AN ELEMENTARY ANALYSIS OF FUSION IMPLOSIONS

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ABSTRACT

The elementary, self-similar solutions of single fluid hydrodynamics have been frequently employed^(1,2) to study diverse aspects of fusion implosions. We have used them for the explicit purpose of attempting empirical correlations among various parameters measured in the laser-fusion experiments conducted by KMSF.

Some of the measurable parameters correlated by these model calculations are: D-T neutron yields; spectra and total energy radiated as X-rays; compression; initial target radii; thermal energy at peak compression; etc. In spite of the over-simplified nature of the physical model, the correlations are surprisingly good.

A two fluid version of the same model - the electron and ion fluids coupled by electrostatic forces - was also investigated perturbation-wise relative to the uncoupled solutions. It was found that substantial, relative to thermal, energy could be stored as plasma oscillations.

Here, I will attempt to address the issue of the importance of some two-fluid properties of a fully ionized plasma. I believe that this is a matter of some significance, since it is expected that some of the energy coupled into the plasma will be stored as electrostatic potential energy - and hence may be unavailable for hydrodynamic compression. This issue can be attacked analytically only in the context of a simple model - and then only with the aid of numerous (largely untestable) approximations.

The mathematical model used here is a slight generalization of the one employed before⁽¹⁾, i.e.,

$$\left(\frac{\partial}{\partial t} + \underline{w}^{E,I} \cdot \underline{\nabla}\right) n^{E,I} + n^{E,I} \underline{\nabla} \cdot \underline{w}^{E,I} = 0 \quad , \quad (1)$$

$$m_{E,I} n^{E,I} \left(\frac{\partial}{\partial t} + \underline{w}^{E,I} \cdot \underline{\nabla}\right) \underline{w}^{E,I} + \underline{\nabla} n^{E,I} \underline{\Phi}^{E,I} = -en^{E,I} \underline{E} \quad (2)$$

$$n^{E,I} \left(\frac{\partial}{\partial t} + \underline{w}^{E,I} \cdot \underline{\nabla}\right) \underline{\Phi}^{E,I} + \frac{2}{3} n^{E,I} \underline{\Phi}^{E,I} \underline{\nabla} \cdot \underline{w}^{E,I} = 0 \quad (3)$$

The notation implies a set of six, coupled equations describing particle, momentum, and energy balance for electrons and ions. The coupling, of course, occurs in the momentum equations (EQ. (2)) via the electric field, $\underline{E}$, which satisfies

$$\underline{\nabla} \cdot \underline{E} = 4\pi e(n^I - n^E) \quad , \quad (4)$$

for a hydrogen-like plasma. The approach to the solution of these equations is also familiar, i.e., assume

$$\underline{w}^E = \underline{r}(\dot{R}_E/R_E) \equiv \hat{r}v\dot{R}_E \quad (5)$$

$$\underline{w}^I = \underline{r}(\dot{R}_I/R_I) \equiv \hat{r}z\dot{R}_I \quad , \quad (6)$$

$$n^E = n^E(y,t) \quad , \quad \phi^E = \phi^E(y,t) \quad , \quad (7)$$

$$n^I = n^I(z,t) \quad , \quad \phi^I = \phi^I(z,t) \quad , \quad (8)$$

in which case Eqs. (1), (2), and (3) become,

$$\frac{\partial n^{E,I}}{\partial t} + 3n^{E,I} (\dot{R}_{E,I}/R_{E,I}) = 0 \quad , \quad (9)$$

$$\frac{\partial \phi^{E,I}}{\partial t} + 2\phi^{E,I} (\dot{R}_{E,I}/R_{E,I}) = 0 \quad , \quad (10)$$

$$M_E n^E \ddot{y} R_E + \frac{1}{R_E} \frac{\partial n^E \phi^E}{\partial y} = -en^E \hat{r} \cdot \underline{E} \quad , \quad (11)$$

$$M_I n^I \ddot{z} R_I + \frac{1}{R_I} \frac{\partial n^I \phi^I}{\partial z} = en^I \hat{r} \cdot \underline{E} \quad . \quad (12)$$

Equations (9) and (10) imply that (assuming separation of variables)

$$n^E = h^E(y)/R_E^3, \quad n^I = h^I(z)/R_I^3, \quad (13)$$

$$\Theta^E = f^E(y)\Theta^E(t), \quad \Theta^I = f^I(z)\Theta^I(t), \quad (14)$$

$$\Theta^E R_E^2 = \text{constant}, \quad (15)$$

$$\Theta^I R_I^2 = \text{constant}. \quad (16)$$

The functions, $h^E(y)$, $h^I(z)$, $f^E(y)$, and $f^I(z)$ are quite arbitrary except that the requirement of energy conservation imposes certain connections among them. Of course, the normalization

$$4\pi \int_0^1 y^2 dy h^E = 4\pi \int_0^1 z^2 dz h^I = N, \quad (17)$$

where N is the number of electrons or ions in the spheres of radius, R_E or R_I , is expected. The energy conservation requirement is

$$\frac{d}{dt}(T + D + V) = 0, \quad (18)$$

where

$$T = \frac{3}{2} \int d^3r (n^E \Theta^E + n^I \Theta^I) \quad (19)$$

is the thermal energy in the plasma,

$$D = \int d^3r \left[\frac{1}{2} M_E n^E (\underline{w}^E)^2 + \frac{1}{2} M_I n^I (\underline{w}^I)^2 \right] , \quad (20)$$

is the energy of directed motion, and

$$V = \int \frac{\rho(\underline{r}) d^3r \rho(\underline{r}') d^3r'}{|\underline{r} - \underline{r}'|} ,$$

$$\rho \equiv e(n^I - n^E) , \quad (21)$$

is the potential energy.

From here on we specialize to the case of constant densities, i.e.,

$$n^E = n^I = 3N/4\pi . \quad (22)$$

This assumption, together with Eq. (18), implies

$$\begin{aligned} f^E &= 1 - y^2 \\ f^I &= 1 - z^2 . \end{aligned} \quad (23)$$

As a consequence of Eq. (22), the electric field is expressible as

$$\underline{E} = Ne\underline{r} \left(\frac{1}{R_I^3} - \frac{1}{R_E^3} \right) , \quad (24)$$

and Eqs. (11) and (12) for momentum balance become,

$$\ddot{R}_E - \frac{\alpha_E^2}{R_E^3} = - \frac{Ne^2}{M_E} R_E \left(\frac{1}{R_I^3} - \frac{1}{R_E^3} \right) , \quad (25)$$

$$\ddot{R}_I - \frac{\alpha_I^2}{R_I^3} = \frac{Ne^2}{M_I} R_I \left(\frac{1}{R_I^3} - \frac{1}{R_E^3} \right) , \quad (26)$$

where

$$\alpha_{E,I}^2 \equiv 2R_{OE,I}^2 \alpha_O^{E,I} / M_{E,I} \equiv R_{OE,I}^2 v_{E,I}^2 \quad (27)$$

Eqs. (18), (19), (20), (21), (25), and (26) along with the initial conditions

$$\dot{R}_{E,I}(0) = 0 , \quad (28)$$

$$R_{E,I}(0) \equiv R_{OE,I} = R_O \quad (29)$$

imply that

$$\begin{aligned} & M_I \left[\dot{R}_I^2 + \alpha_I^2 \left(\frac{1}{R_I^2} - \frac{1}{R_O^2} \right) \right] \\ & + M_E \left[\dot{R}_E^2 + \alpha_E^2 \left(\frac{1}{R_E^2} - \frac{1}{R_O^2} \right) \right] \\ & + 4Ne^2 \left[\frac{1}{R_E} + \frac{R_E^2 - 3R_I^2}{2R_I^3} \right] = 0 , \end{aligned} \quad (30)$$

for $R_E \leq R_I$, and ($E \leftrightarrow I$) for $R_I \leq R_E$.

To proceed further, we linearize Eqs. (25) and (26) according to

$$R_E = R + r_E, \quad r_E/R \ll 1, \quad (31)$$

$$R_I = R + r_I, \quad r_I/R \ll 1, \quad (32)$$

and assume equal electron and ion temperatures to obtain in zeroth order

$$R^2 = R_O^2 [1 + (t/\tau)^2] = R_O^2 (1 + x^2), \quad (33)$$

$$\theta = \theta_O / (1 + x^2), \quad \tau^2 \equiv R_O^2 / v^2. \quad (34)$$

The first order, linear equations for r_E and r_I are

$$\ddot{r}_E + \frac{3\alpha^2}{R^4} r_E = - \frac{3Ne^2}{M_E} \frac{r_E - r_I}{R^3}, \quad (35)$$

$$\ddot{r}_I + \frac{3\alpha^2}{R^4} r_I = \frac{3Ne^2}{M_I} \frac{r_E - r_I}{R^3} \quad (36)$$

Defining $u = r_E + r_I$ and $\mu = r_E - r_I$ we find that

$$\ddot{u} + \frac{3\alpha^2}{R^4} u = \frac{3Ne^2}{R^3} \left(\frac{1}{M_I} - \frac{1}{M_E} \right) \mu, \quad (37)$$

$$\ddot{\mu} + \frac{3\alpha^2}{R^4} \mu = - \frac{3Ne^2}{R^3} \left(\frac{1}{M_I} + \frac{1}{M_E} \right) \mu. \quad (38)$$

Concentrating on Eq. (38) and exploiting $M_I \gg M_E$, we obtain

$$\ddot{\mu} + \left(\frac{3\alpha^2}{R^4} + \frac{3Ne^2/M_E}{R^3} \right) \mu = 0. \quad (39)$$

Introducing the dimensionless independent variable, x , (see Eq. (33)), Eq. (39) may be re-written as ($\dot{\mu} \equiv d\mu/dx$)

$$\ddot{\mu} + \frac{9 + b\sqrt{1+x^2}}{(1+x^2)^2} \mu = 0, \quad (40)$$

where we have defined

$$b \equiv \frac{3Ne^2\tau^2}{M_ER_O^3} = \frac{3Ne^2}{R_OM_EV^2} = \frac{3M_INe^2}{2R_O\Theta_OM_E}, \quad (41)$$

where we have identified $v^2 = 2\Theta_O/M_I$.

I do not know the solution for Eq. (40) as it stands. However, $x = t/\tau \approx 10^{12} t \sqrt{\Theta_O(\text{keV})}$, so it seems that we are only interested in this equation for $x \gg 1$, i.e., $t \gtrsim$ nanoseconds. Furthermore, $b \approx 4 \times 10^9 / \Theta_O(\text{keV}) \gg 1$ for $N = 10^{13}$ particles, so that Eq. (40) may reasonably be approximated by

$$\ddot{\mu} + \frac{b}{x^3} \mu = 0, \quad (42)$$

which has a solution

$$\mu = \sqrt{x} J_1 \left(2\sqrt{\frac{b}{x}} \right). \quad (43)$$

Again, in the interesting parameter range, $\sqrt{b/x} \gg 1$; so that for practical purposes we may use

$$\mu \approx \cos\left(2\sqrt{\frac{b}{x}}\right) . \quad (44)$$

I assume that $\mu \equiv r_E - r_I \approx r_E$, since the electrons oscillate relative to the ions as both surfaces move with the speed, R (see Eq. (33)). Thus Eq. (44) describes electron oscillations of the form

$$r_E \approx r \cos\left(2\sqrt{\frac{b}{x}}\right) . \quad (45)$$

To determine the amplitude of the oscillations, r , we return to Eq. (30) and linearize (keeping in mind that we have set $r_I = 0$).

We find that the maximum for r_E (hence, r) is

$$r_{E\max} = r \approx \left[\frac{4\phi_O R^3}{6Ne^2} \right]^{1/2} = \left[\frac{2\phi_O R_O^3 (1+x^2)^{3/2}}{3Ne^2} \right]^{1/2} \approx \left[\frac{2\phi_O R_O^3 x^3}{3Ne^2} \right]^{1/2} , \quad (46)$$

so that $r/R \approx [\phi_O R_O x / 6Ne^2]^{1/2}$ is satisfactorily small as required.

From Eq. (45) we may calculate dr_E/dt and then estimate the average kinetic energy, $\langle KE \rangle$, stored in these electron oscillations according to

$$\langle KE \rangle \approx \left\langle \left(\frac{r}{R} \right)^2 N \frac{M_E}{2} \left(\frac{dr_E}{dt} \right)^2 \right\rangle , \quad (47)$$

where $3rN/R$ is the maximum fraction of the electrons participating. Using Eqs. (45) and (46) we find that

$$\langle KE \rangle = 3N_0 \left(\frac{M_E}{M_I} \right) \left(\frac{\theta_o R_o}{3Ne^2} \right)^{3/2} b \left\langle \frac{(1+x^2)^{7/4}}{x^3} \sin^2 \left(2\sqrt{\frac{b}{x}} \right) \right\rangle. \quad (48)$$

Thus the ratio of the average kinetic energy of plasma oscillations (which is roughly equal to the stored electrostatic potential energy useless to the hydrodynamics) is given by

$$\langle KE \rangle / N\theta_o \approx 3 \left(\frac{M_E}{M_I} \right) \left(\frac{\theta_o R_o}{3Ne^2} \right)^{3/2} b \left\langle \frac{(1+x^2)^{7/4}}{x^3} \sin^2 \left(2\sqrt{\frac{b}{x}} \right) \right\rangle. \quad (49)$$

The average in Eq. (40) may be estimated as $x_o^{1/2}$, where x_o is an uncertain number, small compared to b . I will guess that $x_o \approx 10^{-2}b$, so that Eq. (40) becomes

$$\frac{\langle KE \rangle}{N\theta_o} = .3 \left(\frac{M_E}{M_I} \right) \left(\frac{\theta_o R_o b}{3Ne^2} \right)^{3/2} = .15 \left(\frac{M_I}{2M_E} \right)^{1/2}, \quad (50)$$

recalling the definition of b in Eq. (41). For DT fuel, Eq. (50) implies that $\langle KE \rangle / N\theta_o \approx 6$. If one were to argue that the fast ions are generated by these oscillating electrons, i.e., equate the energy of the fast ions with $\langle KE \rangle$; the present model calculation yields a result about one order of magnitude too small. Nominal KMSF shot data are 30 joules on target, about 4 joules absorbed - half of which show up in the fast ions, and about 50 millijoules thermal energy in the fuel. From such data one concludes that $\langle FI \rangle / N\theta_o = \langle KE \rangle / N\theta_o \approx 40$. However, an order of magnitude disagreement between measurement and the calculation here described may not be all that bad, considering the simple-mindedness of the analysis employed.

REFERENCES

1. R.K. Osborn and F.J. Mayer, Am. J. Phys. 44, 149 (1976). See also the several references cited therein.
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