zeta) : \begin{cases} g_{i+j} = a_{ij}, & \text{otherwise } = 0, \\ g_{n+i+j} = a_{ij}, & \end{cases} \]

(4.9) \[ g^{*\beta} (\zeta) : \begin{cases} g^{n+j+i} = a_{jk}, & \text{otherwise } = 0, \\ g^{n+i+j} = \bar{a}_{jk}, & \end{cases} \]

we see that \( g_{\alpha \beta} (\zeta) \) and \( g^{*\beta} (\zeta) \) are the convergent power series of \( \zeta^i, \bar{\zeta}^\bar{i}, \ldots, \zeta^{2n} \), and satisfy the relations:

(4.10) \[ g_{\alpha \beta} = g_{\beta \alpha}, \quad g^{*\beta} = g^{*\alpha}, \]

(4.11) \[ A^\alpha_{\beta \tau} = g^{\alpha \rho} \frac{\partial g_{\rho \beta}}{\partial \zeta^\tau}. \]

Therefore the coefficient of connection \( A^\alpha_{\beta \tau} (\zeta) \) and the curvature tensor \( \theta_{\alpha \beta \gamma} (\zeta) \) in our sense can be developed into the power series of \( \zeta^i, \bar{\zeta}^\bar{i}, \ldots, \zeta^{2n} \). Hence with respect to a normal coordinates defined in a neighborhood of any point \( 0 \in M^{(0)} \) the components of the curvature tensor \( \theta_{\alpha \beta \gamma} (P) \) at some point \( P \) on any geodesic \( (\zeta^\alpha = A^\alpha t) \) through 0 are uniquely determined by the numerical values at the origin 0 of \( \theta_{\alpha \beta \gamma} (A^\tau) \) and its successive derivatives with respect to \( t \), that is we have the following expansion:
\begin{equation}
\Theta_{\alpha\beta\gamma\delta}(P) = \Theta_{\alpha\beta\gamma\delta}(0) + \sum_{r=1}^{\infty} \frac{1}{r!} \left( \frac{\partial^r \Theta_{\alpha\beta\gamma\delta}}{\partial t^r} \right)_0 t^r
\end{equation}

and the following equations along the geodesic through 0:

\begin{equation}
\frac{\partial^r \Theta_{\alpha\beta\gamma\delta}}{\partial t^r} = \Theta_{\alpha\beta\gamma\delta;\sigma_1...\sigma_r} A^{\sigma_1} ... A^{\sigma_r}, \quad (r = 1, 2, \ldots).
\end{equation}

From the above considerations and the Theorem 1 we get the following

**Theorem 2.** With respect to a normal coordinates \( \zeta \) defined in a neighborhood of any point \( 0 \in M^{(0)} \), the fundamental differential forms \( \theta_{\alpha}(, \zeta, d\zeta) \) are uniquely determined by (the numerical values of) the curvature tensor \( \Theta_{\alpha\beta\gamma\delta} \) and its successive covariant derivatives at the point 0.

§5. Determination of the fundamental differential form. In the present paragraph, let us try to determine the fundamental metric tensor \( g_{\alpha\beta}(\zeta) \) in terms of the curvature tensor and its successive covariant derivatives. To do this, it is necessary to solve the fundamental differential equations obtained in the last paragraph. Differentiating (4.6) successively with respect to \( t \) and noticing the initial condition (4.7), we easily obtain

\begin{align}
\left( \frac{\partial^2 \theta_{\alpha}}{\partial t^2} \right)_{t=0} &= 0, \\
\left( \frac{\partial^3 \theta_{\alpha}}{\partial t^3} \right)_{t=0} &= \Theta_{(\alpha+\beta\gamma\delta)} A^\gamma A^\delta dA^\alpha, \\
\left( \frac{\partial^4 \theta_{\alpha}}{\partial t^4} \right)_{t=0} &= 2 \Theta_{(\alpha+\beta\gamma\delta\rho)} A^\gamma A^\delta A^\rho dA^\alpha,
\end{align}

therefore the solution of (4.6) is given by

\begin{equation}
\theta_{\alpha}(t, A, dA) = t dA^\alpha + \frac{1}{6} t^3 \Theta_{(\alpha\beta\gamma\delta)} A^\gamma A^\delta dA^\alpha + \frac{1}{12} t^4 \Theta_{(\alpha\beta\gamma\delta\rho)} A^\gamma A^\delta A^\rho dA^\alpha + \ldots.
\end{equation}

Substituting (5.2) into (4.5) and solving it under the initial condition (4.3), we get the solution
\[ \theta^*_\beta(t, A, dA) = \frac{1}{12} t^2 \theta^*_{\beta \alpha \sigma} A^\alpha dA^\alpha + \frac{1}{3} t^3 \theta^*_{\alpha \beta \gamma \rho} A^\beta A^\gamma dA^\rho + \]
\[ + \frac{1}{4!} t^4 \left( 3 \theta^*_{\beta \alpha \gamma \rho} + \theta^*_{\beta \gamma \sigma} + \theta^*_{\alpha \gamma \rho} \right) A^\delta A^\sigma dA^\rho + \ldots \]

Therefore it follows from (4.2) that
\[ \theta_\beta(t, A, dt, dA) = A^\beta dt + t dA^\beta + \frac{1}{6} t^3 \theta^*_{\alpha + \beta \gamma} A^\sigma A^\delta dA^\gamma + \]
\[ + \frac{1}{12} t^4 \theta^*_{\alpha \gamma + \beta \delta \rho} A^\sigma A^\delta A^\epsilon dA^\gamma + \ldots, \]

\[ \theta_{\beta\gamma}(t, A, dt, dA) = \frac{1}{2} t^2 \theta^*_{\beta \gamma \delta} A^\delta dA^\sigma + \frac{1}{3} t^3 \theta^*_{\gamma \delta \beta \rho} A^\gamma dA^\rho + \]
\[ + \frac{1}{4!} t^4 \left( 3 \theta^*_{\gamma \delta \alpha \rho} + \theta^*_{\gamma \delta \sigma} + \theta^*_{\alpha \gamma \rho} \right) A^\delta A^\sigma dA^\rho + \ldots \]

For the normal coordinates \( \zeta^\alpha \) it is evident that (4.1) and the relation
\[ d\zeta^\alpha = tdA^\alpha + A^\alpha dt \]
hold good, hence we get, from (5.4) and (5.5) and noticing (2.8), the solution of the fundamental differential equations
\[ \theta_\beta(\zeta, d\zeta) = d\zeta^\beta + \frac{1}{6} \theta^*_{\alpha + \beta \gamma} \zeta^\sigma d\zeta^\gamma + \]
\[ + \frac{1}{12} \theta^*_{\alpha + \beta \gamma \rho} \zeta^\sigma \zeta^\epsilon d\zeta^\gamma + \ldots, \]

\[ \theta_{\beta\gamma}(\zeta, d\zeta) = \frac{1}{2} \theta^*_{\beta \gamma \delta} \zeta^\delta d\zeta^\sigma + \frac{1}{3} \theta^*_{\gamma \delta \beta \rho} \zeta^\gamma d\zeta^\rho + \]
\[ + \frac{1}{4!} \left( 3 \theta^*_{\gamma \delta \alpha \rho} + \theta^*_{\gamma \delta \sigma} + \theta^*_{\alpha \gamma \rho} \right) \zeta^\delta \zeta^\epsilon d\zeta^\rho + \ldots \]
in terms of curvature tensor and its successive covariant derivatives.

To get the relation between the metric \( ds^2 \) at the point \( 0 \in M^{(w)} \) and the metric \( ds'_5 \) of the tangential Euclid-Hermitian space at the point \( 0 \in M^{(w)} \), we decompose (5.6) into the following forms:
\[ \theta_j = dz^j + \frac{1}{6} \theta^*_{\alpha j + \beta \gamma} \zeta^\sigma d\zeta^\gamma + \frac{1}{12} \theta^*_{\alpha j + \beta \gamma \rho} \zeta^\sigma \zeta^\epsilon d\zeta^\gamma + \ldots, \]

\[ \bar{\theta}_j = d\bar{z}^j + \frac{1}{6} \theta^*_{\alpha j + \beta \gamma} \zeta^\sigma d\zeta^\gamma + \frac{1}{12} \theta^*_{\alpha j + \beta \gamma \rho} \zeta^\sigma \zeta^\epsilon d\zeta^\gamma + \ldots. \]

These two metrics are given respectively by
\[ ds^2 = \sum_j \theta_j d\bar{z}^j, \quad ds'_5 = \sum_j dz^j d\bar{z}^j, \]
hence we get from (5.8) and (5.9)
(5.10) \[ ds^2 = ds^2_0 + \frac{1}{6} \left\{ \theta_{\alpha\beta\gamma} + \frac{1}{2} \theta_{\alpha\beta\gamma;\iota} \zeta^\iota \right\} \zeta^\alpha \zeta^\beta d\zeta^\gamma d\zeta^\iota + \ldots. \]

On the other hand it follows easily that

(5.11) \[ ds^2 = \frac{1}{2} g_{\alpha\beta}(\zeta) d\zeta^\alpha d\zeta^\beta. \]

Comparing these two relations (5.10) and (5.11) and noticing (4.10), we get, as the fundamental metric tensor, the following:

(5.12) \[ g_{\alpha\beta}(\zeta) = \alpha_\alpha + \frac{1}{3} \left( \theta_{\alpha\beta\gamma} + \frac{1}{2} \theta_{\alpha\beta\gamma;\iota} \zeta^\iota \right) \zeta^\alpha \zeta^\beta + \ldots. \]

By virtue of the definition (2.7) of \( \theta_{\alpha\beta\gamma} \) and the property (3.9) of the normal coordinates, it is easily verified that \( g_{\alpha\beta}(\zeta) \) thus determined satisfies the condition (4.8).

§6. Symmetry and parallel transport. Let \( U \) be a neighborhood defined in §3 of a point \( 0 \in M^{(n)} \) and take any point \( P \in U \). Join the two points \( 0 \) and \( P \) by the unique geodesic and take the point \( P' \) on the opposite direction of the geodesic \( \overline{0P} \) such that \( \overline{0P'} = \overline{0P} \). We call the correspondence \( P \to P' \) the symmetry with respect to the point \( 0 \). As a consequence of this, a vector \( \zeta \) at a point \( P \) is transformed into a vector \( \zeta' \) at a point \( P' \).

The symmetry with respect to the point \( 0 \) is expressed in terms of a normal coordinate system \( \zeta \) defined in \( U \) as follows:

(6.1) \[ \begin{align*}
\text{(i)} & \quad \zeta^a \to \zeta'^a = -\zeta^a \\
\text{(ii)} & \quad X^a \to X'^a = -X^a
\end{align*} \]

where \( X^a \) and \( X'^a \) are respectively the components of vectors \( X \) and \( X' \) with respect to the natural frames associated to the normal coordinate.

Now at a point \( P' \), let \( X'' \) be a vector of equal magnitude to \( X' \) and of opposite direction to \( X' \). Then the components \( X''^a \) of \( X'' \) with respect to the natural frame are given by

(6.2) \[ X''^a = -X'^a = X^a. \]

On account of the vanishing of the parameters of connection \( A_{\alpha\beta}(\zeta) \) at the origin \( 0 \) of a normal coordinate, it follows that the two vectors \( X \) at a point \( P \) and \( X'' \) at \( P' \) are parallel along the geodesic \( \overline{P0P} \). We call the correspondence \( X \to X'' \) the transport by symmetry.
As a consequence of the last arguments, it follows that the transport by symmetry coincides with the parallel transformation along a geodesic within the infinitesimals of the third and higher order. Following E. Cartan, we observe the infinitesimal of the third order.

We take a unitary orthogonal frame $R_a$ at the point 0, and displace it parallely along the geodesic $\hat{OP}$ to any point $P \in U$ thus we get an adapted family of frames $(R_p)$ in the neighborhood $U$.

Let $X$ be a vector at the point 0 whose components with respect to the natural frame associated to the normal coordinates are $d\zeta$, say. Displace the vector $X$ from 0 to any point $P(\zeta') \in U$ along a geodesic $\hat{OP}$. The vector $X'$ thus obtained has the components $\theta_\beta(\zeta', d\zeta)$ with respect to the unitary orthogonal frame $(R_p)$. Since the components of a vector with respect to the adapted frames are unaltered by paralleled displacement along a geodesic, the components of the vector $X''$, obtained by parallel displacement of the vector $X'$ along a geodesic $\hat{P}0\hat{P}$ from $P$ to $P'$, with respect to the frame $(R_p)$ are $\theta_\beta(\zeta', d\zeta)$. On the other hand, the components of the vector $X'''$ at the point $P'$, obtained from the vector $X'$ at the point $P$ by transport by symmetry with respect to 0, with respect to the frame $(R_p)$ are $-\theta_\beta(-\zeta', -d\zeta) = \theta_\beta(-\zeta', d\zeta)$. Thus we get from (5.6)

$$\theta_\beta(\zeta', d\zeta) - \theta_\beta(-\zeta', d\zeta) = \frac{1}{6} \Theta_{\alpha\beta\gamma\delta\epsilon\zeta} \zeta^\alpha \zeta^\beta \zeta^\gamma \zeta^\delta \zeta^\epsilon + \ldots .$$

Hence, in order that a parallel displacement along a geodesic coincides with a transport by symmetry including the infinitesimals of the third order, it is necessary and sufficient that

$$\Theta_{\alpha\beta\gamma\delta\epsilon\zeta} \zeta^\alpha \zeta^\beta \zeta^\gamma \zeta^\delta \zeta^\epsilon = 0 \quad \text{for any} \quad \zeta'$$

hold good. Noticing (2.8), it follows that these conditions are equivalent to

$$\Theta_{\alpha\beta\gamma\delta\epsilon\zeta} + \Theta_{\alpha\beta\gamma\delta\epsilon\zeta} + \Theta_{\alpha\beta\gamma\delta\epsilon\zeta} + \Theta_{\alpha\beta\gamma\delta\epsilon\zeta} + \Theta_{\alpha\beta\gamma\delta\epsilon\zeta} + \Theta_{\alpha\beta\gamma\delta\epsilon\zeta} = 0 .$$

Using the Bianchi's identities (2.12) suitably, we get from (6.5)

$$\Theta_{\alpha\beta\gamma\delta\epsilon\zeta} + \Theta_{\alpha\beta\gamma\delta\epsilon\zeta} + \Theta_{\alpha\beta\gamma\delta\epsilon\zeta} = 0 .$$

It follows from (6.6) by virtue of the identities (2.8) and (2.12) that

$$\Theta_{\alpha\beta\gamma\delta\epsilon\zeta} = \Theta_{\alpha\beta\gamma\delta\epsilon\zeta}$$

which shows that the tensor $\Theta_{\alpha\beta\gamma\delta\epsilon\zeta}$ is unaltered by the permutations of indices $\beta \leftrightarrow \gamma$ and $\delta \leftrightarrow \epsilon$. We get easily from (6.7), (2.8) and
\[(2.14) \text{ that} \]
\[\theta_{\alpha\beta\gamma\delta;\varepsilon} = -\theta_{\alpha\beta\delta\varepsilon;\gamma}.\]

Using relations (6.8) and (2.12 i) we can get at last
\[(6.9) \]
\[\theta_{\alpha\beta\gamma\delta;\varepsilon} = 0.\]

Thus we obtain the following

**Theorem 3.** In a Kählerian space $M^{(n)}$, in order that a parallel displacement along a geodesic coincides with a transport by symmetry including the infinitesimals of the third order, it is necessary and sufficient that $\theta_{\alpha\beta\gamma\delta;\varepsilon} = 0$.

We proceed to prove the following

**Theorem 4.** In a Kählerian space $M^{(n)}$, if $\theta_{\alpha\beta\gamma\delta;\varepsilon} = 0$ hold good, then a parallel displacement along a geodesic coincides precisely with a transport by symmetry.

**Proof.** Take a point $0 \in M^{(n)}$ as an origin of a normal coordinate system $\zeta^n$ defined in a neighborhood $U$. On account of the assumption $\theta_{\alpha\beta\gamma\delta;\varepsilon} = 0$, along a geodesic $\zeta^n = A^n t$ through a point $0$ we have
\[
\left( \frac{d\theta_{\alpha\beta\gamma\delta}}{dt} \right)_0 = \theta_{\alpha\beta\gamma\delta;\varepsilon} A^\varepsilon = 0,
\]
\[
\left( \frac{d^2\theta_{\alpha\beta\gamma\delta}}{dt^2} \right)_0 = \theta_{\alpha\beta\gamma\delta;\sigma\tau} A^\sigma A^\tau = 0,
\]

etc.,

hence it follows that
\[\theta_{\alpha\beta\gamma\delta}(t) = \theta_{\alpha\beta\gamma\delta}(0) = \text{const.},\]

which shows that $\theta_{\alpha\beta\gamma\delta}(\zeta)$ are absolute constants in the neighborhood $U$ of the origin $0$.

For the proof of the Theorem 4, noticing the relations (4.2), it is sufficient to show that $\theta_{\alpha}(\zeta, d\zeta) = \theta_{\alpha}(-\zeta, d\zeta)$ or
\[(6.10) \]

\[\theta_{\alpha}(t, A, dA) = \theta_{\alpha}(t, -A, dA)\]

hold precisely. Now if we put
\[(6.11) \]

\[
\begin{align*}
\text{(i) } & \quad \theta_{\alpha}(t, A, dA) - \theta_{\alpha}(t, -A, dA) = \tilde{\theta}_{\alpha}, \\
\text{(ii) } & \quad \theta_{\alpha}(t, A, dA) + \theta_{\alpha}(t, -A, dA) = \tilde{\theta}_{\alpha},
\end{align*}
\]

then it follows from (4.5) that
\begin{align}
\left\{ \begin{array}{l}
\frac{\partial \tilde{\theta}_\beta}{\partial t} = A^T \tilde{\theta}_{\gamma a+\beta}, \\
\frac{\partial \tilde{\theta}_{\beta \gamma}}{\partial t} = \theta_{\beta \gamma} \delta A^\delta \tilde{\theta}_\alpha.
\end{array} \right. \\
(6.12)
\end{align}

At the origin \( t = 0 \), it holds that \( \tilde{\theta}_\beta = 0 \), \( \tilde{\theta}_{\beta \gamma} = 0 \) by virtue of (4.3). Therefore the unique solution of (6.12) is given by

\( \tilde{\theta}_\beta = 0 \), \( \tilde{\theta}_{\beta \gamma} = 0 \),

which completes the proof.

**Theorem 5.** In a Kählerian space \( M^{(\nu)} \), in order that a symmetry with respect to any point \( 0 \in M^{(\nu)} \) be an isometric transformation, it is necessary and sufficient that \( \theta_{\alpha \beta \gamma \delta \zeta} = 0 \).

**Proof.** (I) Suppose that \( \theta_{\alpha \beta \gamma \delta \zeta} = 0 \).

(i) By virtue of Theorem 4, it holds that \( \theta_{\alpha}(\zeta, d\zeta) = \theta_{\alpha}(\zeta, d\zeta) \), hence

\( \theta_{\alpha}(\zeta, d\zeta) \theta_{\alpha \beta \gamma \delta \zeta}(\zeta, d\zeta) = \theta_{\alpha}(\zeta, d\zeta) \theta_{\alpha \beta \gamma \delta \zeta}(\zeta, d\zeta) \).

(ii) It is evident that \( \theta_{\alpha}(\zeta, -d\zeta) = -\theta_{\alpha}(\zeta, d\zeta) \), therefore we have

\( \theta_{\alpha}(\zeta, -d\zeta) \theta_{\alpha \beta \gamma \delta \zeta}(\zeta, d\zeta) = \theta_{\alpha}(\zeta, d\zeta) \theta_{\alpha \beta \gamma \delta \zeta}(\zeta, d\zeta) \).

Therefore it follows from (i) and (ii) that

\( \theta_{\alpha}(\zeta, -d\zeta) \theta_{\alpha \beta \gamma \delta \zeta}(\zeta, -d\zeta) = \theta_{\alpha}(\zeta, d\zeta) \theta_{\alpha \beta \gamma \delta \zeta}(\zeta, d\zeta) \),

which shows that a symmetry with respect to a point 0 is an isometry.

(II) Suppose that a symmetry with respect to a point \( O \) is an isometry, then we see from (6.13) that \( g_{\alpha \beta}(-\zeta) = g_{\alpha \beta}(\zeta) \), from which it follows that \( \theta_{\alpha \beta \gamma \delta \zeta}(\zeta) = \theta_{\alpha \beta \gamma \delta \zeta}(\zeta) \) and consequently \( D^{\theta_{\alpha \beta \gamma \delta \zeta}}(-\zeta) = -D^{\theta_{\alpha \beta \gamma \delta \zeta}}(\zeta) \). Therefore at the origin 0, we have \( (D^{\theta_{\alpha \beta \gamma \delta \zeta}})_{\alpha} = 0 \). This means that \( D^{\theta_{\alpha \beta \gamma \delta \zeta}} = 0 \) i.e. \( \theta_{\alpha \beta \gamma \delta \zeta} = 0 \) at any point of the space \( M^{(\nu)} \).

§7. **Symmetric Kählerian space.** A Kählerian space \( M^{(\nu)} \) is said to be symmetric if a symmetry with respect to any point \( 0 \in M^{(\nu)} \) is always an isometry. Suppose that \( M^{(\nu)} \) be symmetric. Then on account of the Theorem 5, it holds \( \theta_{\alpha \beta \gamma \delta \zeta} = 0 \). Therefore in the neighborhood \( U \) of a normal coordinate with center \( 0 \in M^{(\nu)} \), \( \theta_{\alpha \beta \gamma \delta \zeta} \) are all constants, and satisfy the identities:
\[
\begin{aligned}
\begin{cases}
(i) & \theta_{\alpha T\delta} = -\theta_{\beta T\delta} = -\theta_{\alpha B T}, \\
(ii) & \theta_{\alpha T\delta} + \theta_{\beta T\delta} + \theta_{\alpha B T} = 0, \\
(iii) & \theta_{\alpha B T\delta} = \theta_{\beta B T},
\end{cases}
\end{aligned}
\]

of which (iii) is a consequence of (i) and (ii). The assumption \(D^{\alpha T\delta} = 0\) leads us to the relation

\[
\theta_{\alpha T\delta} \theta_{\alpha T\delta} + \theta_{\alpha T\delta} \theta_{\beta T\delta} + \theta_{\alpha B T} \theta_{\alpha B T} + \theta_{\alpha B T} \theta_{\beta B T} = 0
\]
on account of \(d^{\alpha T\delta} = 0\) along a geodesic \(\zeta^\alpha = A^\alpha t\) through a point \(0 \in M^{(x)}\). Differentiating the last relation with respect to a parameter \(t\) and noticing (4.5) we get

\[
H_{\alpha B T\delta\rho} = \theta_{\alpha B T\delta} \theta_{\alpha T\delta + \rho} + \theta_{\alpha T\delta} \theta_{\beta T\delta + \rho} + \theta_{\alpha B T} \theta_{\beta T\delta + \rho} + \theta_{\alpha B T} \theta_{\beta T\delta + \rho} = 0.
\]

Now we proceed to prove the following

**Theorem 6.** For a symmetric Kählerian space, the curvature tensor \(\theta_{\alpha T\delta}\) is absolute constent in the neighborhood \(U\) of normal coordinate and satisfies the identities (7.1) and (7.2). Conversely, for constants \(\theta_{\alpha T\delta}\) satisfying these identities, there corresponds a symmetric Kählerian space with \(\theta_{\alpha T\delta}\) as its curvature tensor.

**Proof.** It remains to prove the converse. For the purpose it is sufficient to show that a solution \(\theta_{\alpha T}^*, \theta_{\beta T}^*\) of the differential equations (4.5) satisfies the equations of structure. In other words it remains for us to prove that if we put

\[
\begin{aligned}
\dfrac{d\theta_{\alpha T}^*}{dt} &= \theta_{\alpha T}^* \wedge \theta_{\alpha T\delta + \rho} + \epsilon_{\beta}, \\
\dfrac{d\theta_{\beta T}^*}{dt} &= \theta_{\beta T}^* \wedge \theta_{\beta T\delta + \rho} + \frac{1}{2} \theta_{\beta T\delta} \theta_{\alpha T}^* \wedge \theta_{\beta T}^* + \epsilon_{\beta T},
\end{aligned}
\]

then \(\epsilon_{\beta} = \epsilon_{\beta T} = 0\) hold identically.

Now as the exterior differentiation and the partial differentiation with respect to \(t\) are commutative, we have

\[
\begin{aligned}
\begin{cases}
(i) & \dfrac{\partial}{\partial t} (d^\alpha T\delta) = d\left(\dfrac{\partial \theta^\alpha T\delta}{\partial t}\right), \\
(ii) & \dfrac{\partial}{\partial t} (d^\beta T\delta) = d\left(\dfrac{\partial \theta^\beta T\delta}{\partial t}\right).
\end{cases}
\end{aligned}
\]

From (7.4i) it follows easily by using (7.3), (4.5) and the identity (7.1) that

\[
\begin{aligned}
(\text{i}) & \quad \dfrac{\partial \epsilon_{\beta}}{\partial t} = A^\gamma \epsilon_{T\delta + \beta}.
\end{aligned}
\]

From (7.4 ii) it follows by the same calculation that
\[ (7.6) \quad \frac{\partial \epsilon_{\beta T}}{\partial t} = A^\delta \theta_{\beta T \delta \alpha} \epsilon_{\alpha} + A^\rho \theta_{\theta \rho} \epsilon_{\theta} \phi_{\beta T \rho}, \]

where we put for brevity that
\[ (7.7) \quad \phi_{\beta T \rho} = \theta_{\alpha T \rho} \theta_{\beta \alpha}^{\ast} + \theta_{\beta T \rho \alpha} \theta_{\alpha}^{\ast} + \theta_{\beta T \rho \alpha} \theta_{\alpha}^{\ast} + \theta_{\beta T \rho \alpha} \theta_{\alpha}^{\ast}. \]

However it is evident that \( \phi_{\beta T \rho} = 0 \) for \( t = 0 \) because of (4.3). On the other hand, as \( \theta_{\beta T \rho} \) are constants we have using (4.5) and (7.2)
\[ \frac{\partial}{\partial t} \phi_{\beta T \rho} = H_{\beta T \rho \mu T} A^\tau \theta_{\epsilon \mu} = 0. \]

Therefore we have identically \( \phi_{\beta T \rho} = 0 \). Hence (7.6) reduces to
\[ (7.5) \quad (ii) \quad \frac{\partial \epsilon_{\beta T}}{\partial t} = A^\delta \theta_{\beta T \delta \alpha} \epsilon_{\alpha}. \]

For \( t = 0 \) it is evident that \( \epsilon_{\beta} = \epsilon_{\beta T} = 0 \) because of the initial conditions (4.3). Hence the unique solution of (7.5 i, ii) is given by \( \epsilon_{\beta} = \epsilon_{\beta T} = 0 \). This completes the proof of Theorem 6.

Bibliography