Laser Original

**Quantum Theory of Laser
with a Fabry-Perot Resonator II
— A microscopic theory —**

Masahiko HIROOKA* and Sigenobu SUNAKAWA*

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A microscopic theory of laser has been presented by taking into consideration of the frequency pulling effect, when the laser light is enclosed in a Fabry-Perot resonator. On the basis of the microscopic theory, the validity of the average-field theory developed in the preceding paper has been discussed.

1. Introduction

In the preceding paper¹⁾ (hereafter referred to as I), the present authors have presented an average-field theory of laser for the case when standing waves are formed in the cavity, by taking into account of the frequency pulling effect. The theory predicts, however, that no emission of photons occurs at all below threshold. In reality, the laser emits photons even below threshold as a usual lamp. Furthermore, the theory does not give the fluctuation of photon number around its mean value and does not describe correctly the diffusion of the phase of laser light due to spontaneous emission of photons. In order to describe these effects, one needs to develop a microscopic theory of laser by considering in detail elementary interactions between atoms and photons.^{2,3)} The purpose of the present paper is to present such a microscopic theory for the case when the photon field is enclosed in a Fabry-Perot resonator. The frequency pulling effect has been taken into consideration by making use of the method of renormalization in quantum electrodynamics. In the final section, the validity of the average-field

theory is discussed on the basis of the microscopic theory.

2. Formalism

The Hamiltonian of our system is given by I, (2.2). In order to take into consideration of the frequency pulling effect, one needs to rewrite it as

$$H = \sum_{i=1}^N H_A^{(i)} + \hbar \Omega a^* a + i \sum_{i=1}^N (e E s_i) z_i (a - a^*) - \hbar (\Omega - \omega) a^* a, \quad (2.1)$$

by reminding of the method of renormalization in quantum electrodynamics, where Ω stands for the modified frequency of the laser light due to the interaction of the field with laser atoms and is determined later. The factor s_i in (2.1) stands for $\sqrt{2} \sin k X_i$. Transforming the state $\Psi(t)$ in I, (2.3) to that $\Phi(t)$ in the interaction representation by

$$\Phi(t) = e^{i[\sum_i H_A^{(i)} + \hbar \Omega a^* a]t/\hbar} \Psi(t), \quad (2.2)$$

one has

$$i \hbar \frac{\partial \Phi(t)}{\partial t} = H_{\text{int}}(t) \Phi(t), \quad (2.3)$$

* Department of Physics, College of General Education, Osaka University (1-1, Machikaneyama-cho, Toyonaka, Osaka 560, Japan)

where

$$H_{\text{int}}(t) = i \sum_{i=1}^N (e E s_i) z_i(t) [a(t) - a^*(t)] - \hbar(\Omega - \omega) a^*(t) a(t), \quad (2.4)$$

$$z_i(t) = e^{iH_A^{(i)} t / \hbar} z_i e^{-iH_A^{(i)} t / \hbar},$$

$$a(t) = e^{i\Omega a^* a t} a e^{-i\Omega a^* a t} = a e^{-i\Omega t},$$

and

$$a^*(t) = e^{i\Omega a^* a t} a^* e^{-i\Omega a^* a t} = a^* e^{i\Omega t}. \quad (2.5)$$

We now expand the state $\Phi(t)$ as

$$\Phi(t) = \sum_{\alpha_1, \dots, \alpha_n} C_n^{\alpha_1 \dots \alpha_n}(t) u_{\alpha_1} u_{\alpha_2} \dots u_{\alpha_n} |n\rangle, \quad (2.6)$$

where u_{α_i} stands for an eigenstate of $H_A^{(i)}$ and the state $|n\rangle$ indicates a normalized eigenstate of photon number defined by $a^* a |n\rangle = n |n\rangle$ ($n=0, 1, 2, \dots$). Introduction of (2.6) into (2.3) gives

$$\begin{aligned} \frac{dC_n^{\alpha_1 \dots \alpha_n}(t)}{dt} &= \sum_{i=1}^N \left(\frac{e E s_i}{\hbar} \right) \sum_{\beta_i} \langle \alpha_i | z_i(t) | \beta_i \rangle \\ &\times [\sqrt{n+1} e^{-i\Omega t} C_{n+1}^{\alpha_1 \dots \beta_i \dots \alpha_n}(t) - \sqrt{n} e^{i\Omega t} \\ &\times C_{n-1}^{\alpha_1 \dots \beta_i \dots \alpha_n}(t)] + i n (\Omega - \omega) C_n^{\alpha_1 \dots \alpha_n}(t), \end{aligned} \quad (2.7)$$

where

$$\langle \alpha_i | z_i(t) | \beta_i \rangle \equiv \langle u_{\alpha_i} | z_i(t) | u_{\beta_i} \rangle. \quad (2.8)$$

When one writes

$$A_n^{\alpha_1 \dots \alpha_n}(t) = C_n^{\alpha_1 \dots \alpha_n}(t) e^{-i n (\Omega - \omega) t}, \quad (2.9)$$

one has from (2.7)

$$\begin{aligned} \frac{dA_n^{\alpha_1 \dots \alpha_n}(t)}{dt} &= \sum_{i=1}^N \left(\frac{e E s_i}{\hbar} \right) \sum_{\beta_i} \langle \alpha_i | z_i(t) | \beta_i \rangle \\ &\times [\sqrt{n+1} e^{-i\omega t} A_{n+1}^{\alpha_1 \dots \beta_i \dots \alpha_n}(t) \\ &- \sqrt{n} e^{i\omega t} A_{n-1}^{\alpha_1 \dots \beta_i \dots \alpha_n}(t)], \end{aligned} \quad (2.10)$$

where one should note that the frequency Ω in the exponents of (2.7) has been replaced by the mode frequency ω . In order to separate out the interaction between a single atom and the photon field, we define here the particle-derivative* by

$$\begin{aligned} \frac{dA_n^{\alpha_1 \dots \alpha_n}(t)}{dt_i} &\equiv \left(\frac{e E s_i}{\hbar} \right) \sum_{\beta_i} \langle \alpha_i | z_i(t) | \beta_i \rangle \\ &\times [\sqrt{n+1} e^{-i\omega t} A_{n+1}^{\alpha_1 \dots \beta_i \dots \alpha_n}(t) \\ &- \sqrt{n} e^{i\omega t} A_{n-1}^{\alpha_1 \dots \beta_i \dots \alpha_n}(t)]. \end{aligned} \quad (2.11)$$

The total derivative (2.10) is of course given by the sum of (2.11) as

$$\frac{dA_n^{\alpha_1 \dots \alpha_n}(t)}{dt} = \sum_{i=1}^N \frac{dA_n^{\alpha_1 \dots \alpha_n}(t)}{dt_i}. \quad (2.12)$$

Noting I, (3.1) and making use of the rotating wave approximation, one can write (2.11) as

$$\begin{aligned} \frac{dA_n^{\alpha_1 \dots \alpha_n}(t)}{dt_i} &= \left(\frac{\mu E s_i}{\hbar} \right) \sqrt{n+1} e^{i(\omega_s - \omega) t} \\ &\times A_{n+1}^{\alpha_1 \dots \beta_i \dots \alpha_n}(t), \\ \frac{dA_n^{\alpha_1 \dots \alpha_n}(t)}{dt_i} &= - \left(\frac{\mu E s_i}{\hbar} \right) \sqrt{n} e^{-i(\omega_s - \omega) t} \\ &\times A_{n-1}^{\alpha_1 \dots \beta_i \dots \alpha_n}(t). \end{aligned} \quad (2.13)$$

We now define the atomic density matrix for the i -th atom by

$$\begin{aligned} \rho_{n,m}^{aa}(i) &= \sum_{\alpha_1 \dots \alpha_n}^i C_n^{\alpha_1 \dots \alpha_n}(t) C_m^{\alpha_1 \dots \alpha_n}(t)^*, \\ \rho_{n,m}^{bb}(i) &= \sum_{\alpha_1 \dots \alpha_n}^i C_n^{\alpha_1 \dots \alpha_n}(t) C_m^{\alpha_1 \dots \alpha_n}(t)^*, \\ \rho_{n,m}^{ab}(i) &= \sum_{\alpha_1 \dots \alpha_n}^i C_n^{\alpha_1 \dots \alpha_n}(t) C_m^{\alpha_1 \dots \alpha_n}(t)^* \end{aligned}$$

* The particle-derivative d/dt_i is represented in the previous paper³⁾ as d/dt_i , which means the operation that we first take the time-derivative like (2.10) and then retain from the sum concerning i on the right-hand side of (2.10) only a term which describes the transition of state of the specified atom i .

and

$$\rho_{n,m}^{ba}(i) = \sum_{a_1 \cdots a_N}^i C_n^{a_1 \cdots a_N}(t) C_m^{a_1 \cdots a_N}(t)^*,$$

$$\frac{dp_{n,m}^{bb}(i)}{dt_i} = -\left(\frac{\mu E s_i}{\hbar}\right) [\sqrt{n} e^{-i(\omega_s - \omega)t} p_{n-1,m}^{ab}(i) + \sqrt{m} e^{i(\omega_s - \omega)t} p_{n,m-1}^{ba}(i)] \quad (2.19)$$

where the symbol i on the right shoulder of the summation implies that the sum over the states for the i -th atom should be excluded. The photon density matrix is defined by

$$\rho_{n,m}(t) = \sum_{a_1 \cdots a_N} C_n^{a_1 \cdots a_N}(t) C_m^{a_1 \cdots a_N}(t)^* \quad (2.15)$$

Comparison of (2.14) with (2.15) gives a relation

$$\rho_{n,m}(t) = \rho_{n,m}^{aa}(i) + \rho_{n,m}^{bb}(i) \quad (2.16)$$

for an arbitrary i . Introducing (2.9) into (2.14), one has

$$\rho_{n,m}^{aa}(i) = e^{i(n-m)(\Omega - \omega)t} p_{n,m}^{aa}(t),$$

$$\rho_{n,m}^{bb}(i) = e^{i(n-m)(\Omega - \omega)t} p_{n,m}^{bb}(t),$$

$$\rho_{n,m}^{ab}(i) = e^{i(n-m)(\Omega - \omega)t} p_{n,m}^{ab}(t),$$

and

$$\rho_{n,m}^{ba}(i) = e^{i(n-m)(\Omega - \omega)t} p_{n,m}^{ba}(t), \quad (2.17)$$

where

$$p_{n,m}^{aa}(i) = \sum_{a_1 \cdots a_N}^i A_n^{a_1 \cdots a_N}(t) A_m^{a_1 \cdots a_N}(t)^*,$$

$$p_{n,m}^{bb}(i) = \sum_{a_1 \cdots a_N}^i A_n^{a_1 \cdots a_N}(t) A_m^{a_1 \cdots a_N}(t)^*,$$

$$p_{n,m}^{ab}(i) = \sum_{a_1 \cdots a_N}^i A_n^{a_1 \cdots a_N}(t) A_m^{a_1 \cdots a_N}(t)^* \quad (2.18)$$

and

$$p_{n,m}^{ba}(i) = \sum_{a_1 \cdots a_N}^i A_n^{a_1 \cdots a_N}(t) A_m^{a_1 \cdots a_N}(t)^*.$$

With the aid of (2.13), one can readily derive two sets of equations

$$\frac{dp_{n,m}^{aa}(i)}{dt_i} = \left(\frac{\mu E s_i}{\hbar}\right) [\sqrt{n+1} e^{i(\omega_s - \omega)t} p_{n+1,m}^{ba}(i) + \sqrt{m+1} e^{-i(\omega_s - \omega)t} p_{n,m+1}^{ab}(i)],$$

and

$$\frac{dp_{n,m}^{ab}(i)}{dt_i} = -\left(\frac{\mu E s_i}{\hbar}\right) e^{i(\omega_s - \omega)t} \times [\sqrt{m} p_{n,m-1}^{aa}(i) - \sqrt{n+1} p_{n+1,m}^{bb}(i)],$$

$$\frac{dp_{n,m}^{ba}(i)}{dt_i} = -\left(\frac{\mu E s_i}{\hbar}\right) e^{-i(\omega_s - \omega)t} \times [\sqrt{n} p_{n-1,m}^{aa}(i) - \sqrt{m+1} p_{n,m+1}^{bb}(i)]. \quad (2.20)$$

From (2.12), (2.17) and the above equations, the total derivative of the density matrix in (2.17) is written as

$$\frac{d\rho_{n,m}^{\alpha\beta}(i)}{dt} = \sum_{i=1}^N \frac{dp_{n,m}^{\alpha\beta}(i)}{dt_i} e^{i(n-m)(\Omega - \omega)t} + i(n-m)(\Omega - \omega) \rho_{n,m}^{\alpha\beta}(i), \quad (2.21)$$

and thus we define the particle-derivative of $\rho_{n,m}^{\alpha\beta}(i)$ as

$$\frac{d\rho_{n,m}^{\alpha\beta}(i)}{dt_i} \equiv \frac{dp_{n,m}^{\alpha\beta}(i)}{dt_i} e^{i(n-m)(\Omega - \omega)t}. \quad (2.22)$$

This definition of the particle-derivative is reasonable since the exponential factors in (2.17) are not related to the individual atoms but only to the photon field. From the definition (2.22), one can write (2.21) as

$$\frac{d\rho_{n,m}^{\alpha\beta}(i)}{dt} = \sum_{i=1}^N \frac{d\rho_{n,m}^{\alpha\beta}(i)}{dt_i} + i(n-m)(\Omega - \omega) \rho_{n,m}^{\alpha\beta}(i). \quad (2.23)$$

From (2.22), (2.19) and (2.20), one has two sets of equations;

$$\begin{aligned} \frac{d\rho_{n,m}^{aa}(i)}{dt_i} &= \left(\frac{\mu E s_i}{\hbar} \right) [\sqrt{n+1} e^{i(\omega_s - \Omega)t} \rho_{n+1,m}^{ba}(i) \\ &\quad + \sqrt{m+1} e^{-i(\omega_s - \Omega)t} \rho_{n,m+1}^{ab}(i)], \\ \frac{d\rho_{n,m}^{bb}(i)}{dt_i} &= - \left(\frac{\mu E s_i}{\hbar} \right) [\sqrt{n} e^{-i(\omega_s - \Omega)t} \rho_{n-1,m}^{ab}(i) \\ &\quad + \sqrt{m} e^{i(\omega_s - \Omega)t} \rho_{n,m-1}^{ba}(i)] \end{aligned} \quad (2.24)$$

and

$$\begin{aligned} \frac{d\rho_{n,m}^{ab}(i)}{dt_i} &= - \left(\frac{\mu E s_i}{\hbar} \right) e^{i(\omega_s - \Omega)t} \\ &\quad \times [\sqrt{m} \rho_{n,m-1}^{aa}(i) - \sqrt{n+1} \rho_{n+1,m}^{bb}(i)], \\ \frac{d\rho_{n,m}^{ba}(i)}{dt_i} &= - \left(\frac{\mu E s_i}{\hbar} \right) e^{-i(\omega_s - \Omega)t} \\ &\quad \times [\sqrt{n} \rho_{n-1,m}^{aa}(i) - \sqrt{m+1} \rho_{n,m+1}^{bb}(i)]. \end{aligned} \quad (2.25)$$

Comparing the above equations with (2.19) and (2.20), one sees that the mode-frequency ω in (2.19) and (2.20) has been replaced by the modified one Ω .

We next investigate the photon density matrix defined in (2.15). From (2.16) and (2.17), one can write as

$$\rho_{n,m} = e^{i(n-m)(\Omega - \omega)t} [p_{n,m}^{aa}(i) + p_{n,m}^{bb}(i)]. \quad (2.26)$$

Taking the total time derivative of (2.26), and using the relation (2.23), one has

$$\begin{aligned} \frac{d\rho_{n,m}}{dt} &= \sum_{j=1}^N \left\{ \frac{dp_{n,m}^{aa}(i)}{dt_j} + \frac{dp_{n,m}^{bb}(i)}{dt_j} \right\} e^{i(n-m)(\Omega - \omega)t} \\ &\quad + i(n-m)(\Omega - \omega)\rho_{n,m}. \end{aligned} \quad (2.27)$$

With the aid of the notable relation

$$\frac{dp_{n,m}^{aa}(i)}{dt_j} + \frac{dp_{n,m}^{bb}(i)}{dt_j} = \frac{dp_{n,m}^{aa}(j)}{dt_j} + \frac{dp_{n,m}^{bb}(j)}{dt_j}, \quad (2.28)$$

of which proof is given in the appendix and from the definition (2.22), one can rewrite (2.27) as

$$\begin{aligned} \frac{d\rho_{n,m}}{dt} &= i(n-m)(\Omega - \omega)\rho_{n,m} \\ &\quad + \sum_{i=1}^N \left\{ \frac{d\rho_{n,m}^{aa}(i)}{dt_i} + \frac{d\rho_{n,m}^{bb}(i)}{dt_i} \right\}. \end{aligned} \quad (2.29)$$

Inserting (2.24) into the right-hand side of (2.29), one obtains the equation for the photon density matrix as

$$\begin{aligned} \frac{d\rho_{n,m}}{dt} &= i(n-m)(\Omega - \omega)\rho_{n,m} \\ &\quad + \sum_{i=1}^N \left(\frac{\mu E s_i}{\hbar} \right) [\sqrt{n+1} e^{i(\omega_s - \Omega)t} \rho_{n+1,m}^{ba}(i) \\ &\quad + \sqrt{m+1} e^{-i(\omega_s - \Omega)t} \rho_{n,m+1}^{ab}(i)] \\ &\quad - \sum_{i=1}^N \left(\frac{\mu E s_i}{\hbar} \right) [\sqrt{n} e^{-i(\omega_s - \Omega)t} \rho_{n-1,m}^{ab}(i) \\ &\quad + \sqrt{m} e^{i(\omega_s - \Omega)t} \rho_{n,m-1}^{ba}(i)]. \end{aligned} \quad (2.30)$$

3. Basic Equations for a Laser System

Up to this stage, we have regarded the laser system as a closed system described by the hermitian Hamiltonian (2.1). The laser system is not, however, a closed system but an open system. The atoms in the cavity are always excited by pumping mechanism. The atomic states are not restricted two states a and b and the transitions from the levels $\hbar\omega_a$ and $\hbar\omega_b$ to other lower levels take place. These effects can be described in a phenomenological way by adding extra terms to (2.24) and (2.25) as

$$\begin{aligned} \frac{d\rho_{n,m}^{aa}(i)}{dt_i} &= \lambda_a \rho_{n,m} - \gamma_a \rho_{n,m}^{aa}(i) \\ &\quad + \left(\frac{\mu E s_i}{\hbar} \right) [\sqrt{n+1} e^{i(\omega_s - \Omega)t} \rho_{n+1,m}^{ba}(i) \\ &\quad + \sqrt{m+1} e^{-i(\omega_s - \Omega)t} \rho_{n,m+1}^{ab}(i)], \end{aligned}$$

$$\begin{aligned}
\frac{d\rho_{n,m}^{bb}(i)}{dt_i} = & -\gamma_b \rho_{n,m}^{bb}(i) \\
& - \left(\frac{\mu E s_i}{\hbar} \right) [\sqrt{n} e^{-i(\omega_0 - \Omega)t} \rho_{n-1,m}^{ab}(i) \\
& + \sqrt{m} e^{i(\omega_0 - \Omega)t} \rho_{n,m-1}^{ba}(i)]
\end{aligned} \quad (3.1)$$

and

$$\begin{aligned}
\frac{d\rho_{n,m}^{ab}(i)}{dt_i} = & -\gamma \rho_{n,m}^{ab}(i) \\
& - \left(\frac{\mu E s_i}{\hbar} \right) e^{i(\omega_0 - \Omega)t} \\
& \times [\sqrt{m} \rho_{n,m-1}^{aa}(i) - \sqrt{n+1} \rho_{n+1,m}^{bb}(i)], \\
\frac{d\rho_{n,m}^{ba}(i)}{dt_i} = & -\gamma \rho_{n,m}^{ba}(i) \\
& - \left(\frac{\mu E s_i}{\hbar} \right) e^{-i(\omega_0 - \Omega)t} \\
& \times [\sqrt{n} \rho_{n-1,m}^{aa}(i) - \sqrt{m+1} \rho_{n,m+1}^{bb}(i)],
\end{aligned} \quad (3.2)$$

where λ_a stands for the pumping rate for the upper level $\hbar\omega_a$, and the constants γ_a and γ_b indicate the decay rates from respective levels to other lower levels. In (3.2), the rate γ stands for $(\gamma_a + \gamma_b)/2$.

The equation for the photon density matrix (2.30) should be modified as

$$\begin{aligned}
\frac{d\rho_{n,m}}{dt} = & -2\kappa \left(\frac{n+m}{2} \right) \rho_{n,m} \\
& + 2\kappa \sqrt{(n+1)(m+1)} \rho_{n+1,m+1} \\
& + i(n-m)(\Omega - \omega) \rho_{n,m} \\
& + \sum_{i=1}^N \left(\frac{\mu E s_i}{\hbar} \right) [\sqrt{n+1} \rho_{n+1,m}^{ba}(i) \\
& - \sqrt{m} \rho_{n,m-1}^{ba}(i)] e^{i(\omega_0 - \Omega)t} \\
& + [\sqrt{m+1} \rho_{n,m+1}^{ab}(i) - \sqrt{n} \rho_{n-1,m}^{ab}(i)] \\
& \times e^{-i(\omega_0 - \Omega)t}].
\end{aligned} \quad (3.3)$$

The additional two terms on the right-hand side of (3.3) represent the loss effect of photons from the cavity and the expression can be derived with the aid of the reasoning that a heat bath interacting with laser photons is not effectively excited even though the bath absorbs photons.³⁾ The constant κ indicates the loss rate of photons.

In order to solve (3.2) approximately, we put as

$$\rho_{n,m}^{ab}(i) = \hat{\rho}_{n,m}^{ab}(i) e^{i(\omega_0 - \Omega)t}$$

and

$$\rho_{n,m}^{ba}(i) = \hat{\rho}_{n,m}^{ba}(i) e^{-i(\omega_0 - \Omega)t}. \quad (3.4)$$

Introducing (3.4) into (3.2) and ignoring the particle-derivative of $\hat{\rho}_{n,m}^{ab}$ and $\hat{\rho}_{n,m}^{ba}$, one has

$$\begin{aligned}
\rho_{n,m}^{ab}(i) = & - \left(\frac{\mu E s_i}{\hbar} \right) \frac{e^{i(\omega_0 - \Omega)t}}{\gamma + i(\omega_0 - \Omega)} \\
& \times [\sqrt{m} \rho_{n,m-1}^{aa}(i) - \sqrt{n+1} \rho_{n+1,m}^{bb}(i)]
\end{aligned}$$

and

$$\begin{aligned}
\rho_{n,m}^{ba}(i) = & - \left(\frac{\mu E s_i}{\hbar} \right) \frac{e^{-i(\omega_0 - \Omega)t}}{\gamma - i(\omega_0 - \Omega)} \\
& \times [\sqrt{n} \rho_{n-1,m}^{aa}(i) - \sqrt{m+1} \rho_{n,m+1}^{bb}(i)].
\end{aligned} \quad (3.5)$$

Substitution of (3.5) for (3.3) yields

$$\begin{aligned}
\frac{d\rho_{n,m}}{dt} = & -2\kappa \left(\frac{n+m}{2} \right) \rho_{n,m} + 2\kappa \sqrt{(n+1)(m+1)} \\
& \times \rho_{n+1,m+1} + i(n-m)(\Omega - \omega) \rho_{n,m} \\
& + B(\Omega) \sum_i s_i^2 [\sqrt{(n+1)(m+1)} \rho_{n+1,m+1}^{bb}(i) \\
& + \sqrt{nm} \rho_{n-1,m-1}^{aa}(i)] \\
& - \sum_{i=1}^N \left(\frac{\mu E s_i}{\hbar} \right)^2 \left[\frac{n+1}{\gamma - i(\omega_0 - \Omega)} \right. \\
& \left. + \frac{m+1}{\gamma + i(\omega_0 - \Omega)} \right] \rho_{n,m}^{aa}(i) \\
& - \sum_{i=1}^N \left(\frac{\mu E s_i}{\hbar} \right)^2 \left[\frac{n}{\gamma + i(\omega_0 - \Omega)} \right. \\
& \left. + \frac{m}{\gamma - i(\omega_0 - \Omega)} \right] \rho_{n,m}^{bb}(i), \quad (3.6)
\end{aligned}$$

where $B(\Omega)$ is the Einstein coefficient defined in I, (3.10). On the other hand, insertion of (3.5) into (3.1) yields

$$\begin{aligned} \frac{d\rho_{n,m}^{aa}(i)}{dt_i} = & \lambda_a \rho_{n,m} - \gamma_a \rho_{n,m}^{aa}(i) \\ & - B(\Omega) s_i^2 \left[\left(\frac{n+m+2}{2} \right) \rho_{n,m}^{aa}(i) \right. \\ & \left. - \sqrt{(n+1)(m+1)} \rho_{n+1,m+1}^{bb}(i) \right] \\ & - i(\omega_0 - \Omega)(B(\Omega) s_i^2 / 2\gamma)(n-m) \rho_{n,m}^{aa}(i) \end{aligned}$$

and

$$\begin{aligned} \frac{d\rho_{n,m}^{bb}(i)}{dt_i} = & -\gamma_b \rho_{n,m}^{bb}(i) \\ & + B(\Omega) s_i^2 \left[\sqrt{nm} \rho_{n-1,m-1}^{aa}(i) \right. \\ & \left. - \left(\frac{n+m}{2} \right) \rho_{n,m}^{bb}(i) \right] \\ & + i(\omega_0 - \Omega) \times (B(\Omega) s_i^2 / 2\gamma)(n-m) \rho_{n,m}^{bb}(i). \end{aligned} \quad (3.7)$$

For the sake of simplicity, we neglect in the later arguments the last imaginary parts on the right-hand sides of (3.7), since the terms are small in comparison to other real terms on account of the existence of the factor $(n-m)$. In this approximation, the coefficients of the density matrices on the right-hand sides of (3.7) are all real, and thus the atomic density matrices in (3.7) have a same phase with the photon density matrix $\rho_{n,m}$ on the first term of the right-hand side of the first equation of (3.7). This situation indicates that the phases of all matrices are kept constant in time.

4. Determination of Modified Frequency Ω

We are now ready to investigate the properties of the laser light. Let us first determine the modified frequency Ω . The expectation value $\langle a(t) \rangle$ is expressed in terms of the photon density

matrix as

$$\langle a(t) \rangle = \sum_{n=1}^{\infty} \sqrt{n} \rho_{n,n-1} e^{-i\Omega t}. \quad (4.1)$$

Writing as

$$\langle \hat{a} \rangle = \sum_{n=1}^{\infty} \sqrt{n} \rho_{n,n-1}, \quad (4.2)$$

we determine the frequency Ω so that $\langle \hat{a} \rangle$ does not oscillate quickly. Making use of the equation (3.6), one has after some rearrangements an equation

$$\begin{aligned} \frac{d\langle \hat{a} \rangle}{dt} = & \sum_{n=1}^{\infty} \sqrt{n} \frac{d\rho_{n,n-1}}{dt} \\ = & i \left[(\Omega - \omega) \langle \hat{a} \rangle \right. \\ & - (\omega_0 - \Omega) \sum_{i=1}^N \sum_{n=1}^{\infty} \frac{(\mu E s_i / \hbar)^2}{\gamma^2 + (\omega_0 - \Omega)^2} \sqrt{n} \\ & \left. \times \{ \rho_{n,n-1}^{aa}(i) - \rho_{n,n-1}^{bb}(i) \} \right] \\ & - \kappa \langle \hat{a} \rangle \\ & + \frac{1}{2} B(\Omega) \sum_{i=1}^N \sum_{n=1}^{\infty} s_i^2 \sqrt{n} [\rho_{n,n-1}^{aa}(i) - \rho_{n,n-1}^{bb}(i)]. \end{aligned} \quad (4.3)$$

The first term on the right-hand side of (4.3) gives an oscillatory behavior of $\langle \hat{a} \rangle$, and Ω should be determined so that the term vanishes;

$$\begin{aligned} (\Omega - \omega) \langle \hat{a} \rangle = & (\omega_0 - \Omega) \sum_{i=1}^N \sum_{n=1}^{\infty} \frac{(\mu E s_i / \hbar)^2}{\gamma^2 + (\omega_0 - \Omega)^2} \\ & \times \sqrt{n} [\rho_{n,n-1}^{aa}(i) - \rho_{n,n-1}^{bb}(i)], \end{aligned} \quad (4.4)$$

which corresponds to the condition I, (4.5). In virtue of the approximation which ignores the imaginary terms in (3.7), the matrix-elements $\rho_{n,n-1}$, $\rho_{n,n-1}^{aa}(i)$ and $\rho_{n,n-1}^{bb}(i)$ have a same phase and thus the frequency Ω is determined as a real solution of (4.4). Under the condition (4.4), the equation (4.3) can be written as

$$\begin{aligned} \frac{d\langle \hat{a} \rangle}{dt} = & -\kappa \langle \hat{a} \rangle \\ & + \frac{1}{2} B(\Omega) \sum_{i=1}^N \sum_{n=1}^{\infty} s_i^2 \sqrt{n} [\rho_{n,n-1}^{aa}(i) - \rho_{n,n-1}^{bb}(i)]. \end{aligned} \quad (4.5)$$

In the case of resonance, i.e. $\omega = \omega_0$, one has from (4.4)

$$\Omega = \omega \quad (4.6)$$

which coincides with I, (4.6). In the stationary state of photons, we have from (4.5) one more condition

$$\begin{aligned} \kappa < \hat{a} > &= \frac{1}{2} B(\Omega) \sum_{i=1}^N \sum_{n=0}^{\infty} s_i^2 \sqrt{n} \\ &\times [\rho_{n,n-1}^{aa}(i) - \rho_{n,n-1}^{bb}(i)]. \end{aligned} \quad (4.7)$$

Combination of (4.7) and (4.4) yields

$$\Omega = \frac{\gamma\omega + \kappa\omega_0}{\gamma + \kappa}. \quad (4.8)$$

This is the frequency pulling effect mentioned in the preceding paper I.

5. Distribution of Laser Photons

We next consider the diagonal elements of $\rho_{n,n}$ and investigate the number distribution of photons around its mean value. In this case, the equation (3.6) reduces to

$$\begin{aligned} \frac{d\rho_{n,n}}{dt} &= -2\kappa n \rho_{n,n} + 2\kappa(n+1) \rho_{n+1,n+1} \\ &+ B(\Omega)(n+1) \sum_{i=1}^N s_i^2 [\rho_{n+1,n+1}^{bb}(i) - \rho_{n,n}^{aa}(i)] \\ &+ B(\Omega)n \sum_{i=1}^N s_i^2 [\rho_{n-1,n-1}^{aa}(i) - \rho_{n,n}^{bb}(i)]. \end{aligned} \quad (5.1)$$

On the other hand, (3.7) becomes

$$\begin{aligned} \frac{d\rho_{n,n}^{aa}(i)}{dt_i} &= \lambda_a \rho_{n,n} - \gamma_a \rho_{n,n}^{aa}(i) \\ &- B(\Omega) s_i^2 (n+1) [\rho_{n,n}^{aa}(i) - \rho_{n+1,n+1}^{bb}(i)], \\ \frac{d\rho_{n,n}^{bb}(i)}{dt_i} &= -\gamma_b \rho_{n,n}^{bb}(i) \\ &+ B(\Omega) s_i^2 n [\rho_{n-1,n-1}^{aa}(i) - \rho_{n,n}^{bb}(i)]. \end{aligned} \quad (5.2)$$

Since the decay rates of atoms are much larger than the photon loss-rate, the atomic system reaches rapidly to the stationary state. In this situation, the solution of (5.2) is given by

$$\rho_{n,n}^{aa}(i) = \frac{\lambda_a [\gamma_b + B(\Omega) s_i^2 (n+1)]}{\gamma_a \gamma_b + 2\gamma B(\Omega) s_i^2 (n+1)} \rho_{n,n}$$

and

$$\rho_{n+1,n+1}^{bb}(i) = \frac{\lambda_a B(\Omega) s_i^2 (n+1)}{\gamma_a \gamma_b + 2\gamma B(\Omega) s_i^2 (n+1)} \rho_{n,n} \quad (5.3)$$

Substituting (5.3) for (5.1), one has

$$\begin{aligned} \frac{d\rho_{n,n}}{dt} &= n \left[-C \rho_{n,n} \right. \\ &\quad \left. + \alpha\beta \left(\frac{1}{N} \sum_{i=1}^N \frac{s_i^2}{\beta + 2n s_i^2} \right) \rho_{n-1,n-1} \right] \\ &- (n+1) \left[-C \rho_{n+1,n+1} \right. \\ &\quad \left. + \alpha\beta \left(\frac{1}{N} \sum_{i=1}^N \frac{s_i^2}{\beta + 2(n+1) s_i^2} \right) \rho_{n,n} \right], \end{aligned} \quad (5.4)$$

where use has been made of the parameters α , β and C given in I, (4.12). When the laser atoms distribute uniformly in the cavity, one can take the sum in (5.4) in the same way as I, (4.14), and can write as

$$\begin{aligned} \frac{d\rho_{n,n}}{dt} &= \left[-C n \rho_{n,n} + \alpha\beta \cdot \frac{(\beta + 2n) - \sqrt{\beta(\beta + 2n)}}{\beta + 2n} \right. \\ &\quad \left. \times \rho_{n-1,n-1} \right] - \left[-C(n+1) \rho_{n+1,n+1} \right. \\ &\quad \left. + \alpha\beta \cdot \frac{\{\beta + 2(n+1)\} - \sqrt{\beta\{\beta + 2(n+1)\}}}{\beta + 2(n+1)} \right. \\ &\quad \left. \times \rho_{n,n} \right]. \end{aligned} \quad (5.5)$$

Although the appearance of this equation is quite different from the corresponding equation

$$\begin{aligned} \frac{d\rho_{n,n}}{dt} &= n \left[-C \rho_{n,n} + \frac{\alpha\beta}{\beta + n} \rho_{n-1,n-1} \right] \\ &- (n+1) \left[-C \rho_{n+1,n+1} + \frac{\alpha\beta}{\beta + (n+1)} \rho_{n,n} \right] \end{aligned} \quad (5.6)$$

for the running wave, the essential feature of the photon distribution which results from (5.5) is same with that from (5.6). In order to show this, consider the stationary solution of (5.5). It is obtained from the recurrence formula

$$\rho_{n,n} = \left(\frac{\alpha\beta}{C} \right) \frac{(\beta+2n) - \sqrt{\beta(\beta+2n)}}{n(\beta+2n)} \rho_{n-1,n-1}. \quad (5.7)$$

Below threshold ($\alpha < C$), $\beta \gg \bar{n}$, and thus one can approximate as $[(\beta+2n) - \sqrt{\beta(\beta+2n)}] \cong \beta[(1+2n/\beta) - (1+n/\beta)] = n$ and $(\beta+2n) \cong \beta$. In this case, one has

$$\rho_{n,n} = \left(\frac{\alpha\beta}{C} \right)^n \frac{n!}{n! \beta^n} \rho_{0,0} = \left(\frac{\alpha}{C} \right)^n \rho_{0,0} \quad (5.8)$$

From the normalization condition $\sum_{n=0}^{\infty} \rho_{n,n} = 1$, we get $\rho_{0,0} = 1 - (\alpha/C)$. The mean photon number is given by

$$\bar{n} = \sum_{n=0}^{\infty} n \left(\frac{\alpha}{C} \right)^n \rho_{0,0} = \left(\frac{\alpha}{C} \right) / \left(1 - \frac{\alpha}{C} \right). \quad (5.9)$$

In contrast to the result $n=0$ in the average-field theory and the semi-classical theory, the present theory gives a correct mean photon number for the laser light below threshold. Furthermore, we see that the light is chaotic one, since (5.8) and (5.9) give a distribution

$$\rho_{n,n} = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}} \quad (5.10)$$

Far above threshold ($\alpha \gg C$), on the other hand, $n \gg \beta$, and then one can approximate in (5.7) as $[(\beta+2n) - \sqrt{\beta(\beta+2n)}] \cong 2n$ and $(\beta+2n) \cong 2n$. Therefore, we have

$$\rho_{n,n} = \left(\frac{\alpha\beta}{C} \right)^n \frac{2^n n!}{n! 2^n n!} \rho_{0,0} = \left(\frac{\alpha\beta}{C} \right)^n \frac{1}{n!} \rho_{0,0}. \quad (5.11)$$

The normalization condition gives $\rho_{0,0} = \exp[-(\alpha\beta/C)]$ and one obtains the Poisson distribution

$$\rho_{n,n} = \frac{(\alpha\beta/C)^n}{n!} e^{-\alpha\beta/C}. \quad (5.12)$$

The mean number of photons is given by $n = (\alpha\beta/C)$. All the results from (5.5) and (5.6) do coincide with each other.

6. Off-Diagonal Elements of the Photon Density Matrix

In order to investigate the diffusion of the phase of the laser light, we evaluate the off-diagonal elements of the photon density matrix. For the sake of simplicity, we consider here only the case of resonance; $\omega = \omega_0$. In this case, one has from (4.6) and (3.6)

$$\begin{aligned} \frac{d\rho_{n,m}}{dt} = & -2\kappa \left(\frac{n+m}{2} \right) \rho_{n,m} \\ & + 2\kappa \sqrt{(n+1)(m+1)} \rho_{n+1,m+1} \\ & + B_0 \sum_{i=1}^N s_i^2 \left[\sqrt{(n+1)(m+1)} \rho_{n+1,m+1}^{bb}(i) \right. \\ & \quad \left. + \sqrt{nm} \rho_{n-1,m-1}^{aa}(i) \right. \\ & \quad \left. - \left(\frac{n+m+2}{2} \right) \rho_{n,n}^{aa}(i) - \left(\frac{n+m}{2} \right) \rho_{n,m}^{bb}(i) \right], \end{aligned} \quad (6.1)$$

where $B_0 = B(\omega_0)$. In the stationary state of the atomic system for $\omega = \omega_0$, one has from (3.7)

$$\rho_{n,m}^{aa}(i) \cong \frac{\lambda_a [\gamma_b + B_0 s_i^2 (n+m+2)/2]}{\gamma_a \gamma_b + \gamma B_0 s_i^2 (n+m+2)} \rho_{n,m}$$

and

$$\rho_{n+1,m+1}^{bb}(i) \cong \frac{\lambda_a B_0 s_i^2 \sqrt{(n+1)(m+1)}}{\gamma_a \gamma_b + \gamma B_0 s_i^2 (n+m+2)} \rho_{n,m}, \quad (6.2)$$

where a factor proportional to $(n-m)^2$ which appears in the denominators of (6.2) has been neglected. Inserting (6.2) in (6.1) and making the same approximation with (6.2), (6.1) is written as

$$\frac{1}{C} \frac{d\rho_{n,m}}{dt} = - \left(\frac{n+m}{2} \right) \rho_{n,m}$$

$$\begin{aligned}
& + \sqrt{(n+1)(m+1)} \rho_{n+1, m+1} \\
& - \left(\frac{\alpha_0 \beta_0}{C} \right) \left[\frac{1}{N} \sum_{i=1}^N \frac{s_i^2 (n+m+2)/2}{\beta_0 + s_i^2 (n+m+2)/2} \right] \rho_{n, m} \\
& + \left(\frac{\alpha_0 \beta_0}{C} \right) \left[\frac{1}{N} \sum_{i=1}^N \frac{s_i^2 \sqrt{nm}}{\beta_0 + s_i^2 (n+m)/2} \right] \rho_{n-1, m-1},
\end{aligned} \quad (6.3)$$

where the parameters in (6.3) are defined by

$$\alpha_0 = \Lambda_a B_0 / \gamma_a, \quad \beta_0 = \gamma_a \gamma_b / 2 \gamma B_0, \quad C = 2\kappa. \quad (6.4)$$

Below threshold ($\alpha_0 < C$), the mean photon number $\bar{n}$ is much smaller than β_0 and (6.3) can be approximated as

$$\begin{aligned}
\frac{1}{C} \frac{d\rho_{n, m}}{dt} = & - \left(\frac{n+m}{2} \right) \rho_{n, m} + \sqrt{(n+1)(m+1)} \\
& \times \rho_{n+1, m+1} - \left(\frac{\alpha_0}{C} \right) \left(\frac{n+m+2}{2} \right) \rho_{n, m} \\
& + \left(\frac{\alpha_0}{C} \right) \sqrt{nm} \rho_{n-1, m-1},
\end{aligned} \quad (6.5)$$

where use has been made of the result

$$\frac{1}{N} \sum_{i=1}^N s_i^2 = \frac{1}{L} \int_0^L 2 \sin^2 kx dx = 1.$$

Assuming that the solution of (6.5) has the form as

$$\rho_{n, m}(t) = (\alpha_0/C)^{(n+m)/2} T_{n-m}(t), \quad (6.6)$$

one has from (6.5)

$$\begin{aligned}
\frac{1}{C} \frac{d\rho_{n, m}}{dt} = & - \frac{1}{2} (\sqrt{n} - \sqrt{m})^2 (\alpha_0/C)^{(n+m)/2} T_{n-m} \\
& - \frac{1}{2} (\sqrt{n+1} - \sqrt{m+1})^2 (\alpha_0/C) (\alpha_0/C)^{(n+m)/2} T_{n-m} \\
& \cong - \frac{1}{2} (\sqrt{n} - \sqrt{m})^2 \rho_{n, m}.
\end{aligned} \quad (6.7)$$

In the last equality, a higher order term concerning $\alpha_0/C (\ll 1)$ has been omitted. Writing as $m = n \pm \nu$ and expanding as $\sqrt{m} \cong \sqrt{n} \pm (\nu/2\sqrt{n})$, one gets a simple equation

$$\frac{d\rho_{n, m}}{dt} = - \frac{C\nu^2}{8n} \rho_{n, m}, \quad (6.8)$$

of which solution is given by

$$\rho_{n, m}(t) = \rho_{n, m}(0) e^{-(C\nu^2/8n)t}. \quad (6.9)$$

Replacing n in the exponent of (6.9) by its average number $\bar{n} \cong (\alpha_0/C)$ below threshold, one obtains

$$\rho_{n, m}(t) \cong \left(\frac{\alpha_0}{C} \right)^{(n+m)/2} \exp \left[- \frac{C}{8} \left(\frac{C}{\alpha_0} \right) \nu^2 t \right] \quad (6.10)$$

If one uses $C \sim 10^6 \text{ sec}^{-1}$, one can see that the off-diagonal elements vanish in the order of time $\tau \sim 10^{-5} \text{ sec}$, and the life-time of $\rho_{n, m}$ becomes smaller for larger ν . This situation means that the phase of $\langle \hat{a} \rangle$ diffuses very rapidly.

Far above threshold ($\alpha_0/C \gg 1$), the mean value $\bar{n}$ is much larger than β_0 and (6.3) can be approximated as

$$\begin{aligned}
\frac{1}{C} \frac{d\rho_{n, m}}{dt} = & - \left(\frac{n+m}{2} \right) \rho_{n, m} + \sqrt{(n+1)(m+1)} \\
& \times \rho_{n+1, m+1} - \left(\frac{\alpha_0 \beta_0}{C} \right) \rho_{n, m} + \left(\frac{\alpha_0 \beta_0}{C} \right) \\
& \times \frac{\sqrt{nm}}{(n+m)/2} \rho_{n-1, m-1}.
\end{aligned} \quad (6.11)$$

Writing the solution of (6.11) in the form

$$\rho_{n, m}(t) = \frac{(\alpha_0 \beta_0 / C)^{(n+m)/2}}{\sqrt{n! m!}} T_{|n-m|}(t), \quad (6.12)$$

one obtains

$$\begin{aligned}
\frac{1}{C} \frac{d\rho_{n, m}}{dt} = & - \left[\left(\frac{n+m}{2} \right) - \frac{nm}{(n+m)/2} \right] \\
& \times \frac{(\alpha_0 \beta_0 / C)^{(n+m)/2}}{\sqrt{n! m!}} T_{|n-m|} = - \frac{(n-m)^2}{2(n+m)} \rho_{n, m}.
\end{aligned} \quad (6.13)$$

If one writes as $m = n \pm \nu$, (6.13) becomes

$$\frac{1}{C} \frac{d\rho_{n, m}}{dt} = - \frac{\nu^2}{4n} \rho_{n, m} \quad (6.14)$$

and its solution is expressed as

$$\rho_{n, m}(t) = \rho_{n, m}(0) e^{-(C\nu^2/4n)t} \quad (6.15)$$

The replacement of n in the exponent by its mean value $\bar{n} = \alpha_0 \beta_0 / C$ in (6.15) gives

$$\rho_{n,m}(t) = \frac{(\alpha_0 \beta_0 / C)^{(n+m)/2}}{\sqrt{n! m!}} \cdot e^{-\alpha_0 \beta_0 / C} \times e^{-(C/4\beta_0)(C/\alpha_0)\nu^2 t} \quad (6.16)$$

Thanks to the appearance of a large constant $\beta_0 \sim 10^7$ in the exponent of the last factor in (6.16) in contrast to the case of (6.10), the life-time of the off-diagonal element increases extraordinary in comparison to that of the off-diagonal element below threshold, and its value becomes the order of 10 min. Therefore, one can effectively regard that the photon density matrix above threshold is that for the coherent state.⁴⁾

7. Validity of the Average-Field Theory

With the aid of the microscopic theory developed above, we finally make clear the approximations tacitly used in the average-field theory.

Let us first consider the equations (5.2) for the density matrices for the i -th atom. The matrix-elements defined by

$$\rho^{aa}(i) = \sum_{n=0}^{\infty} \rho_{n,n}^{aa}(i), \quad \rho^{bb}(i) = \sum_{n=0}^{\infty} \rho_{n,n}^{bb}(i) \quad (7.1)$$

represent the probabilities to find the i -th atom in the upper level $\hbar\omega_a$ and the lower level $\hbar\omega_b$, irrespective of the photon number, respectively. Taking the sum over n in (5.2), one obtains

$$\begin{aligned} \frac{d\rho^{aa}(i)}{dt_i} &= \lambda_a - \gamma_a \rho^{aa}(i) \\ &\quad - B(\Omega) s_i^2 \sum_{n=0}^{\infty} (n+1) [\rho_{n,n}^{aa}(i) - \rho_{n+1,n+1}^{bb}(i)] \\ &= \lambda_a - \gamma_a \rho^{aa}(i) - B(\Omega) s_i^2 \sum_{n=0}^{\infty} n [\rho_{n,n}^{aa}(i) \\ &\quad - \rho_{n,n}^{bb}(i)] - B(\Omega) s_i^2 \rho^{aa}(i) \end{aligned}$$

and

$$\begin{aligned} \frac{d\rho^{bb}(i)}{dt_i} &= -\gamma_b \rho^{bb}(i) + B(\Omega) s_i^2 \sum_{n=0}^{\infty} n \\ &\quad \times [\rho_{n,n}^{aa}(i) - \rho_{n,n}^{bb}(i)] + B(\Omega) s_i^2 \rho^{aa}(i). \end{aligned} \quad (7.2)$$

The last terms on the right-hand sides of (7.2) indicate the spontaneous emission of photons due to the transition of the i -th atom from the level $\hbar\omega_a$ to the lower level $\hbar\omega_b$. If one neglects these terms and can write

$$\begin{aligned} \sum_{n=0}^{\infty} n (\rho_{n,n}^{aa}(i) - \rho_{n,n}^{bb}(i)) &= [\rho^{aa}(i) - \rho^{bb}(i)] |\langle \hat{a} \rangle|^2, \end{aligned} \quad (7.3)$$

one has

$$\begin{aligned} \frac{d\rho^{aa}(i)}{dt_i} &= \lambda_a - \gamma_a \rho^{aa}(i) \\ &\quad - B(\Omega) s_i^2 [\rho^{aa}(i) - \rho^{bb}(i)] |\langle \hat{a} \rangle|^2 \end{aligned}$$

and

$$\begin{aligned} \frac{d\rho^{bb}(i)}{dt} &= -\gamma_b \rho^{bb}(i) \\ &\quad + B(\Omega) s_i^2 [\rho^{aa}(i) - \rho^{bb}(i)] |\langle \hat{a} \rangle|^2, \end{aligned} \quad (7.4)$$

which are nothing but the equations I, (3.9) in the average-field theory. From the above arguments, one sees first that the average-field theory has neglected the effects of spontaneous emission of photons, and thus that the theory does not describe correctly the properties of the laser light below threshold, where the effects of spontaneous emission play a main role. We next discuss the relation (7.3). As will be seen soon after, the relation (7.3) is valid only in the case when the photon state is a coherent state⁴⁾ and thus the average-field theory tacitly assumes that the solution of I, (4.1) is given by the coherent state, without solving it explicitly. For simplicity, we show this situation only for the case of resonance $\omega = \omega_0$. In this case, the density matrix of coherent state is expressed as

$$\rho_{n,m} = \frac{(\alpha_0 \beta_0 / C)^{(n+m)/2}}{\sqrt{n! m!}} \cdot e^{-\alpha_0 \beta_0 / C} \quad (7.5)$$

by omitting the time-dependent factor in (6.16).

By making use of (7.5), one can readily show that

$$\begin{aligned} \langle \hat{a} \rangle &= \sum_{n=1}^{\infty} \sqrt{n} \rho_{n,n-1} = \sqrt{\alpha_0 \beta_0 / C}, \\ \langle \hat{a} \rangle^* &= \sum_{n=1}^{\infty} \sqrt{n} \rho_{n-1,n} = \sqrt{\alpha_0 \beta_0 / C} \end{aligned} \quad (7.6)$$

and

$$\langle a^* a \rangle = (\alpha_0 \beta_0 / C) = |\langle \hat{a} \rangle|^2.$$

On the other hand, one has from (5.3) and (7.5)

$$\begin{aligned} \sum_{n=1}^{\infty} n \rho_{n,n}^{aa}(i) &= \sum_{n=1}^{\infty} n \cdot \frac{\lambda_a [\gamma_b + B_0 s_i^2 (n+1)]}{\gamma_a \gamma_b + 2 \gamma B_0 s_i^2 (n+1)} \rho_{n,n} \\ &= \left(\frac{\alpha_0 \beta_0}{C} \right) \sum_{n=0}^{\infty} \frac{\lambda_a [\gamma_b + B_0 s_i^2 (n+2)]}{\gamma_a \gamma_b + 2 \gamma B_0 s_i^2 (n+2)} \\ &\quad \times \frac{(\alpha_0 \beta_0 / C)^n}{n!} e^{-\alpha_0 \beta_0 / C} \\ &\cong \left(\frac{\alpha_0 \beta_0}{C} \right) \sum_{n=0}^{\infty} \rho_{n,n}^{aa}(i) = \left(\frac{\alpha_0 \beta_0}{C} \right) \rho^{aa}(i). \end{aligned} \quad (7.7)$$

In the last equality, we have replaced $(n+2)$ by $(n+1)$ on treating the effect of spontaneous emission indifferently. In a quite same way, one can show that

$$\sum_{n=1}^{\infty} n \rho_{n,n}^{bb}(i) = \left(\frac{\alpha_0 \beta_0}{C} \right) \rho^{bb}(i). \quad (7.8)$$

From (7.6), (7.7) and (7.8), one has the relation (7.3).

We next investigate the equation (4.5) for the light amplitude. In the case of resonance $\omega = \omega_0$, one can write

$$\begin{aligned} \frac{d\langle \hat{a} \rangle}{dt} &= -\kappa \langle \hat{a} \rangle \\ &+ \frac{1}{2} B_0 \sum_{i=1}^N s_i^2 \sum_{n=1}^{\infty} \sqrt{n} [\rho_{n,n-1}^{aa}(i) - \rho_{n,n-1}^{bb}(i)]. \end{aligned} \quad (7.9)$$

Inserting (6.2) into the right-hand side of (7.9) and making use of (7.5), one has

$$\begin{aligned} \sum_{n=1}^{\infty} \sqrt{n} \rho_{n,n-1}^{aa}(i) &= \sum_{n=1}^{\infty} \frac{\lambda_a [\gamma_b + B_0 s_i^2 (2n+1)/2]}{\gamma_a \gamma_b + \gamma B_0 s_i^2 (2n+1)} \\ &\quad \times \frac{\sqrt{n} (\alpha_0 \beta_0 / C)^{n-(1/2)}}{\sqrt{n! (n-1)!}} e^{-\alpha_0 \beta_0 / C} \\ &\cong \sum_{n=0}^{\infty} \frac{\lambda_a [\gamma_b + B_0 s_i^2 (n+1)]}{\gamma_a \gamma_b + 2 \gamma B_0 s_i^2 (n+1)} \\ &\quad \times \frac{(\alpha_0 \beta_0 / C)^n}{n!} e^{-\alpha_0 \beta_0 / C} \cdot \sqrt{\frac{\alpha_0 \beta_0}{C}} \\ &= \sum_{n=0}^{\infty} \rho_{n,n}^{aa} \sqrt{\alpha_0 \beta_0 / C} = \rho^{aa}(i) \langle \hat{a} \rangle, \end{aligned} \quad (7.10)$$

by replacing $(2n+3)$ indifferently to $2(n+1)$. In the same approximations, one has from (6.2)

$$\sum_{n=1}^{\infty} \sqrt{n} \rho_{n,n-1}^{bb}(i) = \rho^{bb}(i) \langle \hat{a} \rangle. \quad (7.11)$$

Introduction of (7.10) and (7.11) into (7.9) yields the equation

$$\begin{aligned} \frac{d\langle \hat{a} \rangle}{dt} &= -\kappa \langle \hat{a} \rangle \\ &+ \frac{1}{2} B_0 \sum_{i=1}^N s_i^2 [\rho^{aa}(i) - \rho^{bb}(i)] \langle \hat{a} \rangle, \end{aligned} \quad (7.12)$$

which is nothing but the equation I, (4.4) for the case of $\omega = \omega_0$.

From the arguments mentioned above, one can conclude that the average-field theory and thus the well-known semi-classical theory are valid only in the case when the photon state is given by a coherent state and the mean value of photon number is much larger than the number of photons produced by spontaneous emission.

Appendix

In this appendix, we give a proof of the relation (2.28). From the definitions in (2.18) and the equations (2.13), one has

$$\begin{aligned}
\frac{dp_{n,m}^{aa}(i)}{dt_j} &= \sum_{\alpha_1 \cdots \alpha_j \cdots \alpha_N}^i \left[\frac{dA_n^{\alpha_1 \cdots a \cdots a \cdots \alpha_N}}{dt_j} A_m^{\alpha_1 \cdots a \cdots a \cdots \alpha_N^*} \right. \\
&\quad \left. + A_n^{\alpha_1 \cdots a \cdots a \cdots \alpha_N} \frac{dA_m^{\alpha_1 \cdots a \cdots a \cdots \alpha_N^*}}{dt_j} \right] \\
&= \sum_{\alpha_1 \cdots \alpha_N}^{i,j} \left[\frac{dA_n^{\alpha_1 \cdots a \cdots a \cdots \alpha_N}}{dt_j} A_m^{\alpha_1 \cdots a \cdots a \cdots \alpha_N^*} \right. \\
&\quad + \frac{dA_n^{\alpha_1 \cdots a \cdots b \cdots \alpha_N}}{dt_j} A_m^{\alpha_1 \cdots a \cdots b \cdots \alpha_N^*} \\
&\quad + A_n^{\alpha_1 \cdots a \cdots a \cdots \alpha_N} \frac{dA_m^{\alpha_1 \cdots a \cdots a \cdots \alpha_N^*}}{dt_j} \\
&\quad \left. + A_n^{\alpha_1 \cdots a \cdots b \cdots \alpha_N} \frac{dA_m^{\alpha_1 \cdots a \cdots b \cdots \alpha_N^*}}{dt_j} \right] \\
&= \left(\frac{\mu E s_j}{\hbar} \right) \sum_{\alpha_1 \cdots \alpha_N}^{i,j} \left[\sqrt{n+1} e^{i(\omega_0 - \omega)t} \right. \\
&\quad \times A_{n+1}^{\alpha_1 \cdots a \cdots b \cdots \alpha_N} A_m^{\alpha_1 \cdots a \cdots a \cdots \alpha_N^*} \\
&\quad - \sqrt{n} e^{-i(\omega_0 - \omega)t} A_{n-1}^{\alpha_1 \cdots a \cdots a \cdots \alpha_N} A_m^{\alpha_1 \cdots a \cdots b \cdots \alpha_N^*} \\
&\quad + \sqrt{m+1} e^{-i(\omega_0 - \omega)t} \\
&\quad \times A_n^{\alpha_1 \cdots a \cdots a \cdots \alpha_N} A_{m+1}^{\alpha_1 \cdots a \cdots b \cdots \alpha_N^*} \\
&\quad \left. - \sqrt{m} e^{i(\omega_0 - \omega)t} A_n^{\alpha_1 \cdots a \cdots b \cdots \alpha_N} A_{m-1}^{\alpha_1 \cdots a \cdots a \cdots \alpha_N^*} \right] \quad (A.1)
\end{aligned}$$

and

$$\begin{aligned}
\frac{dp_{n,m}^{bb}(i)}{dt_j} &= \sum_{\alpha_1 \cdots \alpha_j \cdots \alpha_N}^i \left[\frac{dA_n^{\alpha_1 \cdots b \cdots b \cdots \alpha_N}}{dt_j} A_m^{\alpha_1 \cdots b \cdots b \cdots \alpha_N^*} \right. \\
&\quad \left. + A_n^{\alpha_1 \cdots b \cdots b \cdots \alpha_N} \frac{dA_m^{\alpha_1 \cdots b \cdots b \cdots \alpha_N^*}}{dt_j} \right] \\
&= \sum_{\alpha_1 \cdots \alpha_N}^{i,j} \left[\frac{dA_n^{\alpha_1 \cdots b \cdots b \cdots \alpha_N}}{dt_j} A_m^{\alpha_1 \cdots b \cdots b \cdots \alpha_N^*} \right. \\
&\quad + \frac{dA_n^{\alpha_1 \cdots b \cdots a \cdots \alpha_N}}{dt_j} A_m^{\alpha_1 \cdots b \cdots a \cdots \alpha_N^*} \\
&\quad + A_n^{\alpha_1 \cdots b \cdots b \cdots \alpha_N} \frac{dA_m^{\alpha_1 \cdots b \cdots b \cdots \alpha_N^*}}{dt_j} \\
&\quad \left. + A_n^{\alpha_1 \cdots b \cdots a \cdots \alpha_N} \frac{dA_m^{\alpha_1 \cdots b \cdots a \cdots \alpha_N^*}}{dt_j} \right]
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\mu E s_j}{\hbar} \right) \sum_{\alpha_1 \cdots \alpha_N}^{i,j} \left[\sqrt{n+1} e^{i(\omega_0 - \omega)t} \right. \\
&\quad \times A_{n+1}^{\alpha_1 \cdots b \cdots a \cdots \alpha_N} A_m^{\alpha_1 \cdots b \cdots b \cdots \alpha_N^*} \\
&\quad - \sqrt{n} e^{-i(\omega_0 - \omega)t} A_{n-1}^{\alpha_1 \cdots b \cdots b \cdots \alpha_N} A_m^{\alpha_1 \cdots b \cdots a \cdots \alpha_N^*} \\
&\quad + \sqrt{m+1} e^{-i(\omega_0 - \omega)t} A_n^{\alpha_1 \cdots b \cdots a \cdots \alpha_N} A_{m+1}^{\alpha_1 \cdots b \cdots b \cdots \alpha_N^*} \\
&\quad \left. - \sqrt{m} e^{i(\omega_0 - \omega)t} A_n^{\alpha_1 \cdots b \cdots b \cdots \alpha_N} A_{m-1}^{\alpha_1 \cdots b \cdots a \cdots \alpha_N^*} \right] \quad (A.2)
\end{aligned}$$

Collecting (A.1) and (A.2) and carrying out the summation over the states α_i , one obtains

$$\begin{aligned}
\frac{dp_{n,m}^{aa}(i)}{dt_j} + \frac{dp_{n,m}^{bb}(i)}{dt_j} &= \left(\frac{\mu E s_j}{\hbar} \right) \sum_{\alpha_1 \cdots \alpha_i \cdots \alpha_N}^j \left[\sqrt{n+1} e^{i(\omega_0 - \omega)t} \right. \\
&\quad \times A_{n+1}^{\alpha_1 \cdots a_i \cdots b \cdots \alpha_N} A_m^{\alpha_1 \cdots a_i \cdots a \cdots \alpha_N^*} \\
&\quad - \sqrt{n} e^{-i(\omega_0 - \omega)t} A_{n-1}^{\alpha_1 \cdots a_i \cdots a \cdots \alpha_N} A_m^{\alpha_1 \cdots a_i \cdots b \cdots \alpha_N^*} \\
&\quad + \sqrt{m+1} e^{-i(\omega_0 - \omega)t} A_n^{\alpha_1 \cdots a_i \cdots a \cdots \alpha_N} A_{m+1}^{\alpha_1 \cdots a_i \cdots b \cdots \alpha_N^*} \\
&\quad \left. - \sqrt{m} e^{i(\omega_0 - \omega)t} A_n^{\alpha_1 \cdots a_i \cdots b \cdots \alpha_N} A_{m-1}^{\alpha_1 \cdots a_i \cdots a \cdots \alpha_N^*} \right] \\
&= \left(\frac{\mu E s_j}{\hbar} \right) \left[\sqrt{n+1} e^{i(\omega_0 - \omega)t} p_{n+1,m}^{ba}(j) \right. \\
&\quad - \sqrt{n} e^{-i(\omega_0 - \omega)t} p_{n-1,m}^{ab}(j) \\
&\quad + \sqrt{m+1} e^{-i(\omega_0 - \omega)t} p_{n,m+1}^{ab}(j) \\
&\quad \left. - \sqrt{m} e^{i(\omega_0 - \omega)t} p_{n,m-1}^{ba}(j) \right] \quad (A.3)
\end{aligned}$$

On the other hand, we see from (2.19)

$$\begin{aligned}
\frac{dp_{n,m}^{aa}(j)}{dt_j} &= \left(\frac{\mu E s_j}{\hbar} \right) \left[\sqrt{n+1} e^{i(\omega_0 - \omega)t} p_{n+1,m}^{ba}(j) \right. \\
&\quad \left. + \sqrt{m+1} e^{-i(\omega_0 - \omega)t} p_{n,m+1}^{ab}(j) \right],
\end{aligned}$$

$$\begin{aligned} \frac{dp_{n,m}^{bb}(j)}{dt_j} = & - \left(\frac{\mu E s_j}{\hbar} \right) \left[\sqrt{n} e^{-i(\omega_s - \omega)t} p_{n-1,m}^{ab}(j) \right. \\ & \left. + \sqrt{m} e^{i(\omega_s - \omega)t} p_{n,m-1}^{ba}(j) \right]. \end{aligned} \quad (\text{A.4})$$

Comparison of (A.3) with (A.4) gives the relation

$$\frac{dp_{n,m}^{aa}(i)}{dt_j} + \frac{dp_{n,m}^{bb}(i)}{dt_j} = \frac{dp_{n,m}^{aa}(j)}{dt_j} + \frac{dp_{n,m}^{bb}(j)}{dt_j}. \quad (\text{A.5})$$

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