研 究

Establishing Plane Hypotheses from 3D Contours
for Scene Recognition of Visual Intelligence

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Key Words

planar surface, plane hypothesis, local plane, connectivity, coplanarity

Abstract

A robust method to derive hypotheses for planar surfaces of 3D objects from noisy contours, is reported. It is a step to establish 3D scene model and a part of general vision system. A key concept is the fact that the object contour, which forms the geometrical shape of a 3D object, should be connected and that the connection forms a hypothesis for a plane which can be a part of a planar surface.

The method works as follows;

- 1) Make a "local" plane from two line segments that have high connectivity.
- 2) Group local planes that have good coplanarity into a plane hypothesis.
- 3) Add the remaining coplanar segments into plane hypotheses.

Results from synthetic data are presented and applications to the establishment of planar surfaces of objects are discussed.

1 Introduction

Vision becomes one of the most important, useful, and direct method to recognize outer world and take active action for any intelligent machine systems. Typical vision system may have such steps;

- 1) Image acquisition and description (in 2D): Edge detection, etc.
- 2) Extraction of 3D description: Stereo matching and reconstruction, etc.
- 3) Establishing and maintaining 3D model of the scene.

4) Recognition of the 3D model.

The topic of this paper is a task in the step 3, extraction of plane hypotheses to derive planar surfaces which are important features of the 3D description. In the step 2, the 3D description, the input for the step 3, is usually represented by a set of 3D contours. But it is not an easy task to group 3D contours which form planar surfaces. The contours, which form the geometrical shape of objects as well as the planar surfaces, are mixed with the texture contours, which do not represent the geometrical form of objects directly. Moreover, output contour from the stereo matching and reconstruction processes is

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usually erroneous, in particular, along the depth direction. Then the questions are, how to group these contours by their planarity, and how to compute the plane equations in the sense of optimization. In this paper, we discuss these two questions to derive plane hypotheses that involve planar surfaces, and show some experimental results.

1.1 The Basic Strategy

Between planes and contours in the planes, there are several fundamental relations that can be useful to form a planar surface hypothesis, namely,

- (i) The direction of a contour is always perpendicular to the plane normal that the contour belongs to.
- (ii) A contour can be shared by up to two planes.
- (iii) The object contours connect to each other by their end points.
- (iv) The texture contours are surrounded by the object contours.

From (i), the projection of the contour's possible plane normal draws a great circle on the unit sphere. Therefore one can derive the plane normal by gathering intersections of these great circles and get plane hypotheses immediately, [Weiss 90].

But if the contour set has rich texture, this method gives too many false plane hypotheses. According to (ii), every contour can vote for only two plane normals. However this method gives every contour a circle to vote. Hence the intersections on the unit sphere increase rapidly. We want to point out here that it is very important to keep the number of arising possible plane hypotheses low, since analyzing and verifying the hypotheses using geometrical restrictions and relations become too messy and complicated if there is a high amount of false hypotheses.

If the object contour and texture are

distinguishable, (iii) and (iv) can give a reduced and robust set of the hypotheses. A contour connection forms a *local plane*, a part of a planar surface. Furthermore one can derive the plane hypotheses by gathering these local planes. We take this approach, namely,

- (1) Finding the local planes by estimating connectivity.
- (2) Computing and estimating the plane equation from (erroneous) coplanar contours. A coplanarity measure is also necessary to select better hypotheses and to seek for remaining coplanar contours at step 4.
- (3) Grouping the local planes.
- (4) Seeking for all residual coplanar contours that do not satisfy the condition of the local plane at step 1.

Under noisy conditions, the object contours may not connect together precisely on a point, they may still have adjacent end points. Then one must estimate *connectivity* of contours to apply (iii). Another problem is separation of the object contour and the texture, i.e., their strict and stable separation isn't easy. But rough separation is possible by thresholding on the length of the contours.

As input to the algorithm in practice, we take a set of contours formed by chains of 3D points. A line segment is represented by two end points and a (curved) contour by a chain of short line segments, (fig. 1-1).

Fig.1-1 A 3D line segment and a 3D contour. Both are represented by 3D points. a line segment has two and a contour has several points.

This representation has a potential merit in that it is possible to handle the curved contour and the line segment in the same, consistent way. Thus we only mention about the line segment in later sections.

2 Finding the Local Planes

As described before, we assume that a local plane is derived from a pair of line segments that have high connectivity and that could also be object contours. Hence the estimation conditions are two; the connectivity between two segments, and the length of each segment. The connectivity is based only on the distance between the end points of two segments. The latter can tell whether the segment is an object contour or a texture. It has another advantage, namely that shorter segments generally cause bigger errors in the stereo matching and reconstruction steps, and thus are less reliable. On the other hand, there is a possibility that this approach too easily ignores short but important object contours. But the most important thing is getting a part of the plane hypothesis. Actually, our experiments have shown that though some of the object contours are lacking due to their shortness, it is enough if some of them can be detected.

The algorithm estimates the connectivity between all segments whose length are longer than a given threshold. (eq. 2-1 and 2-2)

$$l_i, l_j > \text{length-tolerance} \quad (2-1)$$

$$\frac{(\text{distance between two end point})^2}{l_i * l_j} < \text{connectivity-tolerance} \quad (2-2)$$

where l_i and l_j are the length of segments. If a segment pair satisfies both conditions, a measure of coplanarity is estimated and if it is good, a new local plane is established.

3 Establishing and Arranging Plane Hypotheses

3.1 Grouping the Local Planes

We divide the grouping of the local planes into two steps: grouping first by the plane normals and secondly by the plane depth. It gives the groups of parallel planes, which are quite useful information to the following processes. In both cases, the resolution of the grouping are given as fixed values.

3.1.1 Grouping the Plane Normals

We introduce a polyhedron of tessellations of the equilateral triangles, the tessellated "*Gaussian sphere*", [Gård. 89]. It composes a multi-resolutional hierarchy of polyhedra beginning from a icosahedron, which is the smallest anti-symmetric polyhedron formed by the equilateral triangles. Each triangle is divided into four by going down the hierarchy. Antisymmetry is preferable to cancel vectors of opposite direction.

3.1.2 Grouping by the Plane Depth

After performing the grouping of the plane normals, grouping of the plane depths begins. The plane depth is measured from the coordinate origin. An ordinary histogram method is the most suitable technique.

3.2 Searching "Missing" Segments

After grouping the local planes into plane hypotheses, there still remain some coplanar segments that are not included in these hypotheses. Because these hypotheses are composed of only the object contours that satisfy both the connectivity and coplanar conditions, the texture contours and some object contours which are short or far from other segments are still out of the hypotheses. Then each plane hypothesis needs to be searched for possible (coplanar) missing segments from *remaining* segments, to make its list of contained segments perfect.

To perform it, we use the recursive

estimation of coplanarity, which appears in §4.3. It checks every combination of remaining segment and plane hypothesis whether the segment can satisfy the coplanarity criterion.

4 Computation and Estimation of Planes

4.1 Plane Computation

The input is a set of coplanar line segments including some error, in particular along the depth direction. In this case, the input data is a set of segments represented by their end points, we thus minimize the distance between the plane and the end points of the given line segments, (eq. 4-1).

$$\chi^2 = \sum \Delta \mathbf{P}_i^T \mathbf{C}^{-1} \Delta \mathbf{P}_i \rightarrow \min \quad (4-1)$$

where $\Delta \mathbf{P}_i$ is the distance between the plane and point $\mathbf{P}_i (1 \leq i \leq N)$. $\mathbf{C}_i$ is the covariance matrix of $\mathbf{P}_i$. It is assumed that the errors have a normal distribution. But solving eq. 4-1 for a general $\mathbf{C}_i$ computationally is quite difficult. To simplify this equation, we assume $\mathbf{C}_i$ is constant, i.e.

$$\mathbf{C}_i = \sigma^2 \mathbf{I} \quad (4-2)$$

Using the plane equation $ax + by + cz + d = 0$ and (x_i, y_i, z_i) an $\mathbf{P}_i$, eq. 4-1 becomes the least square estimation that the singular value decomposition (SVD) gives the solution, (Appendix A).

4.2 Coplanarity Estimation of Planes

For the measure of goodness of fit, or coplanarity, we take the standard deviation of the correction, the deviation of all end points of segments from the plane. The least square solution also gives its uniqueness.

1) The measure of coplanarity

The square sum of the correction of all points (χ^2) can be obtained directly from the solution by the SVD. Using it, the coplanarity measure, the standard deviation of the error

is

$$\text{Coplanarity} = \sqrt{\frac{\chi^2}{N}} \quad (4-3)$$

This quantity tells us how the given segment set deviates from a plane. If it is small enough, we can admit this set as coplanar.

2) Estimation of uniqueness of the solution

Concerning uniqueness of the solution, there are three situations which should be critical: whether the segment set is not only coplanar but also colinear or "copoint". In such cases, there are more than one possible normals as a solution. Thus one can estimate the uniqueness of the solution by the standard deviations of the second and the third best solutions. The standard deviations of correction X_j ($j=1-3$) are similar to eq. 4-3,

$$X_j = \sqrt{\frac{\chi_j^2}{N}} \quad (4-4)$$

X_3 corresponds to the coplanarity measure. These values are evaluated in the following way.

1) $X_3 = \text{coplanarity measure} < \text{coplanarity-tolerance}$

The fit is good enough to regard the given segments set as coplanar.

2) $X_2 < \text{coplanarity-tolerance}$

The segment set is not only coplanar but also *colinear*. This situation should feed back to the earlier stages to investigate whether these segments are indeed one split long segment.

3) $X_1 < \text{coplanarity-tolerance}$

A quite unusual (and impractical) case, the set is "copoint", i.e. close to a point.

4) $(X_2 - X_3) / (X_2 + X_3) < \text{coplanarity-tolerance}$

The second best solution is equally good as the best solution, i.e. the solution

is not unique. In this case, the given segment set is insufficient to decide a plane equation and it is regarded as “*not coplanar*”.

The coplanarity estimate using the result of the SVD is quite robust since the SVD is very stable to numerical errors, even if the input data set is close to singular. But the SVD isn't a cheap calculation, in particular, testing the coplanarity of segments sets.

4.3 Recursive Estimation of the Coplanarity Test

An efficient recursive estimation of the coplanarity test for adding a new segment into a given set of coplanar segments is necessary to pick up “missing” coplanar segments after forming the plane hypotheses. But rewriting the problem into recursive formulation isn't easy, since adding a new segment into a given segments set may change the calculated plane normal and the error estimate of the existing coplanar segments ($\Delta \mathbf{P}_i$ in eq. 4-1). Thus we need to test it using the expensive SVD, updating the whole plane equation.

However, some approximation is possible. Before using the SVD, one can roughly estimate it by assuming that the given segments set has sufficiently many segments in it, and that the tested segment is close to parallel to the plane. In this situation, the new segment doesn't change the plane normal, only the plane depth may be changed, (Appendix B).

5 Tests and Results

We tested the algorithm to a synthesized 3D line segments drawing of cubic objects with noisy texture segments and random errors. Unfortunately, experiments on real 3D segment data obtained from real stereo image has not been possible yet. The results

of available stereo matching and reconstruction algorithms have been too erroneous, and it has been hard to recognize which segments correspond to which edges in the 2D image even for human observer.

The input data is the drawing of two cubes with some line markings (texture) on the front surface. There are 40 segments in it. All endpoints of the segments are randomized to give ± 9 pixels error at maximum. The upper surface of the right cube is concave. This randomization changes the direction of the shorter segments relatively more than the longer ones, which often happens in real data.

We pay our attention on following points ;

- (i) Are arising plane hypotheses proper, i. e. they contain correct segments?
- (ii) Are local planes correctly selected and is the number of them is not unnecessarily high, with respect to changes of the threshold parameters in the algorithm?
- (iii) Is the grouping of local planes correct?

This algorithm has five threshold parameters and the influence of their variations has to be estimated and tuned to the best. Parameters are,

- (1) Minimum length of segment to test the connectivity, shown in eq. 2-1. [20 pixels]
- (2) Connectivity tolerance, shown in eq. 2-2. [0.1]
- (3) Coplanarity tolerance, discussed in §4.2. [2 pixels]
- (4) Hierarchy depth of the gaussian sphere, discussed in §3.1.1. [3]
- (5) Histogram bin to group the plane depth, discussed in §3.1.2. [5 pixels]

The numbers in [] s are *default values*, the most stable parameters used without notation in the following experiments. These

values are decided on the basis of several experiments in §5.2 and §5.3.

5.1 Result of Plane Hypotheses and Their Accuracy

The following pictures show the plane hypotheses obtained by the default parameters. Local planes which are not grouped with others are deleted. Six plane hypotheses are detected. Fig. 5-1 shows the input line drawing, fig. 5-2a)-f) show the each plane hypothesis by highlighted lines.

All expected planes are successfully detected. It is worth to note the occluded planes in fig. 5-2 e) and f), the "table plane" in e) and the plane between the two cubes in f). These occluded planes are very important to recognize the 3D scene, but not easy to detect from a 2D scene analysis

approach. Also it is very interesting to see how such occluded planes are used in following processes.

5.2 Stability of the Algorithm to Variations in Threshold Parameters

We estimated the influence of the variations in threshold parameters to the forming and the grouping of the local planes. In the forming of them, the number of the local planes is the criterion to check whether the connectivity and coplanarity estimation works as expected. In grouping them, the grouping schema and the restriction by the geometrical relation between plane and contour are tested.

5.2.1 Threshold Parameters with Respect to Forming the Local Planes

The number of the local planes shows, 1)

Fig. 5-1 The input drawing. Numbers are the IDs of segments. There are two blocks. Right one has some texture on the front surface and the top surface is concave. Segment # 39 is background line, and # 40 is shadow of blocks. All segments have certain amount of errors in the end points.

Fig. 5-2 a) : Top plane

Fig. 5-2 b) : Concave in the right block

Fig. 5-2 c) : Front plane

Fig. 5-2 d) : Side plane of right block

Fig. 5-2 e) : Table plane

Fig. 5-2 f) : Plane between two blocks

Fig.5-2 a)-f) Obtained plane hypotheses. (Shown by highlighted lines.) All expected planes are detected. Some planes contain unnecessary segments, but there is no lack of necessary segment. planes in e) and f) are occluded planes, which are difficult to detect for approaches using 2D drawing.

Table 5-1 Number of local planes and obtained plane hypotheses

Connect.	Min. Length = 50			20			5		
0.05	33 5-0	34 5-0	34 5-0	33 5-0	34 5-0	34 5-0	33 5-1	34 5-0	34 5-0
0.1	39 5-1	56 6-0	57 6-0	39 5-1	56 6-0	57 6-0	39 5-1	56 6-0	57 6-0
0.5	43 5-1	65 6-0	67 6-0	46 5-1	68 6-1	70 6-1	60 6-1	73 6-1	70 6-1
Coplan.	1	2	5	1	2	5	1	2	5

L
C-W

L Number of local planes.

C Number of correct plane hypotheses.

W Number of wrong plane hypotheses.

how the short segments are eliminated, 2) how the possible object contour segments are selected properly. To test the number of local planes, two parameters for grouping are fixed to the default values (the hierarchy depth=4 and the histogram bin=5).

It shows ;

- i) The number of correct plane hypotheses is very stable to the variance of these parameters even though the number of the local planes changes.
- ii) The number of local planes is also stable to the variance of length restriction and *connectivity-tolerance*, in particular when both two are small.
- iii) Too small *coplanarity-tolerance*(=1) causes some false hypotheses.

Several false plane hypotheses appear when the coplanarity tolerance is small. On the other hand, some important local planes which consist of very long and stable segments are discarded by this criterion. In general, as the error in the segments becomes worse, this algorithm requires more correct local planes to get a correct and exact plane

hypothesis. (Here “correct” means choosing segments pair that are part of planar surface.)

5.2.2 Threshold Parameters with Respect to Grouping the Local Planes

The grouping thresholds determine the width of histogram bin. We test how the variance of the grouping thresholds influence the results with fixing the three parameters for the arising of local planes to their default values (minimum length of segment=20, connectivity tolerance=0.1, coplanarity tolerance=2).

- i) As expected, too narrow cells cause more false, split hypotheses.
- ii) But wider cells don't create very many wrong hypotheses.
- iii) The coarser grouping of the plane normals doesn't cause wrong hypotheses unlikely to the coarser grouping of the plane depth.

Indeed our test data have only three plane normals which are perpendicular to each other, the grouping of them is very easy and there is less possibility to accidentally merge

Table 5-2 Number of obtained plane hypotheses

Hierarchy depth	Hist. bin = 3	5	20
5 (5120 hedron)	4-5 (0:5)	4-2 (1:1)	4-1 (0:1)
4 (1280 hedron)	5-2 (0:2)	6-0 (0:0)	4-1 (1:0)
3 (320 hedron)	4-3 (1:2)	6-0 (0:0)	5-1 (1:0)
1 (top; icosah.)	4-3 (1:2)	6-0 (0:0)	5-1 (1:0)

C - W (I : S)

C Number of correct plane hypotheses.

W Number of wrong plane hypotheses. ($W = I + S$)

I Number of wrong hypotheses incorrectly own segments.

S Number of wrong hypotheses split into several.

wrong directions. This is the reason of good result in iii). But still there are several errors when the hierarchy depth is 1 or 3. Such errors happen since there are a few false local planes, which are formed by short (and relatively more erroneous) segments and have wrong normal directions.

6 Discussion

6.1 Efficiency and Advantage of the Method

This method has a drawback in that searching the local planes from a set of segments (N pieces) costs order of N^2 . But we don't think it is such a serious problem compared to other methods. Searching the local planes is not a difficult task ; just some algebra, but throwing away fake planes in terms of both statistical and geometrical consistency is more complicated and it thus consumes much computation time.

The test data have 40 segments in it. Then, the possible number of combinations of two segments is 780 ($40 \times 39 / 2$), but as shown in table 5-1, the number of formed local planes is about 30 to 70, which decreases the complicated tasks to about 1/10 to 1/20. On this point this method has its strength ; picking up only robust parts of planes yields

robust plane hypotheses more quickly.

6.2 Robust Plane Hypotheses

In §4.2. we define and use the coplanarity measure. But it is a measurement of the goodness of fit of the *given* set of segments to a plane hypothesis and it tells only about the *given* set of segments, not about any other possible sets. Also, a smaller set of segments often gives better coplanarity because a larger set often contains more errors. Then, as a result, planes easily split into fewer segment sets. This tells us that the coplanarity measure doesn't show the relation between plane hypotheses, in other words, it doesn't tell about how the whole system is relaxed by grouping coplanar segments.

Another problem is that our plane computation algorithm solves the least square problem to the given segment sets piecewise. Every segment has the possibility to be shared by two planes and such a segment appears as a part of the intersection of two planes. However, the algorithm solves planes piecewise and totally ignores how each segment is fit to another plane.

These problems tell us that establishing the plane hypotheses should be solved on a more global level, making the whole system status

more relaxed. One possible idea is to describe the system entropy, which takes into account the number of whole plane hypotheses.

7 Conclusions

A method to establish plane hypotheses from noisy 3D contours from a line drawing is developed, which is based on both the geometrical relation of contours and planes, and statistical stability of solutions.

Two fundamental relations between coplanar contours, and between contours and planes are employed, namely i) object contours forming a planar surface of an object connect to each other, ii) a contour can be shared by up to two planes. The former is used to derive reduced and robust local planes, that are parts of planar surfaces. The latter is used to verify plane hypotheses and throw away contours from planes that incorrectly include them.

A least squares solution of the plane equation is introduced to minimize the distances (error correction) of end points of given set of line segments from the plane. It also gives a measure of how each point is fit to the plane. This measure is used to evaluate coplanarity of individual segments to the plane, and its standard deviation is defined as the coplanarity of the given set of segments.

The task is divided into several steps. 1) Find local planes by evaluating the connectivity of object contours. To discriminate object contours from texture contours, line segments longer than certain threshold are evaluated. 2) Group these local planes that satisfy both coplanarity and the geometrical criteria, and establish plane hypotheses. 3) Add remaining coplanar contours into hypotheses.

The method is examined on synthetic data

consisting of line drawings which include some noise. It shows all expected planes with a few unexpected line segments, but there is no lack of elemental object contours. Obtained local planes are reduced to about 1/10 to 1/20 of possible combinations of segments, thus it can reduce computation time for the more complex ensuing procedures. Results are stable to varying the threshold parameters. Also it finds some occluded planes for example the bottom plane of the object. How to classify such planes is a task for the following step, the establishing of the planar surfaces of the object.

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Appendix A: The solution of the plane

equation

Using the plane equation $ax + by + cz + d = 0$ and (x_i, y_i, z_i) as P_i eq.3-1 becomes the least square estimation, (eq. A-1).

$$\chi^2 = \sum \frac{(ax_i + by_i + cz_i + d)^2}{\sigma} \rightarrow \min \quad (A-1)$$

$$a^2 + b^2 + c^2 = 1 \quad (A-2)$$

Solving this equation for d , one gets immediately that the plane includes the center of gravity of these points G , (eq. A-3).

$$G = \left(\frac{\sum x_i}{N}, \frac{\sum y_i}{N}, \frac{\sum z_i}{N} \right) \quad (A-3)$$

Using this fact, we can rewrite the problem the following way: Fitting all vectors $P_i - G$ into a plane passing through the coordinate origin. This is equivalent to reducing the rank of matrix A ;

$$A = \begin{bmatrix} P_1 - G \\ \vdots \\ P_N - G \end{bmatrix} \quad (A-4)$$

to 2 and the singular value decomposition (SVD) solves it in the least square manner. The SVD decomposes A to UWV^T , U and V are column-orthogonal matrices and W is a diagonal matrix, i.e.

$$A = U \begin{bmatrix} \omega_1 & & \\ & \omega_2 & \\ & & \omega_3 \end{bmatrix} V^T \quad (A-5)$$

We assume that $\omega_1 \geq \omega_2 \geq \omega_3$, without loss of generality. The rank of W gives the rank of A , since U and V are orthogonal. Putting $\omega_3 = 0$ makes $\text{rank}[A] = 2$ and gives the best fit of these vectors in A into a plane, and the plane normal is the third column of V , V_3 . The correction of each point is just $u_3 \omega_3 v_3^T$, since the correction of each point is the difference of matrix A ,

$$\Delta A = A' - A$$

$$= U \begin{bmatrix} 0 & & \\ & 0 & \\ & & -\omega_3 \end{bmatrix} V^T \\ = -u_3 \omega_3 v_3^T \quad (A-6)$$

Here, ω_3 is just the square sum of corrections of all points χ^2 in eq. A-1. Because the norm of the correction is $\|\Delta A\| = u_3 u_3^T \omega_3^2 v_3 v_3^T$ from eq. A-6, but u_3 and v_3 are orthogonal, therefore $\|\Delta A\| = \omega_3^2$. Then, the *coplanarity measure*, the standard deviation of χ^2 is

$$\text{Coplanarity} = \sqrt{\frac{\chi^2}{N}} \quad (A-7)$$

This quantity tells how the given segment set deviates from a plane.

Appendix B: Approximation of the recursive estimation of the coplanarity

This approximation requires that the new segment is close to being parallel to the given plane. The given segments set has the error correction of each end point to their coplanar plane expressed by $\Delta P_i (1 \leq i \leq N)$ and that of the new segment to the (old) plane expressed similarly by, $\Delta P_{N+1}, \Delta P_{N+2}, \Delta P_i (1 \leq i \leq N+2)$, the correction after adding P_{N+1} and P_{N+2} , is described as;

$$\Delta P'_i = \Delta P_i - \frac{\Delta P_{N+1} + \Delta P_{N+2}}{N} \quad (A-9)$$

The (approximated) sum of the error corrections χ_{app}^2 , is calculated by taking the square sum of these errors;

$$\chi_{app}^2 = \sum_{i=1}^{N+2} |\Delta P'_i|^2 \quad (A-10)$$

Note that all vectors ΔP_i are parallel to the plane normal and all calculation in eq. A-9 and A-10 are scalar calculations, which means they are quite simple. If the obtained χ_{app}^2 is small enough, one applies the real SVD to get the real estimation.