Two-Dimensional and Three-Dimensional Envelope Solitons

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Extensive investigations have recently been carried out for one-dimensional propagation of envelope solitons¹. They are obtained as stationary localized solutions of the nonlinear Schrödinger equation by balancing the effect of group dispersion, $d^2\omega/dk^2$, and that of quadratic nonlinearity in the susceptibility, $\chi_{NL} \sim |\phi|^2$, where ϕ is an appropriately normalized field amplitude. However, when the equation is generalized to two and three dimensions with the same quadratic nonlinearity, localized solutions are found to be nonstationary except for a very special case of measure zero they tend to either collapse or expand indefinitely depending on the initial condition². Question then arises whether higher order nonlinearity can stop the collapse or expansion and admit localized stationary solutions. In this paper, we present such solutions for the case when the nonlinear susceptibility has the form $\chi_{NL} \sim \{\exp[-|\phi|^2] - 1\}$. This form of nonlinear susceptibility arises in the problem of steady propagation of large amplitude high-frequency waves in a plasma. Physically, the nonlinearity arises in this case through local density depression created by ponderomotive force of the wave³. The nonlinearity saturates when most of the plasma particles are expelled from regions of high field intensity. In this letter, we present examples for both

electromagnetic and electrostatic waves confined in a stationary plasma cavity. Although we specifically discuss only the problem of waves in a plasma, the results obtained are quite general and may be regarded as general soliton-like solutions for a certain class of nonlinear wave equations.

We first consider steady propagation (in the z -direction) of an azimuthally-polarized electromagnetic wave which is localized in the radial direction. Let ϕ be the modulus of the azimuthal electric field normalized as $\phi = e|E_\theta|/\omega[m(T_e + T_i)]^{1/2}$, where the notation used is standard. If we normalize the radial distance by the collisionless skin depth, c/ω_p , where ω_p is the plasma frequency in the absence of the wave, the stationary wave equation takes the following form⁴,

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \phi}{\partial r} + [1 - \sigma - \exp(-\phi^2)] \phi = 0, \quad (1)$$

where σ is the normalized frequency mismatch from the plane-wave dispersion relation, $\sigma = (\omega_p^2 + c^2 k^2 - \omega^2)/\omega_p^2$, and k is the wavenumber in the z -direction. We have solved this equation as an eigenvalue problem (for eigenvalue σ) demanding that ϕ decreases to zero monotonically as r increases. From the structure of the equation, we can infer the existence of

such a solution ; namely, near the origin we have a monotonically decreasing solution of the form $\phi = \phi_0[1 - \alpha r^2]$ with $\alpha = [1 - \sigma - \exp(-\phi_0^2)]/4$ if $\sigma < [1 - \exp(-\phi_0^2)]$, whereas at large distances where $\phi \rightarrow 0$ a monotonically converging solution is obtained for $\sigma > 0$ in the form $K_v(\sqrt{\sigma}r)$, K_v being the v -th order modified Bessel function of the second kind.

Numerical results for the solutions are shown in Fig.1.

We see from this figure that for σ exceeding 0.1, ϕ_0 becomes of order or greater than unity, so that the ^{usual} approximation of quadratic nonlinearity breaks down.

Two features are to be observed in the results. First, ϕ_0 increases monotonically with σ (Fig.2), and secondly, the width of the radial profile first decreases as σ increases up to about 0.3 and then starts increasing indefinitely as σ approaches unity, assuming a broad minimum which extends from $\sigma \sim 0.3$ to $\sigma \sim 0.5$. The minimum width at half-intensity maximum ($\phi = \phi_0/\sqrt{2}$) is about three times the collisionless skin depth. To understand the first feature, we neglect the two dimensional character of the profile and replaces $r^{-1}d/dr(rd\phi/dr)$ by $d^2\phi/dr^2$, to find the relation $\sigma = [\exp(-\phi_0^2) + \phi_0^2 - 1]/\phi_0^2$. This is shown by the dashed curve in Fig. 2.

In two-dimensions, there is additional diffraction effect which requires a stronger nonlinearity or higher

amplitude for balancing. Therefore, for a given frequency mismatch σ , the maximum amplitude is greater in two-dimensional cylindrical case than in one-dimensional slab case, as shown in this figure.

The behavior of the width may be interpreted as follows. For small σ , or for small ϕ_0 , the main part of the solution is governed by $K_0(\sqrt{\sigma}r)$ which shrinks toward the origin as σ increases. This simply represents the tendency of collapse with increasing nonlinearity. On the other hand, for large σ , ϕ becomes substantial over a wide range of r ; then neglecting $\exp[-\phi^2]$, one obtains a solution of the form $J_0(\sqrt{1-\sigma}r)$ which expands indefinitely as σ approaches unity. Physically, this represents the fact that as ϕ increases the index of refraction approaches unity (because the plasma is expelled) and hence the solution becomes close to a free-space plane wave which has an infinite width.

So far we have considered only a radial localization of the wave. Such a solution is, however, likely to be unstable against self-modulation of the amplitude along the z -direction. Several mechanisms^{5,6} have been proposed which may cause such an instability. However, as far as $\omega \gg \omega_p$ and the light wave is untrapped in the z -direction, the modulation propagates with a very high speed, so that even a slight inhomogeneity of plasma parameters along

the z -direction can lead to a low-level saturation of the instability because of convective effects. A deleterious situation may occur only when ω is close to ω_p and the light wave is trapped in a locally underdense region.

The electromagnetic filament will then be split into small pieces in the z -direction and the final stationary state will become a three-dimensional soliton localized in all directions. Restricting ourselves to azimuthally-polarized electromagnetic waves, we can obtain such a solution by replacing $k^2 c^2 / \omega_p^2$ in σ of Eq.(1) by $-\partial^2 / \partial z^2$, where z is normalized by the skin depth. The equation then becomes

$$\left[\frac{\partial^2}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \right] \phi + \{ 1 - \sigma - \exp[-\phi^2] \} \phi = 0, \quad (2)$$

where σ now stands for $(\omega_p^2 - \omega^2) / \omega_p^2$. Soliton-like solution of this equation for two limiting cases, $\phi^2 \gg 1$, and $\phi^2 \ll 1$, assumes the following form :

$$\phi \sim \begin{cases} \cos qz J_0(\sqrt{1-\sigma-q^2} r) & (\phi^2 \gg 1) \\ \exp(-\gamma|z|) K_0(\sqrt{\sigma-\gamma^2} r) & (\phi^2 \ll 1) \end{cases} \quad (3)$$

where q and γ are certain positive parameters with $(\gamma^2 + q^2) < 1$ and σ is assumed to have a value between γ^2 and $(1 - q^2)$. The condition that these two solutions be

connected smoothly gives a relation between q and γ .

One can easily infer that they are of the same order.

However, since we are now dealing with a partial differential equation, we have two independent eigenvalues, q and σ . This brings in an arbitrariness in the shape of the soliton. Now, for given wave energy, $I = \int_{-\infty}^{\infty} dz \int_0^{\infty} r dr |\phi|^2$, the amplitude ϕ_0^2 has a maximum at a certain value of eccentricity of the spheroidal shape. For instance, if $\phi^2 \gg 1$ in the main region of soliton, $\phi_0^2 \sim Iq[1-\sigma-q^2]$ which assumes a maximum value at $q = [(1-\sigma)/3]^{1/2}$. This solution may represent the most stable state. The typical size of this solution is again given by the collisionless skin depth. If we further maximize ϕ_0^2 by varying σ , we obtain $\sigma=0$. This corresponds to an electromagnetic soliton in which the field oscillates at the plasma frequency and might represent the final saturation state of the induced Brillouin backscattering at the cut-off density.

We now consider the electrostatic case. As in ref.2, we treat a spherically symmetric electron-plasma oscillation confined in a stationary plasma cavity.

We now take ϕ to be the modulus of the radial derivative of the electrostatic potential $\varphi(r)$ normalized as

$\phi = e |\partial\varphi/\partial r| / [3T_e(T_e + T_i)]^{1/2}$, where the radial distance is now normalized by $\sqrt{3}\lambda_D$, λ_D being the Debye length

($=[T_e/m\omega_p]^2$). Our equation then becomes²

$$\frac{\partial}{\partial r} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \phi + [1 - \sigma - \exp(-\phi^2)] \phi = 0, \quad (4)$$

where $\sigma = (\omega_p^2 - \omega^2)/\omega_p^2$. Localized solutions of this eigenvalue equation have recently been obtained by J. Z. Wilcox and T. J. Wilcox⁷ for the case $\sigma = 1/2$. One distinct feature different from the electromagnetic case is that ϕ now vanishes at the origin like $I_{3/2}(\sqrt{\sigma}r)$. The "ground state" solution then assumes a maximum at some finite distance, say $r = r_m$, and then damps out as $K_{3/2}(\sqrt{\sigma}r)$ for large r , as shown in Fig.3. The dependence of the maximum amplitude, ϕ_m , as well as the width, on σ is similar to the electromagnetic case. Also, r_m behaves in a way similar to the width; it assumes a minimum value of order $5\lambda_D$ when $\sigma \approx 3$. The solution corresponding to this minimum size will most likely be stable against all perturbations described by the same electrostatic wave equation.

In order to apply the above results to realistic plasma situations, one has to take into account various dissipation mechanisms as well as possible instabilities due to coupling of electromagnetic and electrostatic components. For electromagnetic solitons, perturbations in the azimuthal direction produce coupling with electro-

static waves. The minimum radius of the electrostatic soliton is of order $5\lambda_D$. Such a localized oscillation may suffer a strong transit-time damping due to thermal electrons⁸.

However, for the case of spherical solitons, the transit time damping will be reduced as compared with the one-dimensional case, as most of the particles pass sideways through the soliton and do not experience the acceleration by the maximum field, ϕ_m . A preliminary calculation using the straight-orbit approximation for the transit particle shows that the damping rate is reduced by factor 20-40 in the spherical case as compared with the one-dimensional case for the same typical wavelength.

There are many other effects which we have ignored in the above argument. These include the effect of ions trapped in the cavity, that of electrons reflected by the ponderomotive force, long-range interactions among solitons via particles and untrapped waves, possible rotation of the soliton, and so on. These problems are now under investigation and will be discussed elsewhere.

We are grateful to Dr. K. Mima, Mr. A. Ishida and Mr. H. Hojo of Hiroshima University for many fruitful discussions and assistance in the calculation. This work is partially supported (A.H) by the National Science Foundation Grant No. 7412932.

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Fig. 1. Two-dimensional solitons for azimuthally-polarized electromagnetic waves for $\sigma = .1, .3, .5, .7$ and $.9$.

Fig. 2. Relation between σ and ϕ_0 for two-dimensional electromagnetic soliton. Dashed curve denotes the relation for the one-dimensional slab model.

Fig. 3. Soliton profile for spherically symmetric electrostatic waves for $\sigma = .1, .3, .5, .7$ and $.9$.