EFFECT OF RANDOM DENSITY FLUCTUATIONS ON PARAMETRIC
INTERACTIONS IN A PLASMA

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Abstract

We have analytically investigated the influence of quasi-static background random density fluctuations on three-wave parametric coupling processes in a plasma. As an example, the case of stimulated Raman scattering is discussed in detail. After Fourier analysing the time variable and eliminating the amplitude of one of the stimulated waves, the basic equation describing the parametric process reduces to a fourth order differential equation with coefficients which are random

functions of the position co-ordinate. Standard methods from theory of wave propagation in random media can now be used. Assuming $|\epsilon(x)| = |[\eta(x) - \eta_0]/\eta_0| \ll 1$, analytical results correct to order $|\epsilon^2|$ are derived for the limiting cases when the correlation length of turbulence is long or short compared to the wavelength of parametrically interacting waves. It is found that in a statistically homogeneous medium, the effect of fluctuations is to enhance the growth length i.e. to make the growth weaker. We next consider convective parametric amplification in a weakly inhomogeneous medium with random density fluctuations superimposed on it. The fourth order differential equation in position space is now Fourier transformed into a second order differential equation in k space. An expression for the convective amplification factor correct to order $|\epsilon^2|$ is obtained for the case of long wavelength turbulence and shows that it is unaltered by the presence of background random density fluctuations.

1. INTRODUCTION

The parametric interaction of intense electromagnetic waves with plasmas has attracted a great deal of attention recently because of the potential importance of such processes in laser fusion [1,2]. Extensive analytical and computer simulation work has been carried out for homogeneous and weakly inhomogeneous plasmas and has demonstrated the crucial role that may be played by parametric processes like anomalous absorption and anomalous scattering [3-5]. In all these

calculations the background plasma is assumed to be quiescent. However, the plasma produced by laser irradiation of a pellet is likely to be turbulent in general. This turbulence could be the one produced by parametric instabilities themselves (in which event the theory we shall discuss below will be like a 'nonlinear' theory for parametric instabilities) or may arise as a result of some other agency promoting plasma fluctuations (e.g. violent mechanism of plasma production, some hydrodynamic instabilities etc.) The presence of background turbulence can significantly modify the parametric coupling process. As an example, turbulence produces random density fluctuations in the background plasma which may then influence the propagation characteristics of interacting waves and hence the parametric coupling process itself. An important problem, that is perhaps of great relevance to laser fusion, is therefore, the investigation of parametric coupling processes in a plasma with background turbulence.

In this paper we carry out a detailed investigation of the effect of quasi-static random density fluctuations on the parametric coupling process. A preliminary investigation of this type was reported in [6]. For simplicity of presentation we shall discuss in detail only the problem of stimulated scattering of an intense electromagnetic wave off electrostatic plasma waves i.e. the problem of stimulated Raman scattering. The analysis can be readily extended to other parametric processes of interest. We ignore the time-dependence of background density fluctuations i.e. we treat them as quasi-static. This is justified if the time period of their

variation is long compared to the time scales of interest to us e.g. growth time of instability, wave periods etc. The one-dimensional problem now effectively reduces to an investigation of stimulated scattering in a plasma in which the background density is an irregular function of the position variable x . The basic equation for this problem can be put in the form of a fourth order differential equation for the amplitude of electrostatic or electromagnetic wave with coefficients as random functions of x (since the problem is homogeneous in time, the time variable is eliminated by appropriate Fourier analysis). Methods developed by Keller and others [7,8] for the study of wave propagation in random media can now be adopted for an analysis of this equation. Assuming that the degree of random inhomogeneity i.e. $[n(x) - n_0]/n_0 = \epsilon(x)$ is small, one can derive an equation for the ensemble average of the wave amplitude $\langle E \rangle$ which is correct to terms of order $|\epsilon|^2$. The averaging is carried out over an ensemble of similarly prepared systems with the background density as a random function of position.

If the basic plasma is homogeneous, n_0 is independent of x , we can use the equation for $\langle E \rangle$ to derive the modified dispersion relation correct to order $|\epsilon|^2$. This dispersion relation involves the two point correlation function for the turbulence. Assuming a Gaussian correlation function one can solve the dispersion relation in two interesting limits i.e. the limits of long wavelength and short wavelength turbulence. In either case, there is a weakening of the instability but not complete quenching. An extension of the calculation to the full

3-dimensional problem shows that these results are unchanged.

If the basic plasma has a weak inhomogeneity i.e. n_0 is a function of x , the Fourier-transformed equation for $\langle E \rangle$ is a second order differential equation in k -space. This can be examined in the long wavelength limit for convective amplification and shows that the amplification factor is unaltered by the turbulence in the background.

The relevance of the results to the laser fusion problem is discussed in the last section.

2. HOMOGENEOUS PLASMA WITH RANDOM DENSITY FLUCTUATIONS

In this section we investigate the stimulated Raman scattering in a statistically homogeneous plasma with background random density fluctuations. The electron density in such a plasma may be written as

$$n = n_0 [1 + \epsilon(x)] \quad (1)$$

where n_0 is independent of x and $\epsilon(x)$ is a random function. $|\epsilon|$ is obviously a quantitative measure of the magnitude of random density fluctuation. ϵ is taken to be time independent because it is assumed that the time variation of background electron density is very slow compared to the time scales of interest to us. We first consider the one-dimensional problem in which a linearly polarized electromagnetic wave

$$\vec{E}_0 = E_0 \hat{e}_y \cos(k_0 x - \omega_0 t) \quad (2)$$

interacts with an electron plasma wave $E_p \exp(-i\omega t)$ and another electromagnetic wave $E_t \hat{e}_y \exp[-i(\omega - \omega_0)t]$ both propagating along the x -direction. We can Fourier analyse in

time because the plasma parameters are time-independent ; the component of E_t at frequency $\omega + \omega_0$ is off-resonant and hence unimportant for the problem considered here. The equations describing the stimulated scattering process in this situation may then be written [4]

$$-(\omega - \omega_0)^2 E_t + \omega_p^2(x) E_t - c^2 \frac{d^2 E_t}{dx^2} = \frac{e}{m} E_0 \frac{dE_l}{dx} \quad (3a)$$

$$-\omega^2 E_l + \omega_p^2(x) E_l - v_e^2 \frac{d^2 E_l}{dx^2} + v_e^2 \frac{d\epsilon}{dx} \frac{dE_l}{dx} = \frac{e}{m} \frac{\omega_{p0}^2}{\omega_0(\omega - \omega_0)} \frac{d}{dx} (E_0 E_t) \quad (3b)$$

where $\omega_p(x) = [4\pi n(x) e^2 / m]^{1/2}$ is the electron plasma frequency, $-e$ is the electronic charge, m the electron mass, c the velocity of light in vacuum and $v_e = (3T_e/m)^{1/2}$, the electron thermal speed.

Defining $\mathcal{D} \equiv (d/dx)$ and eliminating E_l from eqs. (3a,b) we obtain the operator equation

$$(M - V) E_t = 0 \quad (4)$$

where

$$M = \left\{ (\mathcal{D} - ik_0)^2 + k_1^2 \right\} (\mathcal{D}^2 + k_2^2) + \frac{v_e^2}{c^2} k_d^2 \mathcal{D}^2 \quad (5a)$$

is the non-random part of the differential operator,

$$V = k_d^2 \left\{ \epsilon(x) [(\mathcal{D} - ik_0)^2 + k_1^2] + 2 \frac{d\epsilon}{dx} (\mathcal{D} - ik_0) + \frac{d^2 \epsilon}{dx^2} \right\} \quad (5b)$$

is the random part of the differential operator (terms of order v_e^2/c^2 have been neglected),

$$v_0 = e E_0 / m \omega_0, \quad k_d = \omega_{p0} / v_e$$

is the inverse Debye length,

$$k_1^2 = [(\omega - \omega_0)^2 - \omega_{p0}^2] / c^2 \quad \text{and} \quad k_2^2 = (\omega^2 - \omega_{p0}^2) / v_e^2. \quad (6a,b)$$

Eqn.(4) is a fourth order differential equation with coefficients which are random functions of x .

Keller [7] has shown that from eqn.(4) one can obtain the following equation for $\langle E_t \rangle$ by taking an ensemble average over a very large number of similarly prepared systems :

$$\{ \langle [M-V]^{-1} \rangle \}^{-1} \langle E_t \rangle = 0 \quad (7)$$

Eqn.(7) may be rewritten in the form of an operator expansion

$$[M - \langle V \rangle - \langle V M^{-1} V \rangle + \langle V \rangle M^{-1} \langle V \rangle - \dots] \langle E_t \rangle = 0. \quad (8)$$

From eqn.(5b) we note that $\langle V \rangle = 0$, so that

$$[M - \langle V M^{-1} V \rangle] \langle E_t \rangle = 0 \quad (9)$$

is the correct equation upto order $|\epsilon^2|$. Taking Fourier transform of eqn.(9) gives the dispersion relation

$$M(k) = \frac{\Phi(k)}{2\pi} \int dk' \rho(k-k') \frac{\Phi(k')}{M(k')} \quad (10)$$

where

$$M(k) = (k^2 - k_2^2) [(k - k_0)^2 - k_1^2] - k^2 \lambda^2 \quad (11a)$$

$$\Phi(k) = [(k - k_0)^2 - k_1^2] k_d^2, \quad \lambda^2 = k_d^2 v_0^2 / c^2 \quad (11b, c)$$

and $\rho(k)$ is the Fourier transform of the two point correlation function $\langle \epsilon(x) \epsilon(x') \rangle$ for the random density fluctuations.

For statistically homogeneous turbulence with Gaussian correlations, one may write

$$\rho(k) = (2\pi)^{1/2} L_T |\epsilon^2| \exp[-k^2 L_T^2 / 2] \quad (12)$$

where L_T is the correlation length for the turbulence.

Eqn.(10) is our final dispersion relation. When the background turbulence is absent i.e. $|\epsilon^2| = 0$, its roots are given by

$$M(k) \equiv (k^2 - k_2^2) [(k - k_0)^2 - k_1^2] - k^2 \lambda^2 = 0 \quad (13)$$

This is the well known dispersion relation for stimulated Raman scattering in a homogeneous plasma [9]. As an example we look for solutions where the electron plasma wave vector $k \simeq k_2$ and the stimulated electromagnetic wave-vector $k - k_0 \simeq -k$,

where k_1, k_2 are defined by eqs.(6a,b). When $\omega \sim \omega_{p0} \ll \omega_0$, this is satisfied, for example, if $k_2 \sim \frac{\omega_{p0}}{c} \ll \omega_0$ and $k_1 \simeq k_0$. This corresponds to the case of stimulated forward-scatter in a homogeneous plasma. Taking $k = k_2 + iq$ and using the above matching conditions, one obtains:

$$q = (\gamma_0^2 / |V_1 V_2|)^{1/2} \quad (14)$$

where $\gamma_0 = (v_0/c)(\omega_{p0}\omega_0)^{1/2}$ is the homogeneous plasma temporal growth rate and V_1, V_2 are the magnitudes of the group velocities of the electromagnetic wave and plasma wave, respectively. Eqn.(14) is the well-known expression for inverse growth length in a homogeneous plasma. It can also be applied for the case of stimulated backscatter ($k_1 \simeq -k_0$, $k_2 \simeq 2k_0$) convective instability provided that one includes damping terms in eqn.(13) and does not exceed the threshold for absolute instability.

For the case when $|e| \neq 0$, we may solve the dispersion relation (10) iteratively. Since we are looking for solution $k \simeq k_2$, we may approximate $M(k')$ in the integral on right side as

$$M(k') \simeq (k'^2 - k_2^2) \Phi(k') / k_d^2 \quad (15)$$

The k' integration may now be readily carried out (in the limits $k_2 L_T \ll 1$ and $k_2 L_T \gg 1$) to give

$$M(k) = -\frac{i}{2} \frac{k_d^2}{k_2} \Phi(k) [\rho(k - k_2) - \rho(k + k_2)] \quad (16)$$

where $\rho(k)$ is defined by eqn.(12). We now investigate this equation in the two limiting cases :

(a) Long Scale-length Turbulence :

In this case $k_2 L_T \gg 1$ and since $k \simeq k_2$, only the first term in square parenthesis gives a significant

contribution. Taking $k \simeq k_2 + iq$ and the resonant matching condition for backscatter, for example, we get

$$-4k_0 k_2 q^2 - 4k_0^2 \lambda^2 = |\epsilon^2| L_T k_d^4 q (\pi/2)^{1/2} \quad (17)$$

where we have assumed that $q L_T \ll 1$.

This is a quadratic which can be solved for the inverse growth length q . The case of interest is when turbulence reduces q significantly i.e. $|q| \ll (\gamma_0^2 / |V_1 V_2|)^{1/2}$.

This corresponds to the neglect of first term and gives

$$q \simeq \left(\frac{8}{\pi}\right)^{1/2} \frac{\gamma_0^2}{\omega_{p0}^2} \left| \frac{V_2}{V_1} \right| \frac{1}{|\epsilon^2| L_T} \quad (18)$$

Eqn.(18) agrees with the equation derived in [6] on semi-quantitative grounds within a numerical factor. It may therefore be interpreted as before. Long scale-length turbulence is like a series of weak inhomogeneities which introduce random phase mismatch in the three interacting waves. If δK is the change in $K = k_0 - k_1 - k_2$

on encountering a typical plasma density fluctuation, then

$$\Delta \simeq | \langle (\delta K)^2 \rangle L_T | \quad (19)$$

is like a diffusion coefficient for phase change and the modified inverse growth length is given by

$$q \simeq \gamma_0^2 / |V_1 V_2| \Delta \quad (20)$$

For stimulated Raman scattering this agrees with the above equation (18).

(b) Short Scale-length Turbulence :

We next consider the limit $k_2 L_T \ll 1$. Taking $k \simeq k_2 + iq$ and expanding the exponential in the expression for $P(2k_2)$, one gets a quadratic equation for q as before. In the interesting region of significant reduction of inverse growth

length, one obtains

$$|\eta| \approx \left(\frac{8}{\pi}\right)^{1/2} \frac{\gamma_0^2}{\omega_p^2} \left|\frac{V_2}{V_1}\right| \frac{1}{|\epsilon^2| L_T} \left(\frac{1}{k_1^2 L_T^2}\right) \quad (21)$$

Since $k_2 L_T \ll 1$, this means that short scale-length turbulence is less effective in weakening the parametric amplification process. Since the correlation length for turbulence is shorter than wavelengths of interacting waves, the waves essentially suffer damping as a result of scattering off the density discontinuities. Eqn.(21) is similar to the corresponding expression derived in [6] on qualitative grounds.

We have extended the above calculations to the full 3-dimensional problem and find that the results are unaltered except for some geometrical numerical factors.

3. CONVECTIVE AMPLIFICATION IN INHOMOGENEOUS PLASMAS

We now consider a plasma in which the electron density is given by

$$n(x) = n_0 \left[1 + \frac{x}{L} + \epsilon(x) \right] \quad (22)$$

where L denotes the scale-length of the monotonic inhomogeneity and $\epsilon(x)$ is a random function as before. One can proceed as in Sec.2 and derive a fourth order differential equation for the amplitude of the electromagnetic wave. Following Keller's method one can write an equation for $\langle E_t \rangle$. Taking its Fourier transform now results in a second order differential equation in k -space, such as the one obtained in [10] for $\epsilon = 0$. This equation may be put in the form

$$\frac{d^2 \psi}{dk^2} + F^2(k) \psi = 0 \quad (23)$$

where

$$\psi(k) = \langle E_t(k) \rangle \exp \left[-\frac{iL}{2\omega_{p0}^2} \int dk \left\{ c^2 [(k-k_0)^2 - k_1^2] + v_e^2 (k^2 - k_2^2) \right\} \right] \quad (24)$$

$$= \langle E_t(k) \rangle \exp(-i\phi(k)) \quad (25)$$

$$F^2(k) = \frac{c^4 L^2}{\omega_{p0}^4} \left[\mu^2 k^2 - \frac{i}{L} \frac{\omega_{p0}^2}{c^2} (k-k_0) + \frac{i}{L} \frac{\omega_{p0}^2}{c^4} v_e^2 k + \right. \\ \left. + \frac{1}{4} \left\{ (k-k_0)^2 - \frac{v_e^2}{c^2} k^2 + Q \right\}^2 + i\eta^2 \left\{ (k-k_0)^2 - k_1^2 \right\} \right] \quad (26)$$

$$\mu^2 = v_0^2 \omega_{p0}^2 / c^4 = \lambda^2 v_e^2 / c^2 \quad (27)$$

$$Q = (2\omega\omega_0 - \omega_0^2) / c^2 \quad (28)$$

$$\text{and } \eta^2 = (\pi/2)^{1/2} (k_d^4 / k_2) |\epsilon^2| L_T. \quad (29)$$

Gaussian correlation for the turbulence has been assumed as before and the equation is written in the limit of long scalelength turbulence i.e. when $k_2 L_T \gg 1$.

For $|\epsilon| = 0$, eqs.(23-29) reduce to the inhomogeneous quiescent plasma equations discussed in [10]. The convective amplification factor in a quiescent plasma may be obtained from these equations in a simple manner. Taking $\omega \sim \omega_{p0} \ll \omega_0$ so that $Q \simeq -\omega_0^2/c^2 \simeq -k_0^2$ and neglecting the v_e^2/c^2 and L^{-1} terms inside the square parenthesis of eqn.(26) one may write (for $\eta = 0$)

$$F^2(k) \simeq \frac{c^4 L^2}{\omega_{p0}^4} \left[\mu^2 k^2 + \frac{1}{4} k^2 (k - 2k_0)^2 \right] \quad (30)$$

$$= (c^4 k^2 L^2 / \omega_{p0}^4) (k - K_1)(k - K_2) \quad (31)$$

where

$$K_{1,2} = 2k_0 \pm 2i\mu \quad (32)$$

are the turning points of $F^2(k)$. The WKB solution of eqn.(23) is given by

$$\psi(k) \sim F^{-1/2} \exp[-i \int F dk]$$

To obtain the amplification factor we use the saddle-point method to asymptotically evaluate the value of $\langle E_t(x) \rangle$ defined by

$$\langle E_t(x) \rangle = \int_c \psi(k) \exp[i\phi(k)] \exp(ikx) dk \quad (33)$$

where $\phi(k)$ is defined by eqn.(25). The amplification factor is the ratio of $\langle E_t(x) \rangle$ for backward and forward propagating waves at $x = -\infty$ and is given by

$$\left| \exp \left\{ \frac{i}{8} \frac{c^2 L}{\omega_{p0}^2} (K_1 + K_2)(K_1 - K_2)^2 \cosh^{-1} \left(\frac{2k_s - K_1 - K_2}{K_1 - K_2} \right) \right\} \right| \quad (34)$$

where $k_s \simeq (2\omega_{p0}^2 x / c^2 L)^{1/2}$ is the saddle point. Substituting for $K_{1,2}$ from eqn.(32) and using the identity $\cosh^{-1}(-ix) = i\frac{\pi}{2} - \sinh^{-1}(x)$, one obtains the amplification factor as

$$\exp \left(\pi \frac{c^2 \mu^2}{\omega_{p0}^2} k_0 L \right) \equiv \exp \left[\pi \frac{\gamma_0^2}{|K' V_1 V_2|} \right] \quad (35)$$

where $K' \simeq (\omega_{p0} / 2 V_2 L)$ is the space derivative of phase mismatch $K = k_0 - k_1 - k_2$ and γ_0, V_1, V_2 are defined just after eqn.(14). Eqn.(35) is the well-known expression for amplification factor. Note that in the above derivation its value is determined by $K_{1,2}$ viz. the roots of $F^2(k) = 0$.

For $|\epsilon^2| \neq 0$, eqn.(30) is replaced by

$$F^2(k) \simeq \frac{c^4 L^2}{\omega_{p0}^4} \left[\mu^2 k^2 + \frac{k^2}{4} (k - 2k_0)^2 + i\eta^2 k (k - 2k_0) \right] \quad (36)$$

This equation may be solved iteratively for the roots.

For $\eta^2 \gg \mu k_0$, the roots are given by

$$\begin{aligned} K_1 &= 2 + 2i\mu^2/\eta^2, & K_2 &= 2 - 2i\frac{\eta^2}{k_0^2} - 2i\frac{\mu^2}{\eta^2} \\ K_3 &= 0, & K_4 &= 2i\frac{\eta^2}{k_0^2} \end{aligned} \quad (37)$$

The amplification factor as obtained from eqn.(33) by saddle point integration is proportional to the x -independent part of

$$\text{Im} \left[\int^{k_s} dk \{ (k-k_1)(k-k_2)(k-k_3)(k-k_4) \}^{1/2} \right] \quad (38)$$

where $k_s \simeq (2\omega_{pe}^2 x / c^2 L)^{1/2}$ is the saddle point. Since $k_s \gg k_1, k_2, k_3, k_4$, the amplification factor can be readily obtained by an expansion procedure. One finds that the amplification factor is unaltered to order $|\epsilon^2|$. This result may be readily interpreted physically. The presence of long wave-length turbulence produces a broadening of the resonant matching condition $K=0$ of order the 'diffusion coefficient',

$$\Delta \simeq \langle (\delta K)^2 \rangle L_T \quad (39)$$

In an inhomogeneous plasma therefore, the three waves continue to interact in a broadened region of size

$$x \sim \langle (\delta K)^2 \rangle L_T / K' \quad (40)$$

This broadened region of interaction exactly compensates for the reduced growth discussed in Sec.2. We thus conclude that the amplification factor in a weakly inhomogeneous plasma is unaltered by the presence of long wave-length background turbulence.

4. DISCUSSION

We have investigated the influence of background random density fluctuations on the parametric coupling processes by taking SRS (Stimulated Raman Scattering) as an example. The results can readily be extended to other laser fusion decay instabilities like SBS (Stimulated Brillouin Scattering), decay into plasma waves and ion acoustic waves etc. We find

that density fluctuations can enhance the growth length but cannot quench the instability. The influence of density fluctuations is stronger on SRS than on SBS. This is because in SBS, density changes leave the ion wave propagation unmodified and only produce a weak influence on the high-frequency electromagnetic wave. Other forms of turbulence e.g. velocity fluctuations, may be more effective in modifying SBS.

In an inhomogeneous plasma with long scale-length turbulence, the amplification factor is found to be unaltered. Since the inverse growth length is reduced (see eqn.18), this suggests that the interaction region is broadened. Thus, if the size of the plasma is less than the broadened region of interaction, the amplification may be reduced and the stimulated scattering effectively quenched. The effect of short wavelength turbulence on amplification factor in an inhomogeneous plasma is yet to be studied.

We have not discussed the effect of time-dependence of background fluctuations on the instabilities. A more general study incorporating this feature will be presented in future.

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